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A model for transonic plasma flow

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A linear, two-dimensional model of a transonic plasma flow in equilibrium is constructed and given an explicit solution in the form of a complex Laplace integral. The solution indicates that the transonic state can be solved as an elliptic boundary value problem, as is done in the numerical code FLOW [Guazzotto *et al.*, Phys. Plasmas **11**, 604 (2004)]. Moreover, the presence of a hyperbolic region does not necessarily imply the presence of a discontinuity or any other singularity of the solution. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4866600]

I. INTRODUCTION

Plasma rotation in fusion devices, both toroidal and poloidal, is of considerable interest nowadays, mostly because of the general understanding that shear flow helps suppress various instabilities.[1] Also, a strong correlation was observed in various tokamaks between sheared poloidal flows and the existence of an internal transport barrier,[2] especially in the Joint European Torus (JET).[2,3] However, our interest in this paper is not the secondary effects of mass flow on the plasma stability and transport properties, but the effect of a strong poloidal flow on the plasma equilibrium itself. By strong flow, we mean that the poloidal flow speed times the ratio of the total magnetic field to the poloidal field is close to or exceeds the plasma's speed of sound. Such strong flows may result from heating the plasma by neutral beam injection, and this was indeed observed in JET.[2]

With flow of such magnitude, the partial differential equation (PDE) governing the *equilibrium* state might show real characteristics, and the flow may become "transonic" and require the presence of shocks and discontinuities, as is the case in classical fluid dynamics.[4] This phenomenon was discovered for magnetohydrodynamic (MHD) plasmas over half a century ago.[5] For an *axisymmetric* tokamak plasma, transonic behavior depends on the *poloidal* flow strength[6,7] in the manner described above. (We will be more precise in the body of the paper.) A number of works have already dealt with strong flows, but mostly in an astrophysical context, since such flows appear in the solar wind,[8] accretion disks,[9] and more. (See a brief review in Ref. 10 and a more extensive one in Ref. 11.) There are only few works dealing with transonic laboratory plasmas, mostly in relation to their stability.[12,13] A notable exception is the work of Guazzotto and Betti, plus collaborators,[14–16] which seeks to calculate numerically rotating equilibrium states, even under transonic conditions. Indeed, this work provided the motivation for the present research.

The origin of the Guazzotto-Betti project is a result by Betti and Freidberg[17] who showed that a low-beta, large aspect ratio tokamak with a transonic equilibrium poloidal flow must have a contact discontinuity coinciding with a magnetic flux surface, across which the pressure, density, and velocity profiles are discontinuous. Following Ref. 17, the numerical code FLOW[14] was constructed in order to calculate tokamak equilibria with flow, including transonic flow. The code, however, suffers from one important potential shortcoming:[2] it solves the governing Grad-Shafranov-like (GS) equation as a boundary value problem, which is appropriate for a subsonic and elliptic PDE, but is perhaps not appropriate for a transonic and mixed elliptic-hyperbolic equation. For the latter case, a boundary condition typically cannot be imposed on part of the boundary. (See, for example, Sec. III of Ref. 18 which develops this result in a plasma physics context, and references therein for a more general mathematical theory.) The question is then whether the numerical results of FLOW are reliable or not. Answering this question is what motivated the work described in this paper, and our conclusion is that the basic approach used in FLOW indeed appears proper.

To answer the question, we construct in Sec. II a *linear* but two-dimensional model equation carefully, after an analysis of the characteristic directions in the exact nonlinear equilibrium equation. This determines which derivatives in the model equation should be multiplied by factors that vanish at points of transition of the transonic-type. In Sec. III, we present an exact solution of the model equation which shows no singularity at any of the transition points, thus suggesting the correctness of solving for the equilibrium state as a boundary value problem, along the lines of the FLOW code.[14]

Additional support for solving the exact equilibrium equation as a boundary value problem is obtained from the exact solution of the one-dimensional "adiabatic compression" problem,[19] which describes a slowly changing equilibrium state. Here too we find a solution with no singular points. It should be mentioned that this is the first analytic solution of an adiabatic compression problem to appear in the literature. (The solution is quite easy to work out, but apparently no one thought there was something to learn from the one-dimensional case.) This solution is described in Sec. IV, and the article concludes with a brief summary in Sec. V.

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II. CONSTRUCTING A MODEL EQUATION

We start by describing the exact system of equations that determines an axisymmetric, toroidal, ideal MHD plasma in equilibrium.[7] The poloidal magnetic field is represented in terms of its poloidal flux function ψ which is determined by a pair of equations:

$$\mathrm{div}\left[\left(1-\frac{\Phi^2}{\rho}\right)\frac{1}{R^2}\nabla\psi\right]+\mathbf{u}\cdot\mathbf{B}\,\frac{d\Phi}{d\psi}+R\rho u_\phi\,\frac{d\Omega}{d\psi}+\frac{1}{R}B_\phi\,\frac{dF}{d\psi}+\rho\,\frac{dH}{d\psi}-\frac{\rho^\gamma}{\gamma-1}\,\frac{dS}{d\psi}=0, \tag{1}$$

$$\frac{\Phi^2}{2\rho^2}\left(B_\phi^2+\frac{1}{R^2}|\nabla\psi|^2\right)-\frac{1}{2}R^2\Omega^2+\frac{\gamma}{\gamma-1}S\rho^{\gamma-1}=H(\psi). \tag{2}$$

