PARALLEL SEARCH FOR INFORMATION IN CONTINUOUS TIME – OPTIMAL STOPPING AND GEOMETRY OF THE PDE

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Abstract. We consider the problem of a decision-maker searching for information on multiple alternatives when information is learned on all alternatives simultaneously. The decision-maker has a running cost of searching for information, and has to decide when to stop searching for information and choose one alternative. The expected payoff of each alternative evolves as a diffusion process when information is being learned. After establishing the well-posedness of the equation, we show that the optimal boundary where search is stopped (free boundary) is star-shaped, and present an asymptotic characterization of the value function and the free boundary. We prove that the distance between the free boundary and each point on the diagonal is logarithmic in the number of alternatives.

Key words : Brownian motion, diffusion processes, free boundary problem, optimal stopping, search theory.

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1. Introduction

In this paper we study the problem of *parallel search* in continuous time, which relies on the analysis of a partial differential equation (PDE) in the unbounded domain $\mathbb{R}^d$. In several situations a decision-maker (DM) has to decide how long to gain information on several alternatives simultaneously at a cost before stopping to make an adoption decision. An important aspect considered here, is that the DM gains information on all alternatives at the same time and cannot choose which alternative to gain information on—which we call parallel search. This can be, for instance, the case of a consumer trying to decide among several products in a product category and passively learning about the product category, or browsing through a web site that compares several products side by side. Another interesting application is a financial option based on several assets, where at the time of exercising the option, the investor decides which asset to take.

The idea goes as follows. If all the alternatives have a relatively low expected payoff, the DM may decide to stop the search, and not choose any of the alternatives. If two or more alternatives have a similar and sufficiently high expected payoff, the DM may decide to continue to search for information until finding out which alternative may be the best. If the expected payoff of the best alternative is clearly higher than that of the second best alternative, the DM may decide to stop the search process and choose the best alternative. To implement the preceding idea, consider $B^x = (B_1^{x_1}(t), \ldots, B_d^{x_d}(t))_{t\geq 0}$, a d-dimensional Brownian motion starting at $x = (x_1, \ldots, x_d)$. Each component of this Brownian motion could be the value of the alternative if the process is stopped. In the consumer learning application, this would be the value of that product at the time when the consumer makes

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the purchase decision. In the financial option application, this would be the value of the asset when the option is exercised. Let $\mathcal{T}$ be a suitable set of stopping times with respect to the natural filtration of B^x. Our goal is to study the following optimal stopping problem:

$$u(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\max\left\{B_1^{x_1}(\tau), \dots, B_d^{x_d}(\tau), 0\right\} - c\tau\right], \tag{1}$$

where $c > 0$ is the cost per unit time, considered as the cost of processing information when learning about different alternatives. We refer to Peskir and Shiryaev [36] for general background on optimal stopping problems.

Traditional research of optimal stopping problems generally fits into two categories: (i) Find the closed-form solution to the optimal stopping problem, see e.g. [1, 19, 21, 22, 36] and [43, Chapter 8]. These exactly solvable optimal stopping problems are often in low dimensions, namely for $d = 1$ or 2. (ii) Characterize the optimal stopping problem as a certain solution to a PDE, and study the regularity and convexity of the solution, see e.g. [2, 3, 12, 13, 35, 38]. The optimal stopping problem (1) does not have a closed-form solution. But it can be shown that it is the unique *viscosity solution* to the corresponding Bellman PDE. Now the question is: what can we say about this optimal stopping problem other than it is the solution to some PDE? The purpose of this paper is to explore the geometric properties of the particular optimal stopping problem (1), especially in the high dimensional regime as $d \to \infty$.

The analysis includes two parts: identifying the optimal stopping problem as the unique viscosity solution to the Bellman PDE (stochastic analysis), and studying the geometric properties of this PDE (analysis of PDE).

- Stochastic analysis: the approach to identify the optimal stopping problem (1) as the unique solution to the Bellman PDE (Theorem 2) is standard. Roughly speaking, we first show that the value function u is a viscosity solution to the corresponding Bellman PDE, and then we complete the argument by proving that this PDE has only one solution. But even in this routine part, there are some subtleties. In most existing literature, optimal stopping problems are considered for a fixed finite time horizon, or with an exponential discount factor (see e.g. [3, 20, 38, 39]). Our problem (1) is slightly different from these standard setups. In particular, the set $\mathcal{T}$ of stopping times should be properly defined so that the resulting value function u is indeed a viscosity solution to the Bellman PDE (Proposition 1). Moreover, a large volume of literature deals with bounded solutions to Bellman PDEs (see e.g. [15, 36]), and unbounded solutions are only considered for optimal stopping problems with an exponential discount factor (see e.g. [20, 38]). We prove by a comparison principle that under some assumptions, the Bellman PDE of a general optimal stopping problem without any discount factor has a unique viscosity solution that has sub-quadratic growth (Theorem 1). This result seems to be novel, see Remark 3.
- Analysis of PDE: the main contribution of this paper is to explore geometric properties of the optimal stopping problem (1) using the Bellman PDE. One important ingredient of the problem is that there is a free boundary where the process is optimal to stop, and this boundary is determined by the solution to the PDE. We prove some geometric properties of the free boundary: it is star-shaped with respect to the origin (Theorem 3); it is close to a columnal surface when all x_i's are large positive

(Proposition 3). Though it is not possible to derive a closed-form expression for the value function or the free boundary, we study the asymptotics of the value function as well as the free boundary when all x_i's are large positive. For $d = 2$, we provide fine estimates of the distance from the free boundary to the line $x_1 = x_2$ as $x_1, x_2 \to \infty$. More interestingly, in the high dimensional setting where d is large, we show that

$$C\sqrt{\ln d} \leq \text{distance from the free boundary to } \{x_1 = \cdots = x_d\} \leq C_\delta (\ln d)^\delta$$

as $x_1, \ldots, x_d \to \infty$ for any $\delta > 1$ (Theorem 4 and Theorem 5). The logarithmic scale is highly non-trivial, and relies on fine analysis of the Bellman PDE. Note that there is a gap between the lower bound and the upper bound. The question of the precise asymptotic distance as the dimension $d \to \infty$ remains open.

To the best of our knowledge, this is one of the few results concerning the asymptotic geometry of the optimal stopping problem in dimension $d \geq 2$. To prove the results, we rely heavily on the PDE machinery: viscosity solutions and the comparison principle. The Bellman PDE, also known as the obstacle problem, is widely studied (see e.g. [7, 8, 17, 37]). While previous works focused mostly on the local regularity of the free boundary, we study asymptotic properties of the free boundary as the dimension d increases to infinity with unbounded solutions and unbounded free boundaries. Since for many decision making problems the state process is high dimensional in the real-world application (see Section 2.1), we hope that this work may trigger further developments of optimal stopping problems in unbounded domains and in high dimensions.

To conclude the introduction, we provide a few references on the search theory. There is some literature on the case of learning about a single alternative in comparison to an outside option, see [5, 18, 33, 41]. The case with a single alternative can be traced back to the discrete costly sequential sampling in [46]. When there is more than one uncertain alternative the problem becomes more complex, as choosing one alternative means giving up potential high payoffs from other alternatives about which the decision maker could also learn more. This paper can then be seen as extending this literature to allow for more than one alternative, which requires the solution to a PDE. Another possibility, considered in [28], is that the DM can choose to search for information on one alternative at a time with alternatives having independent values. That simplifies the analysis because in each region in which one alternative is searched, the value function satisfies an ordinary differential equation on the state of that alternative, keeping the states of the other alternatives fixed. Here, the value function does not satisfy that property as the states of all alternatives move simultaneously. Consequently, the value function is determined by a partial differential equation (with free boundaries) on the state of any alternative. [9] considers which type of information to collect in a Poisson-type model, when the decision maker has to choose between two alternatives, with one and only one alternative having a high payoff. See also [23, 29, 34]. For problems where the DM gets rewards while learning, see [4, 30]. The literature on financial options based on multiple assets (rainbow options) is also related to this paper, see e.g. [6, 26, 42, 45].

Organization of the paper: We present the problem in Section 2, and characterize the optimal strategy by establishing the existence and uniqueness of the solution to the corresponding PDE. Section 3 shows that the optimal solution is star-shaped. Section 4 considers asymptotic results of the solution. We show that in general dimensions and in the region where all x_i are large, the free boundary is close to a columnal surface, and the distance from the free boundary to each point of $x_1 = \ldots = x_d > 0$ is logarithmic in d.

2. The problem setup and the PDE

2.1. **Motivating examples.** Consider a consumer and d products. For each $i = 1, \ldots, d$, the utility U_i of product i is the sum of the utility derived from each attribute of the product

$$U_i = x_i + \sum_{t=1}^{T} a_{it},$$

with x_i the consumer's initial expected utility, and a_{it} the utility of attribute t of product i which is uncertain to the consumer before search. It is also assumed that a_{it} are i.i.d. across t and i, and without loss of generality, $\mathbb{E}a_{it} = 0$. There is an outside option which is worth zero. Each time by paying a search cost c, the consumer checks one attribute a_{it} for all products $i = 1, \ldots, d$. The consumer decides when to stop searching and upon stopping which product to buy so as to maximize the expected utility. After checking t attributes, the consumer's conditional expected utility of product i is

$$X_i(t) = \mathbb{E}_t U_i = x_i + \sum_{s=1}^{t} a_{is}.$$

It is easily seen that $(X_i(t),\, t = 0, 1, \ldots)$ is a random walk which scales to Brownian motion $(B_i^{x_i}(t),\, t \geq 0)$ in the limit. The problem of the consumer is to decide when to stop the process, and then choose the best alternative.

