Hölder Type Inequalities for Non-symmetric Non-square Matrices

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Suppose m, n, and k are positive integers, and let $\langle\cdot,\cdot\rangle$ denote standard inner product on the spaces $\mathbb{R}^p$, $p>0$. We show that if D is an $m\times n$ non-negative real matrix, and u and v are non-negative unit vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, then

$$\langle (DD^t)^k Du, v\rangle \;\geq\; \langle Du, v\rangle^{2k+1}, \tag{1}$$

with equality if and only if $\langle (DD^t)^k Du, v\rangle = 0$, or there exists $\alpha > 0$ such that $Du = \alpha v$ and $D^t v = \alpha u$. This inequality extends to non-symmetric non-square matrices a 1965 result of Blakley and Roy which asserts that if D is a non-negative $n\times n$ symmetric matrix, and $u\in\mathbb{R}^n$ is a non-negative unit vector, then

$$\langle D^k u, u\rangle \;\geq\; \langle Du, u\rangle^k, \tag{2}$$

with equality, when $k\geq 2$, if and only if $\langle D^k u, u\rangle = 0$, or there exists $\alpha > 0$ such that $Du = \alpha u$. The generality of the inequality (1) derives not only from the fact that D is not assumed to be symmetric or square, but from the fact that we admit two unit vectors u and v instead of the single unit vector u appearing in the inequality (2) of Blakley and Roy. We apply our result to verify the conjecture of A. Sidorenko in the non-symmetric case provided that the underlying graph is a path.