Here and in the remainder of this paper, we use units such that the magnetic permeability of free space μ_0 is unity, for ease of notation. Equation (1) should be thought of as the GS equation, while Eq. (2) is the Bernoulli equation which determines the density $\rho=\rho(\psi,|\nabla\psi|^2,R)$, to be substituted into Eq. (1) which is then solved for ψ. Here, R is the radial coordinate (in cylindrical R,ϕ,Z coordinates), where the Z-axis is the axis of symmetry and ϕ is the toroidal angle. (Note that subscript ϕ indicates the ϕ-component of a vector.) The magnetic field $\mathbf{B}$ is represented as

$$\mathbf{B}=\nabla\psi\times\nabla\phi+B_\phi\hat{\boldsymbol{\phi}}, \tag{3a}$$

$$B_\phi=\frac{1}{1-\Phi^2/\rho}\left(\frac{F(\psi)}{R}+R\Phi\Omega\right), \tag{3b}$$

and the plasma velocity field $\mathbf{u}$ is given by

$$\mathbf{u}=\frac{\Phi(\psi)}{\rho}\mathbf{B}+R\Omega(\psi)\hat{\boldsymbol{\phi}}. \tag{4}$$

It is noted that the equilibrium state depends on five arbitrary functions of ψ to be supplied by the solver: $F(\psi),\Phi(\psi)$, $\Omega(\psi),H(\psi)$, and $S(\psi)\equiv p\rho^{-\gamma}$, where p is the pressure and γ is the adiabatic constant. A typical boundary condition for Eq. (1) is $\psi=$ constant (say, $\psi=0$) on the boundary, if this is a perfectly conducting wall.

The "type" of the GS equation (1), whether elliptic or hyperbolic, is discussed in Ref. 7. Since we are interested here in more information, not only the wave speeds (the eigenvalues) but also the information they carry (their eigenvectors), the analysis of Ref. 7 will be expanded here. First, let us define the poloidal Alfvén Mach number M_{Ap}:

$$M_{Ap}\equiv\frac{|\mathbf{u}_p|}{|\mathbf{B}_p|/\sqrt{\rho}}=\frac{|\Phi|}{\sqrt{\rho}}, \tag{5}$$

where subscript p indicates the poloidal components of a vector, such that $|\mathbf{B}_p|/\sqrt{\rho}$ is the poloidal Alfvén speed, while $c_A=|\mathbf{B}|/\sqrt{\rho}$ is the full Alfvén speed. (Note that M_{Ap} is denoted by A in Ref. 7.) Now, the PDE-type of Eq. (1) is determined by the highest (second) order derivatives, including those implicit in $\nabla\rho$, since ρ depends on $|\nabla\psi|^2$. Defining

$$\alpha\equiv 2M_{Ap}^2\,\frac{\rho'}{\rho},\quad \rho'\equiv\frac{\partial}{\partial|\nabla\psi|^2}\,\rho(\psi,|\nabla\psi|^2,R), \tag{6}$$

the second derivatives of ψ are the same (up to a factor $1/R^2$) as in the expression $D\psi$

$$D\psi=(1-M_{Ap}^2+\alpha\psi_R^2)\psi_{RR}+2\alpha\,\psi_R\psi_Z\psi_{RZ}+(1-M_{Ap}^2+\alpha\psi_Z^2)\psi_{ZZ}. \tag{7}$$

Arranging the coefficients of the second derivatives in a symmetric matrix S,

$$S=\begin{bmatrix}1-M_{Ap}^2+\alpha\psi_R^2 & \alpha\psi_R\psi_Z\\ \alpha\psi_R\psi_Z & 1-M_{Ap}^2+\alpha\psi_Z^2\end{bmatrix}, \tag{8}$$

and letting Δ be the determinant of S, we have ellipticity of the GS equation when $\Delta>0$, and hyperbolicity when $\Delta<0$. Now,

$$\Delta=\left(1-M_{Ap}^2\right)\left(1-M_{Ap}^2+\alpha|\nabla\psi|^2\right), \tag{9a}$$

while from Eq. (2), we get by implicit differentiation with respect to $|\nabla\psi|^2$,

$$\alpha=\frac{M_{Ap}^2/R^2}{B_p^2+B_\phi^2/\left(1-M_{Ap}^2\right)-\gamma p/M_{Ap}^2}, \tag{9b}$$

where $B_p=|\mathbf{B}_p|=|\nabla\psi|/R$ and $B^2=B_p^2+B_\phi^2$. Altogether, we have

$$\Delta=-(1-M_{Ap}^2)^2\frac{M_{Ap}^2(B^2+\gamma p)-\gamma p}{M_{Ap}^4B_p^2-M_{Ap}^2(B^2+\gamma p)+\gamma p}. \tag{10}$$

