FINITE ELEMENT METHODS FOR SEMILINEAR ELLIPTIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We study finite element methods for semilinear stochastic partial differential equations. Error estimates are established.

1. INTRODUCTION

In recent years, it has become clear that many phenomena, both in nature and in engineering, which are commonly described by systems of deterministic partial differential equations, may be more fully modeled by systems of stochastic partial differential equations (SPDEs) instead. However, the complexity of the SPDE model is carried over to the solutions themselves, which are no longer simple functions, but instead stochastic processes. This complexity of the solution is the reason that SPDEs are able to more fully capture the behavior of interesting phenomena; it also means that the corresponding numerical analysis of the model will require new tools to model the systems, produce the solutions, and analyze the information stored within the solutions.

Indeed the numerical analysis and simulations of SPDEs has become a highly active research area in the past few years. SPDEs derived from fluid flow and other engineering fields have been studied using Wiener chaos expansions in [4, 7, 13, 16, 20, 22]. In [3, 14], the analysis based on the traditional finite element method was successfully used on SPDEs with random coefficients, using the tensor product between the deterministic and random variable spaces. Numerical methods for SPDEs with white noise and Brownian motion added to the forcing term have also been developed, analyzed, and tested by numerous authors [2, 9, 12, 11, 23, 24, 18, 19].

In this paper, we study finite element methods for the following boundary value problem of a semilinear stochastic elliptic partial differential equation driven by an additive white noise:

\[
\begin{align*}
-\Delta u(x) + f(u(x)) &= g(x) + \hat{W}(x), \quad x \in \Omega, \\
u(x) &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded open set of \( \mathbb{R}^2 \), \( \hat{W}(x) \) is a white noise, \( g \in L^2(\Omega) \), and \( f \) is a continuous function on \( \Omega \) satisfying certain regularity conditions given in §2. The existence and uniqueness of the weak solution for (1.1) have been established in [6] by converting the problem into the Hammerstein integral equation using the Green’s function. The integral equation is
also used as a tool to derive the error estimates of the numerical approximations for problem (1.1) (see [2, 9, 11]).

The difficulty in the error analysis of finite element methods and general numerical approximations for a SPDE is the lack of regularity of its solution. For instance, as shown in [2], the required regularity conditions are not satisfied for the problem (1.1) for the standard error estimates of finite element methods. To overcome this difficulty, Allen, Novosel, and Zhang [2] and Du and Zhang [9] consider replacing $W$ with its piecewise constant approximation. Then the solution of the resulting SPDE has the desired regularity for the error estimates of finite element methods for $\Omega = (a, b)$.

The main challenge to carrying out an error analysis of the above finite element approach for the SPDE (1.1) in higher dimensional spaces is the lack of regularity of the Green’s function for the Laplacian operator. When the domain $\Omega = (a, b)$, the Green’s function for the Laplacian operator is a simple Lipschitz-continuous function; but this is not the case when $\Omega$ is a domain in higher dimensional spaces. In this work we provide a Lipschitz-type regularity estimate for the Green’s function of the Laplacian operator in the $L^2$ norm. This allows us to obtain an error estimate for the approximation of (1.1) with discretized white noises. Notice that we allow $\Omega$ to be any convex domain in $R^2$, not just a rectangle. This extension to general domains is one of the major advantages of finite element methods over other methods such as finite difference methods and spectral methods.

Nonlinearity in (1.1) is the other challenge for the error analysis of finite element methods. Because of the lack of regularity of the exact solution, we must use the $L^2$-norm to estimate the errors of the approximate solutions. For linear problems, this can be done easily using a duality argument. Here we shall use the Galerkin projection operator to resolve the difficulty.

The paper is organized as follows. In Section 2, we study the approximation of (1.1) using discretized white noises. We shall establish the estimate of the approximate solution in $H^2$-norm and their error estimate in $L^2$-norm. In Section 3, we study a finite element method of the SPDE with discretized white noises and obtain the $L^2$ error estimate between the finite element solution and the exact solution of (1.1). Finally, in Section 4, we present a numerical simulation using the finite element method constructed in the Section 3.