Another example is concerned with Bayesian learning. Assume that the true value of the alternatives $\widehat{X}(t)$ follows the dynamics $d\widehat{X}(t) = dB(t)$, where $(B(t),\, t \geq 0)$ is a d-dimensional Brownian motion, and that the signal $S(t)$ of $\widehat{X}(t)$ is a d-dimension vector governed by

$$dS(t) = \widehat{X}(t)dt + y\, d\widetilde{B}(t),$$

where $(\widetilde{B}(t),\, t \geq 0)$ is a d-dimensional Brownian motion independent of $(B(t),\, t \geq 0)$, and y is a diagonal matrix, with general element on the diagonal y_{ii}. Also assume that the prior of $\widehat{X}(0)$ is a normal with mean $\widehat{X}(0)$ and variance-covariance $\widehat{\rho}(0)^2$, with $\widehat{\rho}(0)$ being a diagonal matrix, with general element in the diagonal $\widehat{\rho}_{ii}(0)$. Then the posterior mean $X(t)$ of $\widehat{X}(t)$ follows

$$X_i(t) = \widehat{\rho}_{ii}(t)^2 / y_{ii}\, d\overline{B}_i(t) \quad \text{for } 1 \leq i \leq d,$$

where $(\overline{B}(t),\, t \geq 0)$ is a d-dimensional Brownian motion, and $d\widehat{\rho}_{ii}(t)^2/dt = 1 - \widehat{\rho}_{ii}(t)^4/y_{ii}^2$ for $1 \leq i \leq d$. As $t \to \infty$, we get $\widehat{\rho}(t)^2 \to y$. So if $\widehat{\rho}(0)^2 = y$, we have $dX_{ii}(t) = d\overline{B}_i(t)$ for $1 \leq i \leq d$, and the analysis that follows would be done on the process $(X(t),\, t \geq 0)$. In both problems, it boils down to solving the optimal stopping problem (1).

2.2. **General framework and viscosity solutions.** We start with the general framework of the optimal stopping problem (1). Let $\Omega \subset \mathbb{R}^d$ be an open domain, and $\partial\Omega$ be its boundary (then $\partial\Omega = \emptyset$ if $\Omega = \mathbb{R}^d$) and $\overline{\Omega} := \Omega \cup \partial\Omega$. Consider the following stochastic differential equation (SDE):

$$dX^x(t) = b(X^x(t))dt + \sigma(X^x(t))dB(t), \quad X^x(0) = x \in \Omega, \tag{2}$$

where $(B(t),\, t \geq 0)$ is a standard d-dimensional Brownian motion starting from 0, and $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfy the Lipschitz condition. That is, there exists $C > 0$ such that

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq C|x - y|, \quad x, y \in \mathbb{R}^d \times \mathbb{R}^d. \tag{3}$$

It is well known that under this condition, the SDE (2) has a strong solution which is pathwise unique. See e.g. [27, Section 5.2] for background and further developments of the strong solution to SDEs. In the context of parallel search, the vector $X^x(t)$ has as each element i the expected utility obtained if the DM were to decide to stop the search process at time t and choose alternative i. Moreover, let $\mathcal{L}$ be the infinitesimal generator of the SDE (2). That is,

$$\mathcal{L}h(x) = \sum_{i=1}^{d} b_i(x)\frac{\partial h}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma(x)\sigma(x)^T)_{ij}\frac{\partial^2 h}{\partial x_i \partial x_j}(x),$$

for any suitably smooth test function $h : \mathbb{R}^n \to \mathbb{R}$.

Let $T_\Omega := \inf\{t > 0 : X^x(t) \notin \Omega\}$ be the exit time of $(X^x(t),\, t \geq 0)$ from Ω, and let

$$J_x(\tau) := \mathbb{E}\left[\int_0^{\tau\wedge T_\Omega} f(X^x(s))ds + g(X^x(\tau \wedge T_\Omega))\right], \tag{4}$$

where τ is a stopping time, and f, g are suitably smooth reward functions, e.g. continuous functions with polynomial growth, or simply Lipschitz continuous functions. We are interested in the value function

$$u(x) = \sup_{\tau\in\mathcal{T}} J_x(\tau), \tag{5}$$

where $\mathcal{T}$ is a suitable set of stopping times. Heuristics from dynamic programming suggest that the value function u "solve" the following Bellman PDE:

$$\min\{-\mathcal{L}u - f, u - g\} = 0, \quad x \in \Omega, \tag{6}$$

with the boundary condition

$$u = g, \quad x \in \partial\Omega. \tag{7}$$

Equations (6)–(7) are known as an *obstacle problem*, or a *variational inequality*, see e.g. [16, 31]. There are two regimes:

$$-\mathcal{L}u = f \text{ when } u > g, \quad \text{ and } -\mathcal{L}u \geq f \text{ when } u = g.$$

The set $\{x \in \Omega : \, u(x) = g(x)\}$ is called the *contact set*, or *coincidence set*.

It was shown in [3, Chapter 3] that if the value function u defined by (5) is sufficiently smooth (e.g. of class $\mathcal{C}^2$) and some additional conditions on $b(\cdot), \sigma(\cdot), f(\cdot), g(\cdot)$ are satisfied, then u is characterized as the unique solution to the variational inequality (6)–(7). However, the value function u is not necessarily smooth enough for the variational inequality to be defined in the classical sense, see e.g. [13, 44]. Hence, we need a weaker notion of viscosity solutions to characterize the value function u. Below we state the definition of viscosity solutions, and we refer to [10, 11, 24, 25] for this notion.

Definition 1. *Let u be a continuous function on $\overline{\Omega}$ and $x^0 \in \Omega$.*

(1) We say that $-\mathcal{L}u \leq f$ at x^0 in the viscosity sense if for any $\varphi \in \mathcal{C}^2(\Omega)$ which touches u at x^0 from above, we have $-(\mathcal{L}\varphi)(x^0) \leq f(x^0)$. We call u a subsolution to (6) *if $-\mathcal{L}u \leq f$ in the viscosity sense at all points of Ω where $u - g > 0$.*

(2) *We say that $-\mathcal{L}u \geq f$ at x^0 in the viscosity sense if for any $\varphi \in \mathcal{C}^2(\Omega)$ which touches u at x^0 from below, we have $-(\mathcal{L}\varphi)(x^0) \geq f(x^0)$. We call u a supersolution to* (6) *if $u - g \geq 0$ in Ω, and $-\mathcal{L}u \geq f$ in the viscosity sense in Ω.*

(3) *We call u a viscosity solution to* (6) *if and only if u is both a subsolution and a supersolution to* (6)*. We call u a viscosity solution to* (6)*–*(7) *if and only if u is a viscosity solution to* (6)*, and u satisfies* (7) *at all boundary points.*

The connection between the value function of the optimal stopping problem (5) and viscosity solutions to the variational inequality (6)–(7) was established by Øksendal and Reikvam [35]. Their result is recorded in the following lemma.

Lemma 1. *Assume that $b(\cdot)$ and $\sigma(\cdot)$ satisfy the Lipschitz condition* (3)*. Also assume that*

(i) *The boundary $\partial\Omega$ is regular for the process $(X^x(t),\, t \geq 0)$ in the sense that $T_\Omega = \inf\{t > 0 : X^x(t) \notin \Omega\} = 0$ almost surely.*

(ii) *$f(\cdot)$ is a continuous function on $\overline{\Omega}$, and $\mathbb{E}\int_0^{\tau\wedge T_\Omega} |f(X^x(s))|ds < \infty$ for all $x \in \overline{\Omega}$ and τ stopping times.*

(iii) *$g(\cdot)$ is a continuous function on $\overline{\Omega}$, and the family $\{g(X^x(\tau)),\, \tau$ stopping time and $\tau \leq T_\Omega\}$ is uniformly integrable for all $x \in \overline{\Omega}$.*

(iv) *The value function u is continuous on $\overline{\Omega}$.*

Then u is a viscosity solution to the variational inequality (6)*–*(7)*.*

Note that Lemma 1 holds for both bounded and unbounded domains. For our purposes, we take $\Omega = \mathbb{R}^d$. As mentioned in the introduction, the optimal stopping problem in an unbounded domain has been considered either for a fixed finite time horizon, or/and with an exponential discount factor [3, 20]. Here we treat this problem in a slightly different manner by letting $\mathcal{T} := \{\tau \text{ stopping time} : \mathbb{E}e^{3C^2\tau} < \infty\}$, where C is given by (3) so that the value function may well be defined. The following proposition gives simple sufficient conditions on $b(\cdot), \sigma(\cdot), f(\cdot), g(\cdot)$ for Lemma 1 to hold, and provides an estimate of the value function u.