The denominator has two real positive roots when viewed as a quadratic in M_{Ap}^2, corresponding to the fast and slow MHD magnetosonic waves,[20] which we will denote by M_f^2 and M_s^2, respectively. The numerator vanishes when $M_{Ap}=M_c$, where M_c corresponds to the trailing cusp speed in the wavefront diagram (the so-called Friedrichs diagram),[20]

$$M_c^2\equiv\frac{\gamma p}{\gamma p+B^2}=\frac{c_s^2}{c_s^2+c_A^2}, \tag{11}$$

and $c_s^2=\gamma p/\rho$ is the square of the plasma's speed of sound. As shown in Ref. 7,

$$M_c^2<M_s^2<1<M_f^2. \tag{12}$$

We conclude that the Grad-Shafranov equation is elliptic when $0\le M_{Ap}^2<M_c^2$ and $M_s^2<M_{Ap}^2<M_f^2$, while it is hyperbolic for $M_c^2<M_{Ap}^2<M_s^2$ and $M_{Ap}^2>M_f^2$. (Note that $M_{Ap}^2=1$ is not a transition point.)

As M_{Ap} increases from zero, Eq. (1) transitions from elliptic to hyperbolic as M_{Ap} crosses the value of M_c, then becomes elliptic again when $M_{Ap} > M_s$. Typically in fusion experiments, such as in the JET[2,21] and in the National Spherical Torus Experiment (NSTX),[22] the poloidal velocity reaches a significant fraction of $c_s B_p/B$, that is, $M_{Ap} \lesssim c_s/c_A$, so that M_{Ap} remains very much smaller than unity. Thus, we can expect a hyperbolic region only when $M_c < M_{Ap} < M_s$. As seen from Eqs. (10) and (11), both transition values, M_s and M_c, are of the order of $\sqrt{\beta}$, where $\beta = 2p/B^2$ and β is rather small in fusion devices. Indeed,

$$\begin{aligned} M_s^2 &= \frac{1}{2B_p^2}\left\{\gamma p + B^2 - \sqrt{(\gamma p + B^2)^2 - 4\gamma p B_p^2}\right\} \\ &= \frac{1}{2B_p^2}\left\{\gamma p + B^2 - (\gamma p + B^2)\left[1 - \frac{4\gamma p B_p^2}{(\gamma p + B^2)^2}\right]^{1/2}\right\} \\ &= \frac{\gamma p}{\gamma p + B^2} + O\left(\frac{(\gamma p)^2 B_p^2}{(\gamma p + B^2)^3}\right), \end{aligned} \tag{13}$$

where we have expanded the square root according to the formula $(1-\epsilon)^{1/2} = 1 - \epsilon/2 + O(\epsilon^2)$ for $\epsilon \ll 1$. We conclude that

$$M_s^2 = M_c^2 + O(\beta^2 B_p^2/B^2), \tag{14}$$

where in ordinary tokamaks not only is β small but B_p/B is small as well, of the order of the inverse aspect ratio [because the safety factor q is $O(1)$]. Since

$$M_c^2 = \frac{\gamma p}{\gamma p + B^2} < \frac{\gamma}{2}\beta = \frac{5}{6}\beta$$

(for $\gamma = 5/3$), we get that the width of the hyperbolic region in units of M_c^2, that is, in a non-dimensionalized sense, as

$$\frac{M_s^2 - M_c^2}{M_c^2} = O\left(\beta\frac{B_p^2}{B^2}\right) \lesssim O(\beta^2) \tag{15}$$

which is typically very narrow. Figure 1 indicates where this region might be in a typical NSTX transonic equilibrium state, and it should be stressed that the hyperbolic region itself was too narrow to be captured by the calculation, and its size is artificially expanded in Fig. 1 to make it visible; what we have is an upper bound on it based on Eq. (15).

Let us now note what the eigenvalues (λ_j) and eigenvectors ($\mathbf{v}_j$) of the matrix of coefficients S in Eq. (8) are. We have

$$\lambda_1 = 1 - M_{Ap}^2, \quad \lambda_2 = 1 - M_{Ap}^2 + \alpha|\nabla\psi|^2, \tag{16a}$$

$$\mathbf{v}_1 = \frac{1}{|\nabla\psi|}\begin{bmatrix}\psi_Z \\ -\psi_R\end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{|\nabla\psi|}\begin{bmatrix}\psi_R \\ \psi_Z\end{bmatrix}, \tag{16b}$$

so that $\mathbf{v}_1$ is tangential to the surface $\psi = $ const., while $\mathbf{v}_2$ is perpendicular to it. The usefulness of knowing the eigenvectors is that they can be used to transform the independent variables such that the matrix of coefficients is diagonalized. Indeed, let us write the operator D in Eq. (7) symbolically as

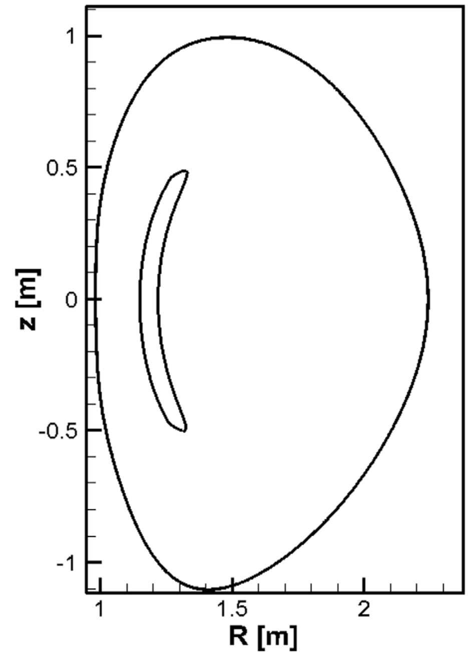

FIG. 1. A typical transonic equilibrium state in NSTX calculated by FLOW. The crescent-shaped area covers the much narrower hyperbolic region not captured by the numerical calculation.