To conclude the introduction, we introduce the notations that will be used throughout the paper. For an integer $m$, we use $H^m(\Omega)$ to denote the Sobolev space whose norm is denoted by $\| \cdot \|_m$. As usual, when $m = 0$, $H^0(\Omega)$ shall be denoted by $L^2(\Omega)$, the space of square integrable functions on $\Omega$. Its inner product and norm are denoted by $(\cdot, \cdot)$ and $\| \cdot \|$, respectively. We also use $H^1_0(\Omega)$ for the subspace of $H^1(\Omega)$ whose elements vanish on the boundary of $\Omega$.

2. THE APPROXIMATE PROBLEM

In this section, we first introduce the approximate problem of (1.1) by replacing the white noise $W$ with its piecewise constant approximation $W_h$. Then we establish the regularity of the solution of the approximate problem and its error estimates. Without loss of generality, we shall assume that $f(0) = 0$. Otherwise, we just replace $f$ by $f - f(0)$ and $g$ by $g - f(0)$. For the simplicity of presentation we will also assume that $\Omega$ is a convex polygonal domain.

Let $\{T_h\}$ be a family of triangulations of $\Omega$ (see [5] for the requirements on $\{T_h\}$), where $h \in (0, 1)$ is the meshsize. We assume the family is quasiuniform, i.e., there exist positive
constants \( \rho_1 \) and \( \rho_2 \) such that
\[
\rho_1 h \leq R_T^{\text{inr}} < R_T^{\text{cir}} \leq \rho_2 h, \quad \forall T \in \mathcal{T}_h, \quad \forall 0 < h < 1,
\]
where \( R_T^{\text{inr}} \) and \( R_T^{\text{cir}} \) are the inradius and the circumradius of \( T \). Write
\[
\xi_T = \frac{1}{\sqrt{|T|}} \int_T 1 \, dW(x)
\]
for each triangle \( T \in \mathcal{T}_h \), where \( |T| \) denotes the area of \( T \). It is well-known that \( \{\xi_T\}_{T \in \mathcal{T}_h} \) is a family of independent identically distributed normal random variables with mean 0 and variance 1 (see [21]). Then the piecewise constant approximation to \( W(x) \) is given by
\[
\hat{W}_h(x) = \sum_{T \in \mathcal{T}_h} |T|^{-\frac{1}{2}} \xi_T \chi_T(x),
\]
where \( \chi_T \) is the characteristic function of \( T \). It is apparent that \( \hat{W}_h \in L^2(\Omega) \) a.s. However, we have the following estimate which shows that \( \|\hat{W}_h\| \) is unbounded as \( h \to 0 \).

**Lemma 1.** There exist positive constants \( C_1 \) and \( C_2 \) independent of \( h \) such that
\[
C_1 h^{-2} \leq E\left( \|\hat{W}_h\|^2 \right) \leq C_2 h^{-2}.
\]

**Proof.** It is easy to see that
\[
E\left( \|\hat{W}_h\|^2 \right) = \sum_{T \in \mathcal{T}_h} \frac{1}{|T|} = \sum_{T \in \mathcal{T}_h} \frac{|T|}{|T|}.
\]
By (2.1), \( 4\pi \rho_1^2 h^2 \leq |T| \leq 4\pi \rho_2^2 h^2 \) for all \( T \in \{\mathcal{T}_h\} \). Thus, we have
\[
E\left( \|\hat{W}_h\|^2 \right) \geq \frac{1}{4\pi \rho_2^2} h^{-2} \sum_{T \in \mathcal{T}_h} |T| = \frac{|\Omega|}{4\pi \rho_2^2} h^{-2},
\]
\[
E\left( \|\hat{W}_h\|^2 \right) \leq \frac{1}{4\pi \rho_1^2} h^{-2} \sum_{T \in \mathcal{T}_h} |T| = \frac{|\Omega|}{4\pi \rho_1^2} h^{-2}.
\]
Hence, (2.3) holds with \( C_1 = \frac{|\Omega|}{4\pi \rho_2^2}, C_2 = \frac{|\Omega|}{4\pi \rho_1^2} \). \( \square \)

Replacing \( \hat{W} \) by \( \hat{W}_h \) in (1.1), we have the following approximate problem:
\[
\begin{cases}
-\Delta u_h(x) + f(u_h(x)) = g(x) + \hat{W}_h(x), & x \in \Omega, \\
u_h = 0, & x \in \partial \Omega.
\end{cases}
\]
Its variational form is: Find \( u_h \in H^1_0(\Omega) \) such that
\[
a(u_h, v) = (F_h, v), \quad \forall v \in H^1_0(\Omega),
\]
where \( F_h = g + \hat{W}_h \), \( (\cdot, \cdot) \) denotes the inner product on \( L^2(\Omega) \), and
\[
a(\phi, \psi) = (\nabla \phi, \nabla \psi) + (f(\phi), \psi).
\]