Proposition 1. *Assume that $b(\cdot), \sigma(\cdot)$ satisfy the Lipschitz condition* (3)*, and are bounded so there exist $K_1, K_2 > 0$ such that*

$$\sum_{i=1}^d \sup_{x\in\mathbb{R}^d} |b_i(x)| < K_1 \quad \textit{and} \quad \max_{i,j} \sup_{x\in\mathbb{R}^d} |\sigma_{ij}(x)| < K_2.$$

Also assume that $f(\cdot)$ is bounded and Lipschitz continuous, and $g(\cdot)$ is nonnegative and Lipschitz continuous so there exists $a > 0$ such that

$$0 \leq g(x) \leq a\left(1 + \sum_{i=1}^d |x_i|\right) \quad \textit{for all } x \in \mathbb{R}^d.$$

Let u be the value function defined by (5)*, with $\Omega = \mathbb{R}^d$ and $\mathcal{T} := \{\tau$ stopping time $: \mathbb{E}e^{3C^2\tau} < \infty\}$, where C is the Lipschitz constant given by* (3)*. Then u is a viscosity solution to the Bellman PDE* (6)*. Furthermore, if there exists $c > K_1$ such that $\sup_{x\in\mathbb{R}^d} f(x) \leq -c$, then we have for some $\gamma > 0$ (independent of d),*

$$g(x) \leq u(x) \leq a\sum_{i=1}^d |x_i| + \gamma d^4 \quad \textit{for all } x \in \mathbb{R}^d. \tag{8}$$

Proof. The first part of the proposition is a consequence of Lemma 1, and we need to check all the conditions therein. For $\Omega = \mathbb{R}^d$, the condition (i) is automatically satisfied. Since f is bounded and $\mathbb{E}\tau < \infty$, we have

$$\mathbb{E}\int_0^{\tau\wedge T_\Omega} |f(X^x(s))|ds \le \sup_{x\in\mathbb{R}^d} |f(x)| \cdot \mathbb{E}\tau < \infty.$$

Combined with the fact that f is Lipschitz continuous, we verify the condition (ii). For the condition (iii), note that

$$\begin{aligned} g(X^x(\tau)) &\le a\left(1+\sum_{i=1}^d |X_i^{x_i}(\tau)|\right) \\ &\le a\left(1+\sum_{i=1}^d |x_i| + \sum_{i=1}^d \left|\int_0^\tau b_i(X^x(s))ds + \sum_{j=1}^d \int_0^\tau \sigma_{ij}(X^x(s))dB_j(s)\right|\right) \qquad (9) \\ &\le a\left(1+\sum_{i=1}^d |x_i| + \sum_{i=1}^d \int_0^\tau |b_i(X^x(s))|ds + \sum_{i=1}^d\sum_{j=1}^d \left|\int_0^\tau \sigma_{ij}(X^x(s))dB_j(s)\right|\right). \end{aligned}$$

Therefore, we have

$$g^2(X^x(\tau)) \le A\left[\left(1+\sum_{i=1}^d |x_i|\right)^2 + \tau\sum_{i=1}^d \sup_{x\in\mathbb{R}^d} b_i^2(x) + \sum_{i=1}^d\sum_{j=1}^d \left|\int_0^\tau \sigma_{ij}(X^x(s))dB_j(s)\right|^2\right],$$

for some $A > 0$, which implies that

$$\begin{aligned} \mathbb{E}g^2(X^x(\tau)) &\le A\left[\left(1+\sum_{i=1}^d |x_i|\right)^2 + \mathbb{E}\tau\sum_{i=1}^d \sup_{x\in\mathbb{R}^d} b_i^2(x) + \sum_{i=1}^d\sum_{j=1}^d \mathbb{E}\left|\int_0^\tau \sigma_{ij}(X^x(s))dB_j(s)\right|^2\right] \\ &\le A\left[\left(1+\sum_{i=1}^d |x_i|\right)^2 + \mathbb{E}\tau\sum_{i=1}^d \sup_{x\in\mathbb{R}^d} b_i^2(x) + \mathbb{E}\tau\sum_{i=1}^d\sum_{j=1}^d \sup_{x\in\mathbb{R}^d} \sigma_{ij}^2(x)\right] < \infty. \end{aligned}$$

This proves the uniform integrability of the family $\{g(X^x(\tau)), \tau$ stopping time and $\mathbb{E}e^{3C^2\tau} < \infty\}$. Finally, by the Lipschitz continuity of f, g, there exists $M > 0$ such that for $x, y \in \mathbb{R}^d$,

$$J_x(\tau) - J_y(\tau) \le M\left(\mathbb{E}\int_0^\tau |X^x(s) - X^y(s)|ds + \mathbb{E}|X^x(\tau) - X^y(\tau)|\right).$$

By a classical Grönwall argument, we have $\mathbb{E}|X^x(s) - X^y(s)| \le e^{3C^2 s}|x-y|$ (see e.g. [32] or [15, Chapter II, Theorem 10.1]). Since $\mathbb{E}e^{3C^2\tau} < \infty$, the value function u is Lipschitz continuous, and hence the condition (iv).

Now we prove the second part of the proposition. By taking $\tau = 0$, we get $u(x) \ge g(x)$. By (9), we have

$$\mathbb{E}g(X^x(\tau)) \le a\left[1+\sum_{i=1}^d |x_i| + \mathbb{E}\left(\sum_{i=1}^d \int_0^\tau |b_i(X^x(s))|ds\right) + \sum_{i=1}^d\sum_{j=1}^d \mathbb{E}\left|\int_0^\tau \sigma_{ij}(X^x(s))dB_j(s)\right|\right].$$

Note that

$$\mathbb{E}\left(\sum_{i=1}^{d}\int_0^\tau |b_i(X^x(s))|ds\right) \le K_1\mathbb{E}\tau,$$

and according to the Burkholder-Davis-Gundy inequality (see [40, Chapter IV]), there exists $L > 0$ such that for any $1 \le i, j \le n$,

$$\mathbb{E}\left|\int_0^\tau \sigma_{ij}(X^x(s))dB_j(s)\right| \le L\mathbb{E}\left[\left(\int_0^\tau \sigma_{ij}^2(X^x(s))ds\right)^{\frac{1}{2}}\right] \le LK_2\sqrt{\mathbb{E}\tau}.$$

Since $\sup_{x\in\mathbb{R}^d} f(x) \le -c$ with $c > K_1$, we get

$$u(x) \le a\left[1+\sum_{i=1}^{d}|x_i| + \sup_{\tau\in\mathcal{T}}\left\{(K_1-c)\mathbb{E}\tau + Ld^2K_2\sqrt{\mathbb{E}\tau}\right\}\right] \le a\left(1+\sum_{i=1}^{d}|x_i| + \frac{L^2d^4K_2^2}{4(c-K_1)}\right),$$

which yields (8). □

Remark 1. Proposition 1 shows that under suitable conditions on $b(\cdot), \sigma(\cdot), f(\cdot), g(\cdot)$, the value function u is a viscosity solution to the variational inequality (6), and grows at most linearly as $|x| \to \infty$. The condition $\mathbb{E}e^{3C^2\tau} < \infty$ for stopping times is to assure the (Lipschitz) continuity of the value function. This is reminiscent of the optimal stopping problem with an exponential discount factor whose exponent is assumed to be sufficiently large, see e.g. [20]. Also note that neither Lemma 1 nor Proposition 1 is meant to be optimal, and it is possible that the value function u is a viscosity solution to the variational inequality under weaker conditions on $b(\cdot), \sigma(\cdot), f(\cdot), g(\cdot)$. Our ultimate goal is to study the geometry properties of the optimal stopping problem (1), and Proposition 1 is adequate for this purpose. Finding minimal conditions to characterize a general optimal stopping problem by the variational inequality is interesting on its own, but we will not pursue this direction.

Lemma 1 or Proposition 1 gives one direction of the program: the value function is a viscosity solution to the variational inequality. To fully characterize the value function, it requires to prove the converse; that is, the variational inequality has a unique viscosity solution which is at most of linear growth. Under the assumptions of Proposition 1, it is well known [20, 38] that for the optimal stopping problem with an exponential discount factor, the associated variational inequality has a unique viscosity solution which is uniformly continuous, or at most of linear growth. However, these results do not imply directly the uniqueness of the solution to the variational inequality (6). In the next subsection, we will show that under fairly general assumptions, the variational inequality of the optimal stopping problem (5) has a unique viscosity solution which has sub-quadratic growth.

2.3. **Uniqueness of the PDE solution.** In this subsection, we prove that the PDE (6)–(7) has a unique viscosity solution among all functions that have sub-quadratic growth, i.e. $\lim_{|x|\to\infty}|u(x)|/|x|^2 = 0$, under the assumption that $\sup_{x\in\Omega}|b(x)||x| < \infty$. This is stronger than the required uniqueness of the viscosity solution that has at most linear growth. We employ the viscosity solution approach.

Theorem 1 (Comparison principle)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open. Let the assumptions in Proposition 1 except $c > K_1$ hold, and further assume that $\sup_{x\in\Omega}|b(x)||x| < \infty$. Let u_1 and u_2 be* *respectively a subsolution and a supersolution to* (6) *in* Ω. *If*

$$\limsup_{|x|\to\infty, x\in\Omega} \frac{\max\{u_1(x), -u_2(x)\}}{|x|^2} \le 0, \tag{10}$$

(with the convention $\sup_\emptyset = -\infty$*), and* $u_2 \ge u_1$ *on* $\partial\Omega$*, then we have* $u_2 \ge u_1$ *in* Ω.