$$\begin{aligned} Du &= au_{RR} + 2bu_{RZ} + cu_{ZZ} \\ &= \begin{bmatrix}\frac{\partial}{\partial R} & \frac{\partial}{\partial Z}\end{bmatrix}\begin{bmatrix}a & b \\ b & c\end{bmatrix}\begin{bmatrix}\partial/\partial R \\ \partial/\partial Z\end{bmatrix}u + \text{1st order derivatives.} \end{aligned} \tag{17}$$

Changing the independent variables R and Z to $\xi(R,Z)$ and $\eta(R,Z)$, we get

$$\begin{aligned} Du = &\begin{bmatrix}\frac{\partial}{\partial \xi} & \frac{\partial}{\partial \eta}\end{bmatrix}\begin{bmatrix}\xi_R & \xi_Z \\ \eta_R & \eta_Z\end{bmatrix}\begin{bmatrix}a & b \\ b & c\end{bmatrix}\begin{bmatrix}\xi_R & \eta_R \\ \xi_Z & \eta_Z\end{bmatrix}\begin{bmatrix}\partial/\partial\xi \\ \partial/\partial\eta\end{bmatrix}u \\ &+ \text{1st order derivatives.} \end{aligned} \tag{18}$$

Denoting the Jacobian matrix

$$Q = \begin{bmatrix}\xi_R & \eta_R \\ \xi_Z & \eta_Z\end{bmatrix}, \tag{19}$$

the transformed coefficients matrix is Q^TSQ, where superscript T indicates the transpose. We take $Q = [\mathbf{v}_1\ \mathbf{v}_2]$ which is an orthogonal matrix such that $Q^TSQ = Q^{-1}SQ$, the result of which is a diagonal matrix with entries λ_1 and λ_2, in this order, along the diagonal.

We have $\mathbf{v}_2$ in Eq. (16b) parallel to $\nabla\psi$, thus η can be identified with ψ. Also, $\nabla\xi$ is perpendicular to $\nabla\eta$, so we can take ξ to be essentially χ, a poloidal angle defined such that curves $\chi = $ const. are perpendicular to $\psi = $ const. and, say, χ varies between 0 and 2π. The operator D in Eq. (18) then takes the form

$$Du = \lambda_1\frac{\partial^2 u}{\partial\chi^2} + \lambda_2\frac{\partial^2 u}{\partial\psi^2} + \text{1st order derivatives.} \tag{20}$$

From definitions (16a) and expressions (9) and (10), we have

$$\lambda_1 = 1 - M_{Ap}^2,$$
$$\lambda_2 = -\frac{(1-M_{Ap}^2)(\gamma p + B^2)(M_{Ap}^2 - M_c^2)}{B_p^2(M_{Ap}^2 - M_s^2)(M_{Ap}^2 - M_f^2)},$$

but $M_{Ap}^2 = O(\beta) \ll 1$ and $M_{Ap}^2 \ll M_f^2 \approx (\gamma p + B^2)/B_p^2$ [an estimate derived from the larger root of the denominator of λ_2 as in Eq. (13)]. We get

$$\lambda_1 \approx 1, \quad \lambda_2 \approx \frac{M_{Ap}^2 - M_c^2}{M_{Ap}^2 - M_s^2}. \tag{21}$$

We are now ready to offer a *linear* model for a transonic MHD plasma. Consider the equation

$$\frac{x - x_0}{x - x_1} u_{xx} + u_{yy} = 0, \tag{22a}$$

where letter subscripts indicate derivatives, x corresponds to the radial variable ψ and y to the poloidal angle χ such that u is periodic in y (with period 2π, say), $x = x_0$ is where $M_{Ap} = M_c$ and $x = x_1$ is where $M_{Ap} = M_s$. Thus, x_0 and x_1 are the elliptic/hyperbolic transition points. Our model, Eq. (22a), is close in spirit to the standard Tricomi equation used to model a single transition of type.[18]

We will assume that x_0 is known, as follows from Ref. 17 based on more global considerations, while $x_1 = x_1(y)$ and has to be determined as part of the solution. For simplicity, and without limiting generality, we take $-1 \le x \le 1$ and u satisfies the boundary conditions

$$u(-1, y) = a(y), \quad u(1, y) = b(y). \tag{22b}$$

We specifically assume that $a(y)$ and $b(y)$ are *not both* constant, for otherwise it would be possible to define another unknown function $v(x, y)$ such that

$$v = u + \frac{1}{2}[(a - b)x - (a + b)] \tag{23}$$

with v satisfying the same equation (22a) as u, while its boundary values at $x = \pm 1$ are both zero. This would suggest that $v(x, y) \equiv 0$ and u is linear in x, thus defeating the purpose of the model to represent a transonic flow based on the coefficients of the second derivatives of u. We are now ready to investigate the behavior of the solution of a transonic flow.