In the remaining of this section, we first show that (2.5) has a unique solution \( u_h \) in \( H^1_0(\Omega) \cap H^2(\Omega) \) and then establish an estimate of the error \( u - u_h \). To these ends, we shall assume that \( f \) satisfies the following conditions:
There is a constant $\alpha < \gamma$ such that

$$f(s) - f(t)(s - t) \geq -\alpha |s - t|^2, \quad \forall s, t \in R.$$  

(A2) There are positive constants $\beta_1$ and $\beta_2$ such that

$$|f(s) - f(t)| \leq \beta_1 + \beta_2 |s - t|, \quad \forall s, t \in R.$$ 

Here $\gamma$ is the positive constant in the Poincaré's inequality (see [1][10]):

$$\|\nabla v\|^2 \geq \gamma \|v\|^2, \quad \forall v \in H^1_0(\Omega).$$ 

When $\beta_1 = 0$, (A2) means that $f$ is Lipschitz continuous, which implies condition (A1). These two conditions can be satisfied when $f$ is a sum of a non-decreasing bounded function and a Lipschitz continuous function.

**Theorem 1.** Under assumptions (A1) and (A2), the variational problem (2.5) has a unique solution in $H^1_0(\Omega) \cap H^2(\Omega)$ a.s. and

$$E(\|u_h\|^2) \leq C_2 h^{-2},$$ 

where $\| \cdot \|_2$ denotes the norm of $H^2(\Omega)$, $C_2$ is a positive constant independent of $h$.

**Proof.** The existence of a unique solution $u_h \in H^1_0(\Omega)$ follows from Proposition 2.9 of [25]. By condition (A1) and the Poincaré's inequality (2.6), we get

$$a(\phi, \phi) \geq \|\nabla \phi\|^2 - \alpha \|\phi\|^2 \geq \frac{\gamma - \alpha}{1 + \gamma} \|\phi\|^2, \quad \forall \phi \in H^1_0(\Omega).$$ 

Then, for $v = u_h$ in (2.5), we have

$$\|u_h\|^2 \leq \frac{1 + \gamma}{\gamma - \alpha} a(u_h, u_h) = \frac{1 + \gamma}{\gamma - \alpha} (F_h, u_h) \leq \frac{1 + \gamma}{\gamma - \alpha} \|F_h\| \|u_h\|,$$

i.e.,

$$\|u_h\| \leq \frac{1 + \gamma}{\gamma - \alpha} \|F_h\|.$$ 

Let $R_h = -f(u_h) + F_h$. Then it follows from (A2) that

$$\|R_h\|^2 \leq 3 \left( \beta_1^2 |\Omega| + (\beta_2^2 (1 + \gamma)^2/(\gamma - \alpha)^2 + 1) \|F_h\|^2 \right).$$

Notice that $u_h$ is the unique weak solution of the boundary value problem

$$\left\{ \begin{array}{l} -\Delta u_h = R_h, \quad \text{in} \ \Omega, \\
 u_h = 0, \quad \text{on} \ \partial\Omega, \end{array} \right.$$ 

Therefore, by the results of the solution regularity of (2.9) (see [10]), we have that $u_h \in H^2(\Omega)$, and

$$\|u_h\|^2 \leq \rho_3 \|R_h\|^2 \leq 3 \rho_3,$$

where $\rho_3$ is a positive constant only dependent on $\Omega$. Then the estimate (2.7) follows from the above inequality, (2.3), and (2.8).

Next we estimate the error between the weak solution $u$ of (1.1) and its approximation $u_h$. Recall that $u$ and $u_h$ are the unique solutions of the following Hammerstein integral equations, respectively (see [6]):

$$u + Kf(u) = Kg + KW,$$

$$u_h + Kf(u_h) = Kg + K\tilde{W}.$$
where

\[ K \phi(x) = \int_{\Omega} G(x, y) \phi(y) dy, \]

\( G(x, y) \) is the Green function of the Laplacian equation with homogeneous Dirichlet boundary condition. It is well-known that

\[ G(x, y) = -\frac{1}{2\pi} \log |x - y| + V(x, y) \]

where \( V(x, y) \) is a Lipschitz continuous function of \( x \) and \( y \) (see §5.14 of [8]). We also have by the Poincaré's inequality (2.6) (see Lemma 2.4 of [6])

\[ (K \phi, \phi) \geq \gamma \|K\phi\|^2, \quad \forall \phi \in L^2(\Omega). \]

The following lemma regarding the regularity of the Green function \( G \) defined in (2.12) will play an important role in the estimate.