Proof. Assume by contradiction that for some $x' \in \Omega$, we have $\delta' := u_1(x') - u_2(x') > 0$. For any $\varepsilon \in (0,1)$, (10) yields that $u_1(x) - \varepsilon|x|^2$ and $-u_2(x) - \frac{\varepsilon}{2}|x|^2$ converge to $-\infty$ as $(\Omega \ni)x \to \infty$. Therefore, we can take $\varepsilon > 0$ to be small enough such that for any $\kappa \in (\frac{1}{2}, 1)$,

$$\begin{aligned}\delta := \sup_{x\in\Omega} u_1(x) - (1-\varepsilon^\kappa)u_2(x) - 2\varepsilon|x|^2 &\ge u_1(x') - (1-\varepsilon^\kappa)u_2(x') - 2\varepsilon\,|x'|^2 \\ &\ge \frac{1}{2}(u_1(x') - u_2(x')) = \frac{\delta'}{2}.\end{aligned} \tag{11}$$

Write $u_2^\varepsilon := (1-\varepsilon^\kappa)u_2$. By the assumption that $f \le -c$ in $\mathbb{R}^d$, and that u_2 is a supersolution to (6), we obtain

$$-\mathcal{L}u_2^\varepsilon \ge (1-\varepsilon^\kappa)f \ge f + c\varepsilon^\kappa \quad \text{and} \quad u_2^\varepsilon \ge (1-\varepsilon^\kappa)g \quad \text{in } \Omega \tag{12}$$

(in the viscosity sense).

Now for a fixed ε and for any $\alpha \ge 1$, define $\Phi_\alpha : \Omega \times \Omega \to \mathbb{R}$ by

$$\Phi_\alpha(x,y) := u_1(x) - u_2^\varepsilon(y) - \varepsilon(|x|^2 + |y|^2) - \alpha|x-y|^2.$$

Using (10), u_1 and u_2 are continuous, and $u_2 \ge u_1$ on $\partial\Omega$, we can find $x_\alpha, y_\alpha \in \Omega$ such that

$$\Phi_\alpha(x_\alpha, y_\alpha) = \sup_{(x,y)\in\Omega^2} \Phi_\alpha(x,y) \ge \sup_{x\in\Omega} u_1(x) - u_2^\varepsilon(x) - 2\varepsilon|x|^2 \ge \frac{\delta'}{2}, \tag{13}$$

where the last inequality is due to (11). Since (10) yields that x_α, y_α are uniformly bounded for all $\alpha \ge 1$, it is clear that $|x_\alpha - y_\alpha| \to 0$ as $\alpha \to \infty$. Moreover, it follows that

$$|x_\alpha - y_\alpha| + \alpha|x_\alpha - y_\alpha|^2 \to 0 \quad \text{as } \alpha \to \infty. \tag{14}$$

Now, due to (12) and that u_1 is a subsolution to (6), the Crandall-Ishii lemma [10, Theorem 3.2] yields that there are symmetric $d \times d$-matrices X_α, Y_α satisfying the following:

$$-6\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \le 6\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{15}$$

and

$$\begin{aligned}&\min\left\{-\frac{1}{2}\mathrm{Tr}(\sigma(x_\alpha)\sigma(x_\alpha)^T X_\alpha) - b(x_\alpha)\cdot p_\alpha - f(x_\alpha),\, u_1(x_\alpha) - g(x_\alpha)\right\} \le 0 \\ &\quad \le \min\left\{-\frac{1}{2}\mathrm{Tr}(\sigma(y_\alpha)\sigma(y_\alpha)^T Y_\alpha) - b(y_\alpha)\cdot q_\alpha - f(y_\alpha) - c\varepsilon^\kappa,\, u_2^\varepsilon(y_\alpha) - (1-\varepsilon^\kappa)g(y_\alpha)\right\},\end{aligned} \tag{16}$$

where

$$p_\alpha := 2\alpha(x_\alpha - y_\alpha) + 2\varepsilon x_\alpha, \quad q_\alpha := 2\alpha(x_\alpha - y_\alpha) - 2\varepsilon y_\alpha. \tag{17}$$

First if $u_1(x_\alpha) \le g(x_\alpha)$, then (12), (13), the definition of Φ_α, and $u_2^\varepsilon(y_\alpha) \ge (1-\varepsilon^\kappa)g(y_\alpha)$ yield

$$g(x_\alpha) - (1-\varepsilon^\kappa)g(y_\alpha) - \varepsilon(|x_\alpha|^2 + |y_\alpha|^2) \ge \Phi_\alpha(x_\alpha, y_\alpha) \ge \frac{\delta'}{2}. \tag{18}$$

Using (14), $g(x) \le a(1+\sum_{i=1}^d |x_i|)$, and g is Lipschitz continuous, we get the right-hand side of the above is no more than

$$\mathrm{Lip}(g)|x_\alpha - y_\alpha| + \varepsilon^\kappa a \left(1 + \sum_{i=1}^d |(y_\alpha)_i|\right) - \varepsilon |y_\alpha|^2 \le \mathrm{Lip}(g)|x_\alpha - y_\alpha| + C_a \varepsilon^{2\kappa-1},$$

due to $\varepsilon^\kappa s - \varepsilon s^2 \le \frac{1}{4}\varepsilon^{2\kappa-1}$ for all $s \ge 0$. However notice that $\delta' > 0$ is independent of ε, α. Therefore, if taking $\varepsilon > 0$ such that $C_a \varepsilon^{2\kappa-1} < \frac{\delta'}{4}$ (since $\kappa \in (\frac{1}{2}, 1)$) and then α to be sufficiently large, we get a contradiction from (18).

Next if $u_1(x_\alpha) > g(x_\alpha)$, by (16), we have

$$-\frac{1}{2}\mathrm{Tr}(\sigma(x_\alpha)\sigma(x_\alpha)^T X_\alpha) - b(x_\alpha)\cdot p_\alpha - f(x_\alpha) \le 0. \tag{19}$$

Multiplying the rightmost inequality in (15) by the following nonnegative symmetric matrix

$$\begin{pmatrix} \sigma(x_\alpha)\sigma(x_\alpha)^T & \sigma(y_\alpha)\sigma(x_\alpha)^T \\ \sigma(x_\alpha)\sigma(y_\alpha)^T & \sigma(y_\alpha)\sigma(y_\alpha)^T \end{pmatrix}$$

and taking traces yields

$$\mathrm{Tr}(\sigma(x_\alpha)\sigma(x_\alpha)^T X_\alpha) - \mathrm{Tr}(\sigma(y_\alpha)\sigma(y_\alpha)^T Y_\alpha) \le 6\alpha\, \mathrm{Tr}((\sigma(x_\alpha) - \sigma(y_\alpha))(\sigma(x_\alpha)^T - \sigma(y_\alpha)^T)).$$

Recall that σ is Lipschitz continuous, and so we get for some $C > 0$,

$$\mathrm{Tr}(\sigma(x_\alpha)\sigma(x_\alpha)^T X_\alpha) - \mathrm{Tr}(\sigma(y_\alpha)\sigma(y_\alpha)^T Y_\alpha) \le C\alpha|x_\alpha - y_\alpha|^2. \tag{20}$$

Using $|b(x)||x| \le C_0$ for some $C_0 > 0$ by the assumption, the Lipschitz continuity of b and f, and (17) yield for some $C > 0$ (independent of ε, α) that

$$\begin{aligned} f(x_\alpha) - f(y_\alpha) &\le C|x_\alpha - y_\alpha|, \\ b(x_\alpha)\cdot p_\alpha - b(y_\alpha)\cdot q_\alpha &\le 2\alpha(b(x_\alpha) - b(y_\alpha))\cdot(x_\alpha - y_\alpha) + 2\varepsilon b(x_\alpha)\cdot x_\alpha + 2\varepsilon b(y_\alpha)\cdot y_\alpha \\ &\le C\alpha|x_\alpha - y_\alpha|^2 + 4\varepsilon C_0. \end{aligned} \tag{21}$$

It then follows from (16), (19), (20), and (21) that

$$\begin{aligned} c\varepsilon^\kappa &\le \frac{1}{2}\mathrm{Tr}(\sigma(x_\alpha)\sigma(x_\alpha)^T X_\alpha) + b(x_\alpha)\cdot p_\alpha + f(x_\alpha) \\ &\qquad - \frac{1}{2}\mathrm{Tr}(\sigma(y_\alpha)\sigma(y_\alpha)^T Y_\alpha) - b(y_\alpha)\cdot q_\alpha - f(y_\alpha) \\ &\le C\left(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|\right) + 4C_0\varepsilon. \end{aligned} \tag{22}$$

However, if we take $\varepsilon > 0$ such that $\varepsilon^{1-\kappa} \le \frac{c}{8C_0}$ (which can be done since $\kappa < 1$), and then α to be large enough, we get a contradiction by (14). The conclusion follows. □