III. AN EXACT SOLUTION OF THE MODEL EQUATION

Let us produce an *exact solution*[23] to a particular case (which does not appear exceptional) of Eq. (22). Instead of specifying boundary conditions (22b), we will assume that the result yields

$$s \equiv x_1(y) - x_0 = \text{const.}, \tag{24}$$

and take the boundary values of the solution, $a(y)$ and $b(y)$ of Eq. (22b), to be such that $s(y)$ actually does not depend on y. This allows for a simple Fourier expansion in y of $u(x, y)$ without the need to use the convolution of s and u. Writing Eq. (22a) as

$$(x - x_0) u_{xx} + (x - x_0 - s) u_{yy} = 0,$$

and expanding u in Fourier series in y,

$$u(x, y) = \sum_{k=-\infty}^{\infty} u_k(x) e^{iky}, \tag{25}$$

we get the one-dimensional equation

$$(x - x_0) u_k'' - k^2(x - x_0 - s) u_k = 0, \tag{26}$$

where the prime denotes d/dx. We note that $x = x_0$ is a regular singularity[24] of Eq. (26), while $x = x_1$ is a regular point. To derive the "indicial equation" related to $x = x_0$, we take as leading order $u_k \sim (x - x_0)^c$ and obtain $c(c - 1) = 0$. Thus, $c_1 = 0$, $c_2 = 1$. Since the two indices differ by an integer, two linearly independent solutions of Eq. (26) are[24] (with $\xi = x - x_0$)

$$u_k^{(1)}(x) = \xi(1 + \alpha_1 \xi + \alpha_2 \xi^2 + \ldots),$$
$$u_k^{(2)}(x) = u_k^{(1)}(x)\log\xi + (\beta_0 + \beta_1 \xi + \beta_2 \xi^2 + \ldots), \tag{27}$$

where the α_j $(j \ge 1)$ and β_j $(j \ge 0)$ are constants. Since the x-derivative of $u_k^{(2)}$ behaves as $\log(x - x_0)$ and becomes infinite as $x \to x_0$, we reject this solution and require that the solution to Eq. (26) be proportional to $u_k^{(1)}$ with the same proportionality constant on both sides of x_0. (This makes the solution continuously differentiable at x_0.) There are two boundary conditions for each k, Eq. (22b), that need to be satisfied, and two available parameters to do so, s itself and the proportionality constant K_k (such that $u_k = K_k u_k^{(1)}$).

An exact solution to Eq. (26) can be constructed[23] by representing it as a Laplace-type integral

$$u_k(x) = \int_C e^{z(x-x_0)} f_k(z)\, dz, \tag{28}$$

where the contour of integration C and the function $f_k(z)$ are to be determined. (Here, z is a complex variable and the contour of integration C is in the complex z-plane). Noting that du_k/dx causes the exponential in Eq. (28) to be replaced by z times itself, and that $(x - x_0) u_k$ can be represented by replacing that exponential by its derivative with respect to z (which is then integrated by parts), we get from Eq. (26)

$$\int_C e^{z(x-x_0)} \left\{ -\frac{d}{dz}(z^2 f_k) + k^2 \frac{df_k}{dz} + sk^2 f_k \right\} dz$$
$$+ (z^2 - k^2) f_k\, e^{z(x-x_0)}|_C = 0, \tag{29}$$

where the last term in Eq. (29) is the difference between the end value (along C) and the beginning value of the indicated function. We get a solution to Eq. (26) if $f_k(z)$ solves

$$\frac{d}{dz}(z^2 f_k) - k^2 \frac{df_k}{dz} - sk^2 f_k = 0, \tag{30}$$

and f_k is such that the last term in Eq. (29) vanishes as well.

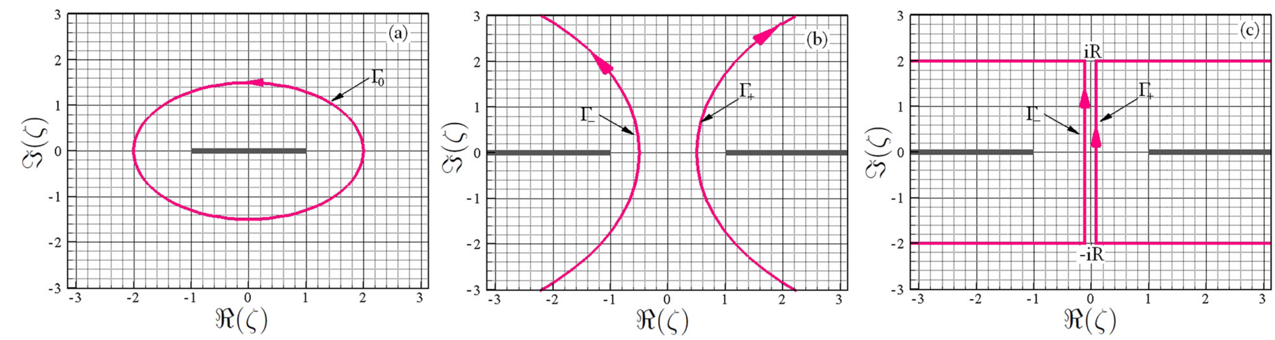

FIG. 2. Contours of integration Γ in the complex ζ-plane (in red). The grey thick lines indicate brunch cuts. (a) Γ_0 gives rise to a smooth solution for all x. (b) Γ_+ and Γ_- give rise to a solution for $x < x_0$ or $x > x_0$ only. (c) Bent contours $\Gamma_\pm$ tending to the imaginary axis as $R \to \infty$.

