1. Suppose $G_1$ is a 3-regular connected simple graph on $n$ vertices.
   a. Find an example of $G_1$ that has a cut-edge and a 1-factor.
   b. Find an example of $G_1$ that has a cut-edge and no 1-factor. (Hint: Tutte’s Theorem with $|S| = 1$ may help you see the structure of such a graph.)
   c. Suppose that $G_1$ has a partition of its edges into sets of size 3, each element of which induces a path of length 3.
      i. In terms of $n$, how many paths are induced by this partition?
      ii. Show that $G_1$ has a 1-factor. (Hint: use (i), and consider choosing the middle edge in each path.)

2. Suppose $G_2$ is a 3-regular connected simple graph on $n$ vertices that has a 1-factor.
   a. Find an example of $G_2$ for which $\chi'(G_2) = 4$.
   b. Show that $G_2$ has a 2-factor.
   c. Show that $G_2$ has a partition of its edges into sets of size 3, each element of which induces a path of length 3. (Hint: Direct the edges in a suitable 2-factor to form directed cycles, and heed the suggestion in (1c(ii)).)

3. Let $G_3$ be a 2x-regular simple graph.
   a. Does $G_3$ necessarily have an Euler tour? Why or why not?
   b. Show that the edges of $G_3$ can be directed so that at each vertex $v$ in the resulting directed graph $D_3$, $d^+(v) = d^-(v)$.
   c. Form a bipartite graph $B_3$ on the vertex set $V(G_3) \times \{1,2\}$ by joining $(v,1)$ to $(w,2)$ if and only if there is an edge in $D_3$ directed from $v$ to $w$.
      i. Show that $B_3$ has a 1-factorization.
      ii. Use this to show that $G_3$ has a 2-factorization.

4. A subgraph of a graph is said to be an odd factor if it is both spanning and all its vertices have odd degree.
   a. By counting the number of edges in $G$ in terms of the degrees of its vertices, show that the number of vertices of odd degree in $G$ is even.
   b. Let $T$ be a tree with an even number of vertices. If $e$ is an edge in $T$ then $e$ is said to be an even edge if when deleted the two remaining components each have an even number of vertices, and is said to be an odd edge otherwise. Show that:
      i. Every odd factor of $T$ must contain every odd edge of $T$, and
      ii. Every odd factor of $T$ contains no even edges of $T$.
      (Hint: Use 4a.)
   c. Find necessary and sufficient conditions in terms of $n$ for a tree $T$ tree on $n$ vertices to have an odd factor.