\textbf{Lemma 2.} There exists a positive number \( \rho_4 \) independent of \( \epsilon \in (0, 1) \) such that

\[ \int_{\Omega} |G(x, y) - G(x, z)|^2 dx \leq \rho_4 \epsilon^{-1} |y - z|^{2-\epsilon}, \quad \forall \; x, z \in \Omega \]

\textbf{Proof.} We only need to show that (2.14) holds for the singular part of \( G \). For \( 0 < \epsilon < 1 \), we have

\[ \int_{\Omega} (\log |x - y| - \log |x - z|)^2 dx \]

\[ = \int_{\Omega} (|x - y| - |x - z|)^{2-\epsilon} \log |x - y| - \log |y - z| \left( \int_0^1 \frac{d\theta}{\theta|x - y| + (1 - \theta)|x - z|} \right)^{2-\epsilon} dx \]

\[ \leq |y - z|^{2-\epsilon} \int_{\Omega} \log |x - y| - \log |y - z| \left( \int_0^1 \frac{d\theta}{\theta|x - y| + (1 - \theta)|x - z|} \right)^{2-\epsilon} dx \]

\[ \leq |y - z|^{2-\epsilon} \int_{\Omega} \log |x - y| - \log |x - z| \left( \frac{1}{|x - y|} + \frac{1}{|x - z|} \right)^{2-\epsilon} dx \]

Using the Hölder inequality with \( p = \frac{3}{\epsilon} \) and \( q = \frac{3}{3-\epsilon} \) we have that

\[ \int_{\Omega} (\log |x - y| - \log |x - z|)^2 dx \]

\[ \leq |y - z|^{2-\epsilon} \left( \int_{\Omega} \log |x - y| - \log |x - z| \right)^{\frac{3}{\epsilon}} \left( \int_{\Omega} \left( \frac{1}{|x - y|} + \frac{1}{|x - z|} \right)^{\frac{3(2-\epsilon)}{3-\epsilon}} dx \right)^{\frac{3-\epsilon}{3}}. \]

Let \( H = \sup_{x,y \in \Omega} |x - y| \). Then we have

\[ \int_{\Omega} |\log |x - y| - \log |x - z||^3 \leq \int_{\Omega} \left( |\log |x - y||^3 + |\log |x - z||^3 \right) dx \]

\[ \leq 2 \int_{|x-y| \leq H} |\log |x - y||^3 dx \]

\[ = 4\pi \int_0^H r |\log(r)|^3 dr \]
and
\[ \int_{\Omega} \left( \frac{1}{|x-y|} + \frac{1}{|x-z|} \right)^{\frac{3(2-\epsilon)}{3-\epsilon}} dx \leq 2 \int_{|x-y| \leq H} \frac{1}{|x-y|^{\frac{3(2-\epsilon)}{3-\epsilon}}} dx \]
\[ \leq 8\pi H^{1-\epsilon} \leq 8\pi \frac{3-\epsilon}{\epsilon} H^{1-\epsilon} \leq 24\pi H^{1-\epsilon}. \]

Combining the above inequalities, we obtain the desired estimate (2.14).

Now we are in a position to establish the error estimate between \( u \) and \( u_h \).

**Theorem 2.** Let \( u \) and \( u_h \) be the solution of (1.1) and (2.4) respectively. If \( f \) satisfies (A1), there is a positive constant \( C_3 \) independent of \( u \) and \( h \) such that
\[ E_k(u - u_h) \leq C_3 \beta_1 |\log(h)|^\frac{1}{2} h + C_4 |\log(h)| h^2. \]

**Proof.** Subtracting (2.11) from (2.10), we get
\[ u(x) - u_h(x) + K(f(u) - f(u_h)) = E_h, \]
where \( E_h = KW - K\tilde{W}_h \).