The general solution of Eq. (30) is

$$f_k(z) = \frac{K(k)}{z^2 - k^2}\left(\frac{z-k}{z+k}\right)^{ks/2},$$

where K is independent of z. To simplify the expression slightly, we set $z = k\zeta$, so that

$$u_k(x) = K\int_\Gamma e^{k(x-x_0)\,\zeta} \times \frac{1}{\zeta^2-1}\left(\frac{\zeta-1}{\zeta+1}\right)^{ks/2} d\zeta \qquad (31)$$

with K replacing K/k and Γ replacing C in the ζ-plane, and we also require that the contour Γ is such that

$$\left(\frac{\zeta-1}{\zeta+1}\right)^{ks/2} \times e^{k(x-x_0)\,\zeta}|_\Gamma = 0. \qquad (32)$$

We now note that the function $[(\zeta-1)/(\zeta+1)]^{ks/2}$ is single-valued in the ζ-plane except for a cut along the real interval between -1 and 1 (see Fig. 2(a)). We can therefore choose Γ to be Γ_0 of Fig. 2(a), a simple closed curve going around the cut. Since Γ_0 is closed and f_k is single-valued, condition (32) is automatically satisfied. We can identify (up to a constant multiple) this solution for $u_k(x)$ with $u_k^{(1)}$ of Eq. (27) since it is obviously analytic in x, in particular, at x_0.

Other possible contours are depicted in Fig. 2(b). For $k > 0$, Γ_+ yields a solution for $x_0 > x$ and Γ_- can serve for $x_0 < x$. If $k < 0$, the roles of Γ_+ and Γ_- are switched. Note that both the integrand of Eq. (31) and expression (32) become exponentially small for large $|\zeta|$. To simplify this solution, we can deform the contours Γ_+ and Γ_- as in Fig. 2(c) such that they coincide with the imaginary axis $\zeta = i\eta$, for $-R \le \eta \le R$. On the horizontal arcs where $Im\,\zeta = \pm R$, the integrand of Eq. (31) is $O(1/R^2)$ and vanishes as $R \to \infty$. It follows that we can replace both $\Gamma_\pm$ by the imaginary axis Γ_1 such that up to a constant multiple, the solution $u_k(x) = w_k(x)$ is valid for *both* $x > x_0$ and $x < x_0$, where

$$w_k(x) = \int_{-\infty}^{\infty} e^{ik(x-x_0)\,\eta}\,\frac{(1-i\eta)^{ks}}{(1+\eta^2)^{1+ks/2}}\,d\eta = \int_{-\infty}^{\infty} e^{ik(x-x_0)\,\eta - iks\tan^{-1}(\eta)}\,\frac{d\eta}{1+\eta^2}. \qquad (33)$$

Clearly, $w_k(x)$ is not differentiable at $x = x_0$ and is rejected as an admissible solution of Eq. (26). The only smooth solution is integral (31) taken along Γ_0. Also, s and K in Eq. (31) are to be chosen such that

$$u_k(-1) = a_k, \quad u_k(1) = b_k, \qquad (34)$$

the Fourier coefficients of a and b in Eq. (22b), or rather, a_k and b_k are to be chosen such that $s =$ const. and independent of k. This only requires

$$\int_{\Gamma_0} e^{-kx_0\zeta}\left(a_k\,e^{k\zeta} - b_k\,e^{-k\zeta}\right)\frac{1}{\zeta^2-1}\left(\frac{\zeta-1}{\zeta+1}\right)^{ks/2} d\zeta = 0, \qquad (35)$$

which determines the ratio $c_k = a_k/b_k$. The multiplying constant $K(k)$ is still available to choose exact values for a_k and b_k. [Parenthetically, we add that if in Eq. (35) ζ and k are replaced by $-\zeta$ and $-k$, respectively, the only change in the expression (after dividing it by a constant) is that c_k is replaced by c_{-k}, so $c_k = c_{-k}$. Since a_k and b_k are Fourier coefficients of real functions, they satisfy $a_k^* = a_{-k}$ and $b_k^* = b_{-k}$. It follows from $c_k = c_{-k}$ that c_k must be real.] The numerical solution of Eq. (26) displayed in Fig. 3 is obtained by using the program MATHEMATICA, after it is realized that a substitution $u_k = \exp[k(x_0 - x)]v_k(x)$ yields that v_k is the confluent hypergeometric function, a solution of which is the Laguerre function[24] which is regular at $x = x_0$, as we require.