Remark 2. If the condition (10) and $\sup_{x\in\Omega} |b(x)||x| < \infty$ are replaced by

$$\limsup_{|x|\to\infty, x\in\Omega} \frac{\max\{u_1(x), -u_2(x)\}}{|x|^\beta} < \infty \quad \text{and} \quad \sup_{x\in\Omega} |b(x)||x|^\gamma < \infty \tag{23}$$

for some $\beta \in [1, 2)$ and $\gamma \in (\frac{\beta}{2}, 1)$, then the conclusion of the theorem still holds. To see this, we need to modify the proof after (20). We use (23) and (13) to get for some $C > 0$,

$$\varepsilon(|x_\alpha|^2 + |y_\alpha|^2) \le u_1(x_\alpha) - u_2^\varepsilon(y_\alpha) \le C(|x_\alpha|^\beta + |y_\alpha|^\beta + 1),$$

and so there exists a possibly different $C > 0$ such that for all $\varepsilon \in (0,1)$ and $\alpha > 0$,

$$\varepsilon(|x_\alpha|^{1-\gamma} + |y_\alpha|^{1-\gamma}) \le C\varepsilon^{\frac{1-\beta+\gamma}{2-\beta}} + C\varepsilon \le C\varepsilon^{\frac{1-\beta+\gamma}{2-\beta}}$$

where we used $\frac{1-\beta+\gamma}{2-\beta} < 1$ by $\gamma < 1$. Then (23) and the Lipschitz continuity of b yield

$$\begin{aligned} b(x_\alpha)\cdot p_\alpha - b(y_\alpha)\cdot q_\alpha &\le 2\alpha\,\mathrm{Lip}(b)|x_\alpha - y_\alpha|^2 + 2\varepsilon(|b(x_\alpha)||x_\alpha| + |b(y_\alpha)||y_\alpha|) \\ &\le C\alpha|x_\alpha - y_\alpha|^2 + C\varepsilon^{\frac{1-\beta+\gamma}{2-\beta}}. \end{aligned}$$

Using this in place of the second estimate in (21), (22) can be replaced by

$$c\varepsilon^\kappa \le C\left(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|\right) + C\varepsilon^{\frac{1-\beta+\gamma}{2-\beta}}.$$

Since $\frac{1-\beta+\gamma}{2-\beta} > \frac{1}{2}$ by $\gamma > \frac{\beta}{2}$, if taking $\kappa \in (\frac{1}{2}, \frac{1-\beta+\gamma}{2-\beta})$, then we get a contradiction as before by taking ε to be sufficiently small and then α to be large.

Remark 3. Consider the optimal stopping problem with an exponential discount factor. If the operator is given by

$$\mathcal{L}u(x) = -\rho(x)u(x) + \sum_{i=1}^d b_i(x)\frac{\partial u}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^d (\sigma(x)\sigma(x)^T)_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(x),$$

with $\rho(\cdot)$ Lipschitz continuous and $\inf_{x\in\mathbb{R}^d}\rho(x) > 0$, then the comparison principle holds without the condition that $\sup_{x\in\Omega}|b(x)||x| < \infty$. See e.g. [38] for details.

With this theorem, we are able to compare sub and supersolutions in $\mathbb{R}^d$. The following theorem provides a complete characterization of the optimal stopping problem (5) under gerenal conditions.

Theorem 2. *Let the assumptions in Proposition 1 hold, and assume that* $\sup_{x\in\Omega}|b(x)||x| < \infty$. *Then the value function u from Proposition 1 is the the unique viscosity solution to the Bellman PDE* (6) *with $\Omega = \mathbb{R}^d$ among all functions that have sub-quadratic growth at infinity. Moreover, there exists $\gamma > 0$ independent of d such that $g(x) \le u(x) \le g(x) + \gamma d^4$ for all $x \in \mathbb{R}^d$.*

Proof. Proposition 1 implies that the value function u is a viscosity solution to the PDE (6) which has at most linear growth. Then Theorem 1 yields that u is the unique viscosity solution to (6) among all functions satisfying $\lim_{|x|\to\infty}|u(x)|/|x|^2 = 0$. □

Once the value function u is determined, then we construct an optimal strategy τ^* by $J_x(\tau^*) = u(x)$. To be more precise, starting at a position $x \in \{u > g\}$, the process will continue until it enters the contact set:

$$\tau^* = \inf\{t > 0 : B^x(t) \in \{u = g\}\}.$$

3. Star-shapedness of the Free Boundary

From now on, we specialize to the optimal stopping problem (1) with $\Omega = \mathbb{R}^d$, $b = 0$, $\sigma = I_d$, and

$$f(x) = -c \quad \text{and} \quad g(x) = \max\{x_1, \dots, x_d, 0\}.$$

The corresponding variational inequality is

$$\min\left\{-\frac{1}{2}\Delta u + c, u - g\right\} = 0, \tag{24}$$

where Δ is the Laplacian operator $\sum_{i=1}^d \partial^2/\partial x_i^2$. By Theorem 2, the value function u of the problem (1) is the unique viscosity solution to the Bellman PDE (6) among all functions that have at most linear growth at infinity. The equation (24) is the focus of analysis in the remaining of the paper.

The *free boundary* of u is defined as $\Gamma(u) := \partial\{x \,|\, u(x) > g(x)\}$. Several regularity results of $\Gamma(u)$ can be found in [8]. In this paper, we are interested in the global geometric property of $\Gamma(u)$. In this section we prove the star-shapedness.

Let $S \subseteq \mathbb{R}^d$. We say that S is *star-shaped* if there exists a point $z \in S$ such that for each point $s \in S$ the line segment connecting s and z lies entirely within S. We say that the free boundary $\Gamma(u)$ is star-shaped with respect to the origin if the set $\{u > g\}$ is star-shaped with $z = 0$. The star-shapedness property of a set rules out holes in the set.

Theorem 3. *Let u be a solution to* (24) *in $\mathbb{R}^d$. The free boundary $\Gamma(u)$ is star-shaped with respect to the origin.*

Proof. To prove that the set $\{u > g\}$ is star-shaped with respect to the origin, we only need to show that if $u(x) = g(x)$ for some $x \in \mathbb{R}^d$, then $u(tx) = g(tx)$ holds for all $t \geq 1$.

Let $v(x) := \frac{1}{t}u(tx)$. We first show that v is a subsolution to (24). In fact, for any $x \in \mathbb{R}^d$, if $v(x) > g(x)$ then

$$u(tx) > tg(x) = t\max\{x_1, \ldots, x_d, 0\} = g(tx).$$

Thus, using that u is a solution to (24) yields

$$-\frac{1}{2}(\Delta u)(tx) \leq -c \quad \text{in the viscosity sense.} \tag{25}$$

To show that $-\frac{1}{2}\Delta v(x) \leq -c$ in the viscosity sense, take any $\varphi \in \mathcal{C}^2$ that touches v at x from above. Then $\varphi^t(\cdot) := t\varphi(\cdot/t)$ touches u at tx from above. It follows from (25) that

$$-\frac{1}{2}(\Delta\varphi^t)(tx) = -\frac{1}{2t}\Delta\varphi(x) \leq -c,$$

which implies $-\frac{1}{2}\Delta\varphi(x) \leq -tc \leq -c$. Therefore $-\frac{1}{2}\Delta v(x) \leq -c$ in the viscosity sense. So we conclude that v is a subsolution, and it follows from the comparison principle that $v \leq u$. Now take $x \in \mathbb{R}^d$ such that $u(x) = g(x)$. From the order of u and v, we get

$$u(tx) \leq tu(x) = tg(x) = g(tx).$$

On the other hand, $u(tx) \geq g(tx)$ by definition, so we must have $u(tx) = g(tx)$. □

Figure 1 shows the continuation and stopping regions, as well as the free boundary separating them in the case of $d = 2$. The figure illustrates the star-shapedness of the free boundaries.

As shown by Figure 1, the optimal search strategy is quite intuitive—roughly speaking, the DM should stop searching and adopt alternative i if and only if x_i is relatively high compared with x_j and the outside option of 0, and she should stop searching and adopt the outside option when both x_1 and x_2 are relatively low. When x_j is relatively low, the DM will continue to search on the two alternatives if and only if x_i is near 0, so as to make a clear

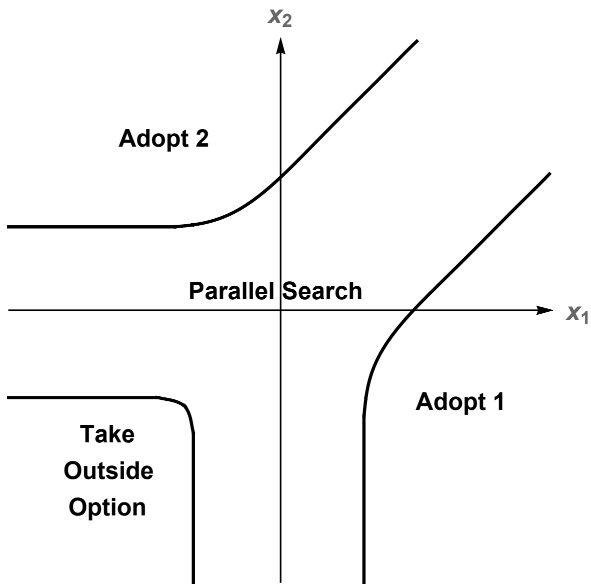

FIGURE 1. Optimal parallel search strategy in two dimensions.

distinction between alternative i and the outside option. When both x_1 and x_2 are relatively high, the DM will continue to search if and only if x_1 and x_2 are close to each other, so as to to make a clear distinction between the two alternatives 1 and 2.