We first prove that there exists a positive constant \( C_5 \) independent of \( h \) such that
\[ E_k(E_h) \leq C_5 |\log(h)| h^2. \]

Using the Ito isometry we have that
\[ E_k\left(E_h^2\right) = E\left(\int_{\Omega} \left[ \int_{\Omega} G(x,y)G(x,y) dy - \int_{\Omega} G(x,y)G(x,y) dy \right]^2 dx \right) \]
\[ = E\left(\int_{\Omega} \left[ \sum_{T \in T_h} \int_{T} G(x,y) dy - |T|^{-1} \sum_{T \in T_h} \int_{T} G(x,z) dz \int_{T} 1 dy \right]^2 dx \right) \]
\[ = E\left(\int_{\Omega} \left[ \sum_{T \in T_h} \int_{T} |T|^{-1} (G(x,y) - G(x,z)) dz dy \right]^2 dx \right) \]
\[ = \int_{\Omega} \left( \sum_{T \in T_h} \int_{T} |T|^{-1} (G(x,y) - G(x,z)) dz \right)^2 dy \ dx. \]
From the Hölder inequality and (2.14) we obtain
\[
E\left(\|K\hat{W} - KW_h\|^2\right) \leq \int_{\Omega} \left( \sum_{T \in T_h} |T|^{-1} \int_T \int_{\Omega} (G(x, y) - G(x, z))^2 dzdy \right) dx
\]
\[
= \sum_{T \in T_h} |T|^{-1} \int_T \int_{\Omega} (G(x, y) - G(x, z))^2 dx dzdy
\]
\[
\leq \sum_{T \in T_h} |T|^{-1} \int_T \int_{\Omega} \rho_4 \epsilon^{-1} |y - z|^{2-\epsilon} dzdy
\]
\[
\leq \rho_4 |\Omega| \epsilon^{-1} h^{2-\epsilon}.
\]
Letting \(\epsilon = 1/|\log(h)|\), we get (2.17) with \(C_5 = e \rho_4 |\Omega|\).

Multiplying (2.11) by \(f(u) - f(u_h)\), we have
\[
(u - u_h, f(u) - f(u_h)) + (K(f(u) - f(u_h)), f(u) - f(u_h)) = (E_h, f(u) - f(u_h)).
\]
Then, by (2.13) and (A1), we obtain
\[
(2.18) \quad -\alpha \|u - u_h\|^2 + \gamma \|K(f(u) - f(u_h))\|^2 \leq \|E_h\| \|f(u) - f(u_h)\|.
\]
Then, from (2.16), we have
\[
(2.19) \quad \|K(f(u) - f(u_h))\|^2 = \|u - u_h - E_h\|^2 \geq \frac{\alpha + \gamma}{2\gamma} \|u - u_h\|^2 - \frac{3\gamma - \alpha}{\gamma - \alpha} \|E_h\|^2,
\]
where we used the following inequality with \(\epsilon = (\alpha + \gamma)/(2\gamma)\):
\[
\|\phi + \psi\|^2 \geq \epsilon \|\phi\|^2 - \frac{2(1 - \epsilon)}{\epsilon} \|\psi\|^2, \quad \forall 0 < \epsilon < 1, \phi, \psi \in L^2(\Omega).
\]
It follows from (A1) that
\[
\|f(u) - f(u_h)\| \leq \beta_1 |\Omega|^{1/4} + \beta_2 \|u - u_h\|.
\]
Thus,
\[
(2.20) \quad \|E_h\| \|f(u) - f(u_h)\| \leq \beta_1 |\Omega|^{1/4} \|E_h\| + \beta_2 \|u - u_h\| \|E_h\|
\]
\[
\leq \beta_1 |\Omega|^{1/4} \|E_h\| + \frac{\alpha - \gamma}{4} \|u - u_h\|^2 + \frac{\beta_2^2}{\alpha - \gamma} \|E_h\|^2.
\]
Combining (2.18)–(2.20), we get
\[
\|u - u_h\|^2 \leq C_6 \beta_1 \|E_h\| + C_7 \|E_h\|^2,
\]
where \(C_6\) and \(C_7\) are positive constants independent of \(\beta_1, u, u_h, \) and \(h\). Then (2.15) follows from the above inequality and (2.17) \(\square\)

3. Finite Element Methods

Let \(V_h\) be a linear finite element subspace of \(H^1_0(\Omega)\) with respect to the triangulation \(T_h\) specified in §2. Define the Galerkin projection operator \(P_h : H^1_0(\Omega) \rightarrow V_h\) by
\[
(\nabla P_h w, \nabla v) = (\nabla w, \nabla v), \quad \forall v \in V_h, \quad w \in H^1_0(\Omega).
\]
It is well-known that (see [5])
\[
(3.1) \quad \|w - P_h w\| + h \|\nabla(w - P_h w)\| \leq C_4 h^2 \|w\|_2, \quad \forall w \in H^2(\Omega) \cap H^1_0(\Omega),
\]
where $C_4$ is a positive constant independent of $h$.