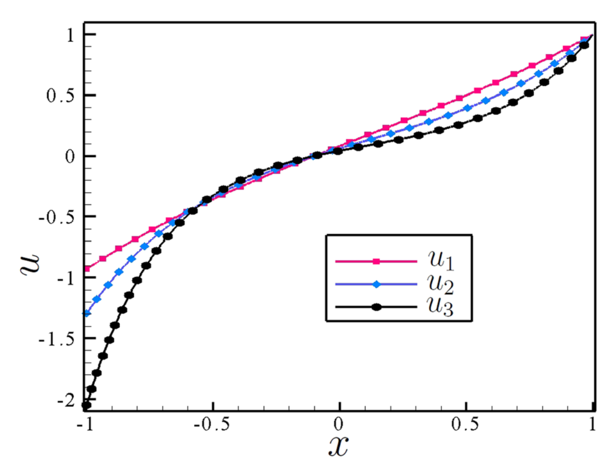

FIG. 3. Solutions u_k of the model problem with $s =$ const. and $k = 1, 2, 3$. We set $u_k(1) = 1$ for all k, so that $u_k(-1) = c_k$.

To conclude, we have demonstrated the existence of a solution $u(x, y)$ to our model equation (22) such that despite the existence of a hyperbolic region $x_0 < x < x_0 + s$, the solution u remains continuously differentiable, as depicted in Fig. 3. This figure shows $u_k(x)$ for $k = 1, 2, 3, s = 0.2$, and $x_0 = -0.1$, where $u_k(1) = 1$ so that $u_k(-1) = c_k$. In Sec. IV, we demonstrate a similar result of continuous differentiability for a *nonlinear*, albeit one-dimensional model equation. But before moving to the nonlinear model, it is a useful exercise to try and calculate directly a numerical solution for Eq. (22a). The purpose is to obtain a first check on whether it is possible to obtain a good approximation to the exact solution of an equation with a narrow hyperbolic region with a numerical solver that does *not* take into account the presence of the hyperbolic region.

The numerical approach is extremely simple. The interval $-1 \leq x \leq 1$ is discretized with a uniform grid of N points (N can be changed in the input and is completely arbitrary). An initial guess for the solution is assigned. Typically, we choose $u = 1$ over the whole domain except the extremes, where the required boundary conditions are assigned. Equation (22a) is discretized with the standard finite-difference second-order approximation for second-order derivatives, after the second derivative with respect to y has been replaced by a factor $-k^2$. We then proceed with a standard successive over-relaxation iterative method similar to the one used in FLOW for a two-dimensional grid. At each iteration, the solution in each point is updated based on the residual of Eq. (22a). No special care is taken at the transition points (which in our test cases are not coincident with grid points). Also as is done in FLOW, a convergence criterion is assigned by requiring that the total residual be less than a specified fraction (e.g., 10^{-7}) of the initial residual. An example of a numerical solution, compared with the exact solution, is shown in Fig. 4 The exact solution is represented with a solid line, the numerical one with a dashed line. In this case, $x_0 = -0.0657354$, $x_1 = 0.116083$ (the values have no special meaning). For the numerical solution, we used 100 grid points. The absolute error, i.e., the difference between exact and numerical solution (not shown) is everywhere $\lesssim 3.5 \times 10^{-9}$. For purpose of illustration, the solution in the figure was obtained by assuming that the boundary condition is given by a single harmonic, specifically $k = 1$. Thus, $a_k = b_k = 0$ for all $k \neq 1$, while $a_1 = -1$ and $b_1 = 1$.

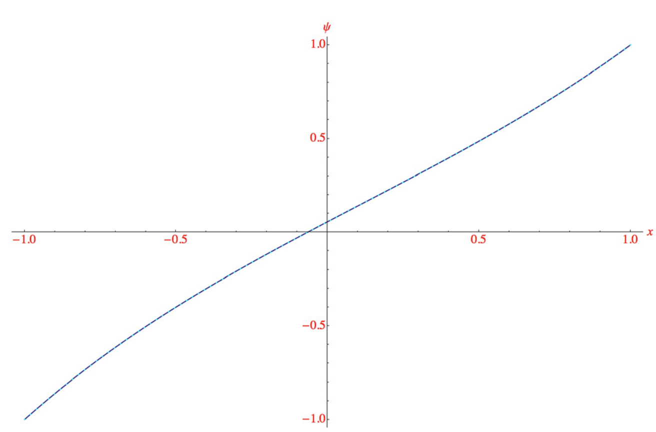

FIG. 4. Solution of the model problem with $s = \text{const}$. Numerical result (dashed line) and exact solution are superimposed.

IV. A NONLINEAR MODEL

Let us solve the problem of an adiabatic compression of an MHD plasma, as formulated variationally by Grad *et al.*,[19] and modified later to include mass flow.[25] We consider only a one-dimensional configuration so as to avoid the complication[19] of using a "volume" coordinate. Instead we use the x-coordinate, and take the magnetic flux function to be $\psi = \psi(x)$. This way, the problem is exactly solvable. Our goal is to minimize the energy (Hamiltonian)

$$H = \int \left(\frac{1}{2} \mathbf{B}^2 + \frac{1}{2} \rho \mathbf{u}^2 + \frac{p}{\gamma - 1} \right) d^3x, \tag{36}$$

where $\mathbf{B} = \nabla\psi \times \hat{\mathbf{z}} + B_z \hat{\mathbf{z}}$ and $p = S(\psi)\,\rho^\gamma$, subject to a given range of ψ, as well as given values in each ψ-surface of

$$\int_\psi \rho\, d^3x = M(\psi) \quad (\text{mass}), \tag{37a}$$

$$\int_\psi B_z\, d^3x = F(\psi) \quad (\text{toroidal flux}), \tag{37b}$$

$$\int_\psi \mathbf{u} \cdot \mathbf{B}\, d^3x = C(\psi) \quad (\text{cross helicity}), \tag{37c}$$

$$\int_\psi \rho\, u_z\, d^3x = L(\psi) \quad (\text{linear momentum}), \tag{37d}$$

where the subscript ψ indicates that the integral is taken within the ψ-surface.