4. ASYMPTOTICS

In this section, we study the asymptotics of the free boundary. We provide a detailed analysis for the case with $d = 2$. For general $d \geq 2$, we show that the distance from the free boundary to each point of $x_1 = \ldots = x_d > 0$ is logarithmic in d, and the free boundary is close to a columnal surface in the region where all x_i are equally large.

4.1. **Dimension of $d = 2$.** Writing $x = (x_1, x_2)$, the PDE (24) specializes to

$$\min\left\{-\frac{1}{2}\Delta u + c, u - \max\{x_1, x_2, 0\}\right\} = 0. \tag{26}$$

The PDE (26) does not have an explicit solution, so it is natural to ask about the properties of the solution, in particular those of free boundaries. There are three interesting regimes of asymptotic behavior:

(1) $x_1 \to 0$ and $x_2 \to -\infty$,
(2) $x_1 \to -\infty$ and $x_2 \to 0$,
(3) $x_1 = x_2 \to \infty$.

The cases (1) and (2) boil down to the search problem of one alternative, since the other alternative has large negative value and thus loses the competition to its counterpart. A classical smooth-pasting technique [28] shows that the distance of the free boundaries to x-axis (resp. y-axis) at $-\infty$ is $\frac{1}{4c}$, as illustrated in Figure 1. The case (3) is subtle, since the values of two products are close so there is a competitive search. One interesting question is to determine the distance from the free boundary to the line $x_1 = x_2$ at infinity.

We first prove a lower bound on the free boundary $\Gamma(u)$ in the region $x_1 + x_2 \geq 0$ by the following lemma.

Lemma 2 (Lower bound of the free boundary)**.** *Let*

$$\eta_c(x_1, x_2) := \begin{cases} \dfrac{x_1 + x_2}{2} + \dfrac{c|x_1 - x_2|^2}{2} + \dfrac{1}{8c} & \text{for } |x_1 - x_2| \leq \frac{1}{2c} \\ \dfrac{x_1 + x_2}{2} + \dfrac{|x_1 - x_2|}{2} & \text{for } |x_1 - x_2| > \frac{1}{2c}. \end{cases} \tag{27}$$

Then η_c is a subsolution to (26)*, and so $u \geq \eta_c$ in $\mathbb{R}^2$. In particular, we have*

$$\Gamma(u) \cap \{x_1 + x_2 \geq 0\} \subseteq \left\{ |x_1 - x_2| \geq \frac{1}{2c} \right\}.$$

Proof. It is direct to check that η_c satisfies (in the sense of viscosity)

$$\min\left\{ -\frac{1}{2}\Delta\eta_c + c, \eta_c - \max\{x_1, x_2\} \right\} = 0 \quad \text{in } \mathbb{R}^2.$$

Thus η_c is a subsolution to (26) and the comparison principle yields $u \geq \eta_c$ in $\mathbb{R}^2$. Notice that $\eta_c = \max\{x_1, x_2\} = g$ whenever $|x_1 - x_2| \geq \frac{1}{2c}$ and $x_1 + x_2 \geq 0$. Therefore, $\Gamma(u) \cap \{x_1 + x_2 \geq 0\}$ lies inside $\{|x_1 - x_2| \geq \frac{1}{2c}\}$. ☐

The last conclusion of Lemma 2 can be viewed as a "lower bound" of $\Gamma(u)$. To obtain an "upper bound", we need the following technical lemma.

Lemma 3. *For any $\varepsilon \in (0, c]$, let*

$$\varphi_\varepsilon(x_1, x_2) := \frac{1}{4c} \max\{1 - \sqrt{2c\varepsilon}(x_1 + x_2), 0\}^2 + \eta_{c-\varepsilon}(x_1, x_2).$$

Then we have $u(x_1, x_2) \leq \varphi_\varepsilon(x_1, x_2)$ for all x_1, x_2 such that $x_1 + x_2 \geq 0$.

Proof. For any $\theta > 0$, define

$$\psi_\theta(t) := \begin{cases} 0 & \text{for } t \leq -\frac{1}{4\theta} \\ \theta\left(t + \dfrac{1}{4\theta}\right)^2 & \text{for } t \in (-\frac{1}{4\theta}, \frac{1}{4\theta}) \\ t & \text{for } t \geq \frac{1}{4\theta}, \end{cases} \tag{28}$$

which is a mollification of $\max\{t, 0\}$. Then $\Psi(x_1, x_2) := \psi_{c/2}(x_1) + \psi_{c/2}(x_2)$ satisfies $\Delta\Psi \leq 2c$ and $g \leq \Psi$ in the viscosity sense. Therefore, Ψ is a supersolution to (26) and so $\Psi \geq u$. Note that $\psi_{c/2}(t) \leq \max\{t, 0\} + \frac{1}{8c}$, and so

$$g(x_1, x_2) \leq u(x_1, x_2) \leq \max\{x_1, 0\} + \max\{x_2, 0\} + \frac{1}{4c}.$$

Now we compare φ_ε with u in the half plane $x_1 + x_2 \geq 0$. On the boundary $(x_1 + x_2 = 0)$,

$$\varphi_\varepsilon(x_1, x_2) = \frac{1}{4c} + \eta_{c-\varepsilon}(x_1, x_2) \geq \frac{1}{4c} + g(x_1, x_2) \geq u(x_1, x_2).$$

It is not hard to see that $\varphi_\varepsilon \in \mathcal{C}^1$ and $\varphi_\varepsilon(x_1, x_2) \geq g(x_1, x_2)$ for all $x_1 + x_2 \geq 0$. Moreover if $|x_1 - x_2| \leq \frac{1}{2(c-\varepsilon)}$, direct computation yields $\Delta\varphi_\varepsilon \leq \frac{(2\sqrt{c\varepsilon})^2}{2c} + 2(c - \varepsilon) \leq 2c$, and if

$|x_1 - x_2| \geq \frac{1}{2(c-\varepsilon)}$, we again get $\Delta\varphi_\varepsilon \leq 2\varepsilon \leq 2c$ in the viscosity sense. Finally, since both u and φ_ε have linear growth at infinity, applying Theorem 1 with $\Omega := \{x_1 + x_2 > 0\}$ yields $u \leq \varphi_\varepsilon$ in $\{x_1 + x_2 \geq 0\}$. □

Lemmas 2 and 3 yield $u(x_1, x_2) \to \eta_c(x_1, x_2)$ as $x_1 + x_2 \to \infty$. Based on this we can obtain a quantitative description about the convergence of $\Gamma(u)$ to $\Gamma(\eta_c) = \{|x_1 - x_2| = \frac{1}{2c}\}$ as $x_1 + x_2 \to \infty$. We denote the Hausdorff distance between the two free boundaries in the region where $x_1 + x_2 \geq T$ as

$$d_H(T) := \max\left\{ \sup_{x \in \Gamma(u)\,\&\, x_1+x_2 \geq T} d(x, \Gamma(\eta_c)), \sup_{x \in \Gamma(\eta_c)\,\&\, x_1+x_2 \geq T} d(x, \Gamma(u)) \right\}.$$

Proposition 2 (Upper bound of the free boundary)**.** *For the 2-dimension problem, we have*

$$d_H(T) \leq (2\sqrt{2}c^3T^2)^{-1} \quad \textit{for all } T \geq c^{-1}.$$

Proof. Lemma 3 yields for all $x_1 + x_2 \geq 0$ that $u\,(x_1, x_2) \leq \varphi_\varepsilon(x_1, x_2)$. Thus if $x_1 + x_2 \geq \frac{1}{\sqrt{2c\varepsilon}}$, we have $u \leq \varphi_\varepsilon = \eta_{c-\varepsilon}$. This, combining with $u \geq \eta_c$ by Lemma 2, shows if $x_1 + x_2 \geq \frac{1}{\sqrt{2c\varepsilon}}$,

$$\begin{aligned}
&u(x_1, x_2) \geq \max\{x_1, x_2, 0\} = g(x_1, x_2) && \text{if } |x_1 - x_2| \geq \frac{1}{2c};\\
&u(x_1, x_2) \geq \frac{c|x_1 - x_2|^2}{2} + \frac{1}{8c} + \frac{x_1 + x_2}{2} > g(x_1, x_2) && \text{if } |x_1 - x_2| < \frac{1}{2c};\\
&u(x_1, x_2) \leq g(x_1, x_2) && \text{if } |x_1 - x_2| \geq \frac{1}{2(c-\varepsilon)}.
\end{aligned}$$

Hence for those (x_1, x_2) we have

$$\begin{aligned}
&u(x_1, x_2) > g(x_1, x_2) \quad && \text{if } |x_1 - x_2| < \frac{1}{2c},\\
&u(x_1, x_2) = g(x_1, x_2) \quad && \text{if } |x_1 - x_2| \geq \frac{1}{2(c-\varepsilon)},
\end{aligned}$$

which implies that the coordinates of $\Gamma(u)$ satisfies $|x_1 - x_2| \in (\frac{1}{2c}, \frac{1}{2(c-\varepsilon)})$ whenever $T := x_1 + x_2 \geq \frac{1}{\sqrt{2c\varepsilon}}$. Now take $\varepsilon := \frac{1}{2cT^2}$. Then for $T \geq c^{-1}$ (so $\varepsilon \leq \frac{c}{2}$), we conclude

$$d_H(T) \leq \frac{1}{\sqrt{2}}\left(\frac{1}{2(c-\varepsilon)} - \frac{1}{2c}\right) = \frac{\varepsilon}{2\sqrt{2}c(c-\varepsilon)} \leq \frac{\varepsilon}{\sqrt{2}c^2} = \frac{1}{2\sqrt{2}c^3T^2}.$$

□

As for the limit $\eta_c(x_1 + x_2, x_1 - x_2)$, the distance of the free boundary to the line of $x_1 = x_2$ is $\frac{1}{2^{3/2}c}$. Note that $\frac{1}{2^{3/2}c} > \frac{1}{4c}$ which is the distance of the free boundaries to x or y-axis at $-\infty$. This means that the search region is larger in case of competition. In other words, people have larger tolerance for search if two products are as good as each other.