The finite element approximation to (2.4) is: Find $U_h \in V_h$ such that

$$ (\nabla U_h, \nabla v) + (f(U_h), v) = (g + \hat{W}_h, v), \quad \forall \ v \in V_h. \quad (3.2) $$

We have the following theorem about the existence of a unique solution $U_h$ of (3.2).

**Theorem 3.** If $f$ satisfies (A1), the approximate variational problem (3.2) has a unique solution $U_h$. In addition, there is a positive constant $C_8$ such that

$$ E(\|U_h\|^2) \leq C_8 h^{-2}. \quad (3.3) $$

**Proof.** The proof is the same as the first part of the proof of Theorem 1. \hfill \Box

For the error estimate between the solution $u$ of (1.1) and its finite element approximation $U_h$.

**Theorem 4.** If $f$ satisfies (A1) and (A2), then there is a positive constant $C_6$ such that

$$ E(\|u - U_h\|^2) \leq C_6 |\log(h)|^{\frac{1}{2}} h. \quad (3.4) $$

**Proof.** It is easy to see that

$$ (\nabla (P_h u_h - U_h), \nabla (P_h u_h - U_h)) + (f(u_h) - f(U_h), P_h u_h - U_h) = 0. $$

Thus, we have by (A1) and (A2)

$$ \|\nabla (P_h u_h - U_h)\|^2 = -(f(u_h) - f(U_h), u_h - U_h) + (f(u_h) - f(U_h), u_h - P_h u_h) $$

$$ \leq \alpha \|u_h - U_h\|^2 + \|\beta_1 + \beta_2\|u_h - U_h\|\|u_h - P_h u_h\| $$

$$ \leq \gamma + \frac{\alpha}{2} \|u_h - U_h\|^2 + \|P_h u_h - U_h\|^2 $$

By Poicaré inequality (2.6), we have

$$ \gamma \|u_h - U_h\|^2 \leq \gamma \|u_h - P_h u_h\|^2 + \gamma \|P_h u_h - U_h\|^2 $$

$$ \leq \gamma \|u_h - P_h u_h\|^2 + \|\nabla (P_h u_h - U_h)\|^2 $$

$$ \leq \gamma + \frac{\alpha}{2} \|u_h - U_h\|^2 + \beta_1 |\Omega|^{\frac{1}{2}} \|u_h - P_h u_h\| + \left(\gamma + \frac{\beta_2}{2(\gamma - \alpha)}\right) \|u_h - P_h u_h\|^2 $$

Hence, we get by (3.1)

$$ \|u_h - U_h\|^2 \leq C_{10}(\beta_1 h^2 \|u_h\|_2 + h^4 \|u_h\|^2_2), $$

where $C_{10}$ is a positive constant independent on $h$ and $\beta_1$. Thus, we have

$$ E(\|u_h - U_h\|^2) \leq C_{10} \left(\beta_1 h^2 E(\|u_h\|_2) + h^4 E(\|u_h\|^2_2)\right) $$

$$ \leq C_{10} \left(\beta_1 h^2 E(\|u_h\|_2^2) + h^4 E(\|u_h\|^2_2)\right) $$

$$ \leq C_{10} \left(\beta_1 C_1^2 h + C_1^2 h^2\right), $$

where Theorem 1 was used. Then by Theorem 2, we obtain

$$ E(\|u - U_h\|^2) \leq C_5 \beta_1 |\log(h)|^{\frac{1}{2}} h + C_6 |\log(h)| h^2 + C_{10} \left(\beta_1 C_1^2 h + C_1^2 h^2\right), $$

which leads to the error estimate (3.4). \hfill \Box
It should be pointed that we also have
\[ E(\|u_h\|^2) \leq C_8h^{-2}. \]
So \( u_h \) and its finite element approximation \( U_h \) have the same bound of order \( h^{-2} \). Although \( u_h \) and \( U_h \) are unbounded, we still have a positive order estimate of the error \( u_h - U_h \) in (3.5). When the nonlinear function \( f \) in (1.1) is Lipschitz continuous, we have a much strong error estimate.