Since all plasma profiles depend on x alone, we can replace d^3x by dx and differentiate constraints (37) with respect to x. Denoting by an overdot $d/d\psi$, and by a prime d/dx, we get

$$\rho = \dot{M}\,\psi', \quad B_z = \dot{F}\,\psi', \quad \mathbf{u} \cdot \mathbf{B} = \dot{C}\,\psi', \quad \rho\, u_z = \dot{L}\,\psi'. \tag{38}$$

Note that the equilibrium velocity field has zero component across the flux surfaces[7] ($u_x = 0$), so $\mathbf{u} \cdot \mathbf{B} = -u_y\,\psi' + u_z\,B_z$. It follows from Eq. (38) that

$$u_y = \frac{\dot{F}\,\dot{L}}{\dot{M}} - \dot{C}, \qquad u_z = \frac{\dot{L}}{\dot{M}}. \tag{39}$$

Substituting in Eq. (36), we get

$$H = \int \left\{ \frac{1}{2}\left(1 + \dot{F}^2(\psi)\right)\psi'^2 + Q(\psi)\,\psi' + \frac{\mu(\psi)}{\gamma - 1}\psi'^{\gamma} \right\} dx, \tag{40}$$

where

$$Q(\psi) = \frac{1}{2}\dot{M}\left[\left(\frac{\dot{L}}{\dot{M}}\right)^2 + \left(\frac{\dot{F}\,\dot{L}}{\dot{M}} - \dot{C}\right)^2\right], \quad \mu(\psi) = S(\psi)\,\dot{M}^{\gamma}. \tag{41}$$

Note that the three functions of $\psi : \dot{F}, Q$, and μ are all prescribed. There are no Lagrange multipliers of unknown values in this formulation.

It is useful to change the unknown function from $\psi(x)$ to $x(\psi)$. Equation (40) then takes the form

$$H = \int \left\{ \frac{1}{2\dot{x}} \left(1 + \dot{F}^2\right) + Q(\psi) + \frac{\mu(\psi)}{\gamma - 1}\, \dot{x}^{1-\gamma} \right\} d\psi \qquad (42)$$

with boundary conditions

$$x(0) = 0, \quad x(\psi_1) = 1, \qquad (43)$$

where the given plasma domain is $0 \leq x \leq 1$, say, and the given range of ψ is $0 \leq \psi \leq \psi_1$. The Euler equation resulting from Eq. (42) is

$$\frac{d}{d\psi}\left(\frac{\partial \mathfrak{H}(\dot{x}, \psi)}{\partial \dot{x}}\right) = 0,$$

where $\mathfrak{H}$ is the integral of (42), and explicitly

$$\frac{1}{2}\left(1 + \dot{F}^2\right)\psi'^2 + \mu \psi'^{\gamma} = C, \qquad (44)$$

where C is a constant. For any $C > 0$, there is a unique solution for ψ' of the form $\psi' = f(\psi, C)$, so that

$$\int_0^{\psi} \frac{d\psi}{f(\psi, C)} = x. \qquad (45)$$

In the special case of $\gamma = 2$,

$$f(\psi, C) = \left\{ \frac{C}{\frac{1}{2}\left(1 + \dot{F}^2\right) + \mu} \right\}^{1/2}. \qquad (46)$$

C is determined by the boundary condition $\psi(1) = \psi_1$.

Obviously, the solution exists and is unique, and even smooth provided that the prescribed functions $\dot{F}$ and μ are continuous. The issue of transonicity does not enter at all. Indeed, the magnitude $|\mathbf{u}|$ of the velocity is incorporated into the prescribed function $Q(\psi)$ which does not play a role in the differential equation. All this supports the conclusion based on our linear two-dimensional model that the presence of a hyperbolic region in MHD still allows for solving the equilibrium equation as a boundary value problem.

V. SUMMARY

The equilibrium code FLOW[14] solves the Grad-Shafranov equation (1) as a boundary value problem, which is appropriate for an elliptic equation, but perhaps not for a mixed elliptic-hyperbolic equation which Eq. (1) becomes when the poloidal Alfvén Mach number M_{Ap} exceeds M_c [as defined in Eq. (11)], which is of the order of the speed of sound divided by Alfvén speed. This objection was raised by Ref. 2 as well as by other colleagues.

To resolve this issue, we have constructed (in Sec. II) a model linear GS equation of mixed type which can be solved exactly. The exact solution does not display any non-smoothness, indicating that treating this equation as a boundary value problem is appropriate. Direct numerical calculation with an algorithm analogous to the one used in FLOW does indeed converge to the correct solution. This conclusion is supported by an exact solution of the "adiabatic compression" problem of Grad *et al.*[19] (but with flow), which again shows no non-smoothness of the solution related to the transonic nature of the flow.

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