4.2. **General dimension.** Now we study the asymptotic properties of the free boundary in the general dimension d. We provide an "upper bound" and a "lower bound" of the distance between the free boundary and each point on the positive diagonal $x_1 = x_2 = \ldots = x_d > 0$. The two estimates show that such distance is logarithmic in d.

For the upper bound, we will show that the free boundary can not be too far away from the set where g is not smooth i.e.

$$N(0) := \Big\{x \in \mathbb{R}^d \,|\, x_i = x_j = \max\{x_1, \ldots, x_d\} > 0 \text{ for some } i \neq j, \\ \text{or } x_i = \max\{x_1, \ldots, x_d\} = 0 \text{ for some } i\Big\}.$$

We denote the d-dimensional cube centered at the origin with side length $2r$ as

$$\mathcal{R}_r := \Big\{x \in \mathbb{R}^d \,|\, x_i \in (-r, r) \text{ for all } i\Big\}.$$

For any $x \in \mathbb{R}^d$, denote $\mathcal{R}_r(x) := x + \mathcal{R}_r$. We write $N(r) := \mathcal{N}_r(N(0))$ as the $\mathcal{R}_r$-neighbourhood of $N(0)$. It turns out that using this $\mathcal{R}_r$-neighbourhood is more convenient. The following theorem presents the first main result of this section.

Theorem 4. *Let $d \geq 2$, and u be the solution to (24) in $\mathbb{R}^d$. For any $\delta > 1$, there exists $C_\delta > 0$ depending only on δ such that $u(x) = g(x)$ for $x \notin N(c^{-1}C_\delta(\ln d)^\delta)$. This implies that $\Gamma(u) \subseteq N(c^{-1}C_\delta(\ln d)^\delta)$.*

Proof. The proof is based on a barrier argument, and we use a mollification of g as the barrier. Let $\alpha := \frac{1}{\delta-1} > 0$. Consider a smooth symmetric modifier $\varphi : \mathbb{R} \to [0, \infty)$ as

$$\varphi(r) := \begin{cases} c_\alpha e^{-\frac{1}{(1-r^2)^\alpha}} & \text{if } |r| \leq 1, \\ 0 & \text{if } |r| > 1, \end{cases}$$

and the numerical constant $c_\alpha := (\int_\mathbb{R} e^{-\frac{1}{(1-r^2)^\alpha}} dr)^{-1}$ ensures normalization. For some $h > 0$ to be determined, set $\varphi_h(r) := h\varphi(hr)$, and then $\Phi_h(x) := \Pi_{i=1}^d \varphi_h(x_i)$. We claim that

$$g_h := \Phi_h * g = \int_{\mathbb{R}^d} \Phi_h(x-y)g(y)dy \tag{29}$$

is a supersolution to (24) if h is small enough. Since g_h is smooth, it suffices to show that $g_h \geq g$ and $\Delta g_h \leq 2c$ in $\mathbb{R}^d$ in the classical sense.

Note that $\Phi_h * x_i = x_i$ for each $i \in \{1, \ldots, d\}$ (since Φ_h is symmetric in x_i-direction), and $g = \max\{x_1, \ldots, x_d, 0\}$. Therefore we have $g_h = \Phi_h * g \geq g$. Next to compute Δg_h, since $\Delta g_h = \nabla\Phi_h * \nabla g$, and $|\nabla g| \leq 1$, we obtain

$$\begin{aligned} \Delta g_h(x) &\leq \int_{\mathbb{R}^d} \max\Big\{ |\varphi_h'(x_i)| \prod_{j\neq i} \varphi_h(x_j),\, 1 \leq i \leq d\Big\} dx \\ &= h \int_{[-1,1]^d} \max\Big\{ |\varphi'(x_i)| \prod_{j\neq i} \varphi(x_j),\, 1 \leq i \leq d\Big\} dx =: hA. \end{aligned} \tag{30}$$

Note that for any $\varepsilon \in (0, 1)$, if $r \in [-1+\varepsilon, 1-\varepsilon]$, direct computation yields

$$|\varphi'(r)| = \frac{2\alpha|r|}{(1-r^2)^{\alpha+1}} c_\alpha e^{-\frac{1}{(1-r^2)^\alpha}} \leq 2\alpha\varepsilon^{-\alpha-1}\varphi(r).$$

Hence, by using the symmetry of φ and $\int_{[-1,1]} \varphi(r)dr = 1$, we obtain

$$\begin{aligned}
A &\leq \int_{[-1+\varepsilon,1-\varepsilon]^d} \max\Big\{ \left|\varphi'(x_i)\right| \prod_{j\neq i} \varphi(x_j),\, 1 \leq i \leq d\Big\} dx \\
&\qquad\qquad + 2d \int_{-1}^{-1+\varepsilon} \varphi'(x_1)dx_1 \int_{[-1,1]^{d-1}} \prod_{2\leq i\leq d} \varphi(x_i)dx_2...dx_d \\
&\leq 2\alpha\varepsilon^{-\alpha-1} \int_{[-1,1]^d} \prod_{i=1}^{d} \varphi(x_i)dx + 2d\varphi(-1+\varepsilon) \int_{[-1,1]^{d-1}} \prod_{2\leq i\leq d} \varphi(x_i)dx_2...dx_d \\
&\leq 2\alpha\varepsilon^{-\alpha-1} + 2dc_\alpha e^{-\frac{1}{(2\varepsilon)^\alpha}}.
\end{aligned}$$

Now picking $\varepsilon := \frac{1}{2}(\ln d)^{-\frac{1}{\alpha}}$ yields $A \leq C_\alpha(\ln d)^{\frac{1+\alpha}{\alpha}} = C_\alpha(\ln d)^\delta$ for some $C_\alpha > 0$ depending only on α. It follows from (30) that $\Delta g_h \leq 2c$ if we pick $h := \frac{2c}{C_\alpha(\ln d)^\delta}$. Thus, with this choice of h, we conclude that g_h is a supersolution and so the comparison principle yields $g_h \geq u$.

Now for any $x \notin N(h^{-1})$ with $h = \frac{2c}{C_\alpha(\ln d)^\delta}$, we have $g_h(x) = g(x)$ because $g(y) = y_i$ for some $i \in \{1,\dots,d\}$ for all $y \in \mathcal{R}_{1/h}(x)$, and Φ_h is supported in $\mathcal{R}_{1/h}$. Hence outside $N(h^{-1})$, we must have $g_h = g \geq u$. However $u \geq g$ by the equation, and so we get $u = g$ for $x \notin N(h^{-1})$, which implies the conclusion with $C_\delta := \frac{C_\alpha}{2}$. $\square$

Since the ray $\{x_1 = x_2 = \ldots = x_d > 0\} \subseteq N(0)$, the theorem implies that for any point on the ray (say $x_0 = (a,\dots,a)$ for some $a > 0$), $\Gamma(u)$ can not be more than $\sim R_\delta := c^{-1}C_\delta(\ln d)^\delta$ away from it. Actually, we can take $x := (a + 2R_\delta, a, \dots, a)$, and then clearly $x \in N(R_\delta)^c$. Thus according to Theorem 4, $u(x) = g(x)$ and so there must be a free boundary point that resides on the line segment connecting x and x_0 (since $u(x_0) > g(x_0)$ by Theorem 5 below). Therefore the distance from x to $\Gamma(u)$ is smaller than $\sqrt{2}R_\delta$ for $d \geq 2$ (since $\left|x - (a + \frac{2R_\delta}{d}, \dots, a + \frac{2R_\delta}{d})\right| = 2R_\delta\sqrt{1 - 1/d}$).