**Theorem 5.** If \( f \) is Lipschitz continuous with the Lipschitz constant \( L \) less than \( \gamma \), then there is a positive constant \( C_{11} \) independent on \( h \) such that
\[ E(\|u - U_h\|^2) \leq C_{11}|\log(h)|h^2. \]

**Proof.** In this case, Assumption (A1) and Assumption (A2) hold for \( \beta_1 = 0, \beta_2 = \alpha = L \). Then (3.7) follows from (3.6).

**Remark 1.** The estimate (3.7) is optimal with respect to the regularity estimate of \( E(\|u_h\|^2) \).

A direct consequence of Theorem 5 is the error estimate when (1.1) is a linear SPDE, i.e., \( f(u) = c(x)u \), where \( c(x) \in L^\infty(\Omega) \) has \(-\alpha\) as its lower bound.

**Remark 2.** The above methodology can also be applied to the one dimensional case (\( \Omega = (a, b) \)) to generalize the results of [2] and [9] to nonlinear problems.

**Remark 3.** We should not expect any estimate of \( E(\|\nabla(u_h - U_h)\|^2) \) with a positive order since \( E(\|u_h\|^2) \) is of order \( -2 \). However, by the proof of Theorem 4, there is a positive constant \( C_{12} \) independent on \( h \) and \( \beta_1 \) such that
\[ E(\|\nabla(P_hu_h - U_h)\|^2) \leq C_{12}(\beta_1h + h^2), \]
which agrees with the property of superconvergence of finite element methods.

**4. NUMERICAL EXAMPLES**

In this section, we present numerical examples to demonstrate our theoretical results in the previous section.

The normal random variables for \( \tilde{W}_h \) are simulated by using the random number generators of GNU Scientific Library (GSL). The sample size is 2000 for all of the following numerical examples. To examine the convergence of our finite element method, we consider the following three types of errors:

\[ e_1(h) = \|\pi - \overline{U}_h\|, \quad e_2(h) = \|\|\pi\| - \|\overline{U}_h\|\|, \quad e_3(h) = \|\overline{U}_h - \overline{U}_{\frac{h}{2}}\|, \]

where \( \overline{U}_h = E(U_h) \). The approximate rates of convergence are are calculated by \( \log(e_i(h)/e_i(h/2))/\log(2) \).

**Example 1.** In this example, we take \( \Omega \) to be the unit square, i.e., \( \Omega = (0, 1) \times (0, 1) \). Let the exact solution be \( \overline{u}(x, y) = \sin(\pi x)\sin(\pi y) \) in the absence of white noise. Then we have that \( \|\overline{u}\| = 0.5 \). The unit square will be partitioned by dividing the unit interval \( [0, 1] \) into \( N \) equal subintervals as shown in Figure 1 with \( N = 4 \). We have that \( h = \sqrt{2}/N = 0.3535534, 0.176777, 0.088388, 0.0441942 \) for \( N = 4, 8, 16, 32 \).

First, consider the linear problem, i.e., \( f(u) = 0 \). The computational errors for sample tests are displayed in Table 1–Table 5.

Next, consider the linear problem, i.e., \( f(u) = \sin(u) \) and \( f(u) = \exp(u) \). In this case, we only display \( E_3 \) for three sample tests in Table 6 and Table 7.
Example 2. In this example, let $\Omega$ be the hexagon inscribed in the unit circle. The initial partition is shown as in Figure 2 and will be refined by dividing each triangle into four equal triangles. We have $h = 1/N = 0.5, 0.25, 0.125, 0.0625$ for $N = 2, 4, 8, 16$. The exact

Table 1. Linear problem: Test 1

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solution is the same as in Example 1 in the absence of white noise, and then we have that \( \| \mathbf{u} \| = 0.910101175 \). The errors are displayed in Table 8–Table 12 and in Table 13 and Table 14 for linear and nonlinear problems, respectively.