Below we show the counterpart of Theorem 4 and conclude that the distance between $\Gamma(u)$ and each point on the ray $x_1 = \ldots = x_d > 0$ is logarithmic in d. For each $r > 0$, we denote the $\mathcal{R}_r$-neighbourhood of the ray as

$$N_0(r) := \mathcal{R}_r(\{x \in \mathbb{R}^d \,|\, x_1 = x_2 = \ldots = x_d > 0\}).$$

Theorem 5. *Let u be the solution to* (24) *in $\mathbb{R}^d$. If d is sufficiently large, we have $u > g$ in $N_0(\frac{1}{3}\sqrt{\ln d})$. This implies that for these d, $\Gamma(u) \subseteq (N_0(\frac{1}{3}\sqrt{\ln d}))^c$.*

Proof. Fix any $a > 0$, and then any $x \in \mathcal{R}_{\sqrt{\ln d}/3}((a,\dots,a))$ (hence $|x_i - a| < \sqrt{\ln d}/3$). To prove the theorem, it suffices to show that $u(x) > g(x)$ if d is sufficiently large (independent of a, x). It is well-known [14, Exercise 3.2.3] that for $Z_1,\dots,Z_d$ i.i.d. $\mathcal{N}(0,1)$, $\mathbb{E}\left[\max_{1\leq i\leq d} Z_i\right] \sim \sqrt{2\ln d}$ as $d \to \infty$. This implies that for all d large enough,

$$\begin{aligned}
\mathbb{E}\left[\max\left(B_1^{x_1}(1),\dots,B_d^{x_d}(1),0\right)\right] &\geq \mathbb{E}\left[\max\left(B_1(1),\dots,B_d(1)\right)\right] + \min\{x_1,\dots,x_d\} \\
&\geq \frac{2}{3}\sqrt{\ln d} + a.
\end{aligned}$$

Now, further assuming that $d > \exp(9c^2)$, we obtain

$$u(x) \geq \mathbb{E}\left[\max\left(B_1^{x_1}(1), \dots, B_d^{x_d}(1), 0\right)\right] - c > a + \frac{1}{3}\sqrt{\ln d} \geq g(x).$$

This concludes the proof. □

Now we are going to show that, for each $d \geq 1$, as $\sum_{i=1}^d x_i \to \infty$, $\Gamma(u)$ is close to a columnal surface, which is the free boundary of the following problem. Let $\rho_d := \max\{x_1, x_2, \dots, x_d\}$, and consider

$$\min\left\{-\frac{1}{2}\Delta w_d + c, w_d - \rho_d\right\} = 0 \quad \text{in } \mathbb{R}^d. \tag{31}$$

Clearly we have $u \geq w_d$ due to $g \geq \rho_d$. When $d = 1, 2$, it is direct to check that

$$w_1(\cdot) = \psi_c(\cdot) \quad \text{and} \quad w_2(\cdot,\cdot) = \eta_c(\cdot,\cdot),$$

where ψ_c is given by (28) and η_c is given by (27).

We write the positive $x_1, \dots, x_d$ directions as $e_1, \dots, e_d$ respectively, and then

$$\tau_d := \frac{\sum_{i=1}^d e_i}{\sqrt{d}}, \qquad H_{\tau_d} := \{x \in \mathbb{R}^d \,|\, x \cdot \tau_d = 0\}. \tag{32}$$

The following lemma shows that $\Gamma(w_d)$ (the set $\partial\{w_d > \rho_d\}$) is a columnal surface.

Lemma 4. *$w_d(x) - \sum_{i=1}^d x_i/d$ is a constant function in τ_d-direction. Therefore $\Gamma(w_d)$ is the surface of one infinitely long columnar with τ_d as its longitudinal axis.*

Proof. We use the cylindrical coordinates. For each $x \in \mathbb{R}^d$, write $x := t(x)\tau_d + y(x)$, where $t(x) := x \cdot \tau_d$, and $y(x) \in H_{\tau_d}$. Then $v(x) := w_d(x) - \frac{t(x)}{\sqrt{d}}$ solves

$$\min\left\{-\frac{1}{2}\Delta v + c, v - \left(\rho_d(x) - \frac{t(x)}{\sqrt{d}}\right)\right\} = 0 \text{ in } \mathbb{R}^d. \tag{33}$$

Notice that shifts in the τ_d-direction preserve the value of $(\rho_d(x) - \frac{t(x)}{\sqrt{d}})$. Therefore by uniqueness of solutions to (33), we have $v(x) = v(x + s\tau_d)$ for all $s \in \mathbb{R}$.

To prove the second claim, take any $x \in \mathbb{R}^d$, and let $y = y(x), t = t(x)$ be as before. We have $w_d(y) = \rho_d(y)$ if and only if $w_d(y) + \frac{t}{\sqrt{d}} = \rho_d(y) + \frac{t}{\sqrt{d}}$, which is then equivalent to (writing $y = (y_1, y_2, \dots, y_d)$)

$$w_d(x) = \max\{y_1, y_2, \dots, y_d\} + \frac{t}{\sqrt{d}} = \max\{x_1, x_2, \dots, x_d\} = \rho_d(x).$$

This implies that $\Gamma(w_d) = \{y + t\tau_d \,|\, y \in \Gamma(w_d) \cap H_{\tau_d},\, t \in \mathbb{R}\}$. □

Finally, we show that $\Gamma(u)$ can be arbitrarily close to $\Gamma(w_d)$ (in Hausdorff distance) in a R-neighbourhood of the ray $\{x_1 = \ldots = x_d > 0\}$ for any $R \geq 1$ when $\sum_{i=1}^d x_i$ is large.

Proposition 3. *There exists a universal constant $\gamma > 0$ such that the following holds for all $d \geq 2$. For any $\varepsilon \in (0,1)$, and $R \geq 1$,*

$$\max\left\{\sup_{\substack{x\in\Gamma(u)\,\&\,\sum_{i=1}^d x_i \geq \frac{\gamma \ln d}{c}\sqrt{\frac{d}{\varepsilon}} \\ |x-(x\cdot\tau_d)\tau_d|\leq R}} d(x, \Gamma(w_d)), \sup_{\substack{x\in\Gamma(w_d)\,\&\,\sum_{i=1}^d x_i \geq \frac{\gamma \ln d}{c}\sqrt{\frac{d}{\varepsilon}} \\ |x-(x\cdot\tau_d)\tau_d|\leq R}} d(x, \Gamma(u))\right\} \leq R\varepsilon.$$

Proof. We start with proving an upper bound of $u-g$, which improves the estimate in (8) for special equations. Let C_2 be from Theorem 4 with $\delta = 2$, and then let g_h with $h := \frac{c}{C_2(\ln d)^2}$ be from (29). Then it follows from the proof of the theorem that $g_h \geq u$. Since $g_h = \Phi_h * g$ where Φ_h is a modifier supported in $\mathcal{R}_{1/h}$, the definitions of g and Φ_h yield for all $x \in \mathbb{R}^d$,

$$|g_h(x) - g(x)| \leq \sup_{y \in \mathcal{R}_{1/h}(x)} |g(x) - g(y)| \leq h^{-1}.$$

Thus, we get

$$u(x) - g(x) \leq g_h - g \leq h^{-1} = c^{-1} C_2 (\ln d)^2 \quad \text{for all } x \in \mathbb{R}^d. \tag{34}$$

Now for any $\varepsilon \in (0,1)$, define $w_d^\varepsilon := (1-\varepsilon)^{-1} w_d((1-\varepsilon)x)$, which then solves

$$\min\left\{ -\frac{1}{2}\Delta w_d^\varepsilon + (1-\varepsilon)c, w_d^\varepsilon - \rho_d \right\} = 0$$

(recall $\rho_d = \max\{x_1, x_2, \ldots, x_d\}$). Next we slightly perturb w_d^ε as

$$\psi_d^\varepsilon(x) := h^{-1}\left(\max\left\{ 1 - T_\varepsilon^{-1}\sqrt{d}\, x \cdot \tau_d, 0 \right\} \right)^2 + w_d^\varepsilon(x),$$

and we claim that ψ_d^ε is a supersolution to (24) in the half hyperplane $\mathcal{D} := \left\{ \sum_{j=1}^d x_j > 0 \right\}$. Indeed, using (34) and $w_d^\varepsilon \geq \rho_d = g$ on $\overline{\mathcal{D}}$, we obtain

$$\psi_d^\varepsilon(x) = h^{-1} + w_d^\varepsilon(x) \geq h^{-1} + g(x) \geq u(x) \quad \text{for any } x \in \partial\mathcal{D} = H_{\tau_d}.$$

Taking $T_\varepsilon := \sqrt{\frac{d}{c\varepsilon h}} = \frac{\ln d}{c}\sqrt{\frac{C_2 d}{\varepsilon}}$, then $\Delta w_d^\varepsilon \leq 2(1-\varepsilon)c$ in $\mathcal{D}$ (in the viscosity sense) yields

$$\Delta \psi_d^\varepsilon(x) \leq 2dh^{-1}T_\varepsilon^{-2} + 2(1-\varepsilon)c = 2c \quad \text{in } \mathcal{D}.$$

Thus ψ_d^ε is a supersolution, and then Theorem 1 with $\Omega := \mathcal{D}$ yields $\psi_d^\varepsilon \geq u$ in $\mathcal{D}$. In particular, if $x \cdot \tau_d \sqrt{d} \geq T_\varepsilon$ (i.e. $\sum_{i=1}^d x_i \geq T_\varepsilon$), we have $\psi_d^\varepsilon = w_d^\varepsilon \geq u \geq w_d$. Writing $\Gamma(w_d^\varepsilon) := \partial\{w_d^\varepsilon > g\}$ as before, we obtain that $\Gamma(u)$ lies between $\Gamma(w_d)$ and $\Gamma(w_d^\varepsilon)$ in $\{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i \geq T_\varepsilon\}$.

Therefore it remains to estimate the distance between $\Gamma(w_d)$ and $\Gamma(w_d^\varepsilon)$. By Lemma 4, it suffices to compare them on H_{τ_d}. From the definition of w_d^ε, for any $R \geq 1$, the Hausdorff distance between $\Gamma(w_d) \cap H_{\tau_d} \cap \{|x| < R\}$ and $\Gamma(w_d^\varepsilon) \cap H_{\tau_d} \cap \{|x| < R\}$ is bounded by $R\varepsilon$. This finishes the proof. □

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