**Figure 2.** The partition of the hexagon

| Table 5. Linear problem: Test 5 |
|----|----|----|----|----|
| \( N \) | \( E_1 \) | rate | \( E_2 \) | rate | \( E_3 \) | rate |
| 4   | \( 7.90E-2 \) | -     | \( 6.98E-2 \) | -     | \( 6.07E-2 \) | -     |
| 8   | \( 2.03E-2 \) | 1.96  | \( 1.78E-2 \) | 1.96  | \( 1.50E-2 \) | 2.01  |
| 16  | \( 5.83E-3 \) | 1.80  | \( 5.25E-3 \) | 1.76  | \( 4.25E-3 \) | 1.82  |
| 32  | \( 1.91E-4 \) | 1.61  | \( 1.59E-3 \) | 1.72  | \( 1.46E-3 \) | 1.54  |

| Table 6. Nonlinear problem: \( f(u) = \sin(u) \) |
|----|----|----|----|----|
| \( N \) | \( E_3 \) | rate | \( E_3 \) | rate | \( E_3 \) | rate |
| 4   | \( 5.85E-2 \) | -     | \( 5.84E-2 \) | -     | \( 5.91E-2 \) | -     |
| 8   | \( 1.62E-2 \) | 1.85  | \( 1.68E-2 \) | 1.79  | \( 1.83E-2 \) | 1.68  |
| 16  | \( 4.63E-3 \) | 1.80  | \( 2.92E-3 \) | 2.52  | \( 3.54E-3 \) | 2.37  |
| 32  | \( 2.09E-3 \) | 1.14  | \( 1.32E-3 \) | 1.14  | \( 2.03E-4 \) | 0.81  |

| Table 7. Nonlinear problem: \( f(u) = \exp(u) \) |
|----|----|----|----|----|
| \( N \) | \( E_3 \) | rate | \( E_3 \) | rate | \( E_3 \) | rate |
| 4   | \( 5.57E-2 \) | -     | \( 5.62E-2 \) | -     | \( 5.75E-2 \) | -     |
| 8   | \( 1.61E-2 \) | 1.78  | \( 1.69E-2 \) | 1.72  | \( 1.42E-2 \) | 2.01  |
| 16  | \( 3.62E-3 \) | 2.15  | \( 3.89E-3 \) | 2.12  | \( 5.47E-3 \) | 1.38  |
| 32  | \( 2.28E-3 \) | 0.66  | \( 1.77E-3 \) | 1.13  | \( 1.30E-4 \) | 2.07  |

| Table 8. Linear problem: Test 1 |
|----|----|----|----|----|
| \( N \) | \( E_1 \) | rate | \( E_2 \) | rate | \( E_3 \) | rate |
| 2   | \( 2.85E-1 \) | -     | \( 2.55E-1 \) | -     | \( 2.22E-1 \) | -     |
| 4   | \( 7.45E-2 \) | 1.94  | \( 6.87E-2 \) | 1.89  | \( 5.81E-2 \) | 1.94  |
| 8   | \( 1.90E-2 \) | 1.97  | \( 1.76E-2 \) | 1.96  | \( 1.51E-2 \) | 1.93  |
| 16  | \( 4.90E-3 \) | 1.95  | \( 4.12E-3 \) | 2.09  | \( 3.97E-3 \) | 1.93  |
Example 3. In this example, let $\Omega$ be the unit disc and the exact solution is $u(x) = \sin(\pi |x|^2)$ in the absence of white noise. Then we have $\|u^*\| = \sqrt{\pi/2}$.

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Table 13. Nonlinear problem: $f(u) = \sin(u)$

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Table 14. Nonlinear problem: $f(u) = \exp(u)$

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5. Conclusions

Our aim in this work is to develop the finite element method for a class of semilinear elliptic stochastic differential equations driven by additive white noises. The previous published work in this area that we are aware of is [2], in which a one dimensional linear problem was studied. In this paper, we substantially extend their work from one dimension two dimension and from linear problems to nonlinear problems. More importantly, we allow the domain to be any convex set with regular boundary, not just a rectangle, which is the main advantage of the finite element method over other methods such as finite difference methods and spectral finite element methods. Both our theoretical analysis and numerical experiments establish the rates of convergence of the finite element approximate solutions. Some of the interesting extensions of the current work include more efficient numerical simulations for white noise and numerical approximations for SPDEs with general nonlinear terms and with general random forcing terms.

References


Table 15. Linear problem: $E_1$

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Table 16. Linear problem: $E_2$

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<td>1.43$E-3$</td>
<td>2.23$E-4$</td>
</tr>
</tbody>
</table>


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