I R A M P
12(2), July 2021, pp. 61-65

ISSN: 2229-39-44

Circular Rydberg States of Relativistic Hydrogenic Ions in Collinear Electric and Magnetic Fields of Arbitrary Strengths

N. KRYUKOV[1] AND E. OKS[2,*]

[1]*Universidad Nacional Autónoma de México, Av. Universidad 3000, col. Ciudad Universitaria, del. Coyoacán, México, DF 04510, Mexico*
[2]*Physics Department, 380 Duncan Drive, Auburn University, Auburn, AL 36849, USA*

ABSTRACT: We study the role of relativistic effects for hydrogenic Rydberg ions under collinear electric and magnetic fields of arbitrary strengths. We show that the primary outcome of the allowance for the relativism is the increase of the critical value of the electric field, at which the ionization occurs.

***Keywords*:** circular Rydberg states; hydrogenic ions; electric and magnetic fields; relativistic effects

1. INTRODUCTION

Circular Rydberg States (CRS) of hydrogenlike systems correspond to $|m| = n - 1 >> 1$, where m and n are magnetic and principal quantum numbers, respectively. CRS have been extensively studied both theoretically and experimentally for several reasons (see, e.g., [1-4] and references therein). First, CRS have long radiative lifetimes and highly anisotropic collision cross sections, thus enabling experimental works on inhibited spontaneous emission, cold Rydberg gases etc. [5-7]. Second, classical CRS correspond to quantal coherent states that are objects of fundamental importance. Third, a classical description of CRS serves as the primary term in the quantal method based on the $1/n$-expansion (see, e.g. [8] and references therein).

In paper [9] one of us focused on the analytical classical description of CRS in collinear electric (**F**) and magnetic (**B**) fields. Exact analytical expressions were obtained in paper [9] for the energy E of the atomic electron under the collinear electric and magnetic fields of arbitrary strengths. In the present paper we consider the same system, but with the allowance for relativistic effects.

2. RESULTS

We study circular Rydberg states of a hydrogen-like system (an atom or an ion) with the nucleus of charge Z, which is at the origin, acted upon by collinear electric and magnetic fields: $\mathbf{F} \,||\, \mathbf{B}$ accordingly. The Oz-axis is in the direction of the angular momentum **L** of the electron. The electric field **F** is assumed to be parallel to **L**, so that $F_z > 0$ in this coordinate system, thus the magnetic field **B** can be either parallel ($B_z > 0$) or antiparallel ($B_z < 0$) to **L**. The electron is in the circular state, and its orbit, whose radius is ρ, is perpendicular to the z-axis, with the center of the orbit having the coordinate z. The classical relativistic Hamiltonian for the electron in this configuration is

$$H = \sqrt{(c\mathbf{P} - e\mathbf{A})^2 + m^2c^4} - mc^2 - \frac{Ze^2}{r} + Fz \tag{1}$$

where **P** is the canonical momentum, **A** is the vector potential, m and e are the electron mass and charge accordingly, and r is the distance of the electron from the nucleus. We will use the atomic units $m = e = \hbar = 1$. We take the vector potential to be $\mathbf{A} = 1/2\ \mathbf{r} \times \mathbf{B}$, use the approximation $P << mc$ and the fact that for unperturbed circular orbits, $P^2 = L^2/(2\rho^2)$, and write the Hamiltonian in cylindrical coordinates, where $r = (z^2 + \rho^2)^{1/2}$:

$$H = (1 - \frac{\Omega L}{c^2})\frac{L^2}{2\rho^2} - \frac{(E_0 + \frac{Z}{\sqrt{z^2+\rho^2}})^2}{2c^2} + \Omega L + \\ + \frac{\Omega^2\rho^2}{2} - \frac{\Omega^2\rho^2(E_0 + \frac{Z}{\sqrt{z^2+\rho^2}})}{2c^2} - \frac{\Omega^2 L^2}{2c^2} - \frac{\Omega^3 L\rho^2}{2c^2} - \frac{\Omega^4\rho^4}{8c^2} - \frac{Z}{\sqrt{z^2+\rho^2}} + Fz \quad (2)$$

where $\Omega = B/(2c)$ and E_0 is the energy of the non-relativistic unperturbed electron. Using the scaled quantities

$$w = \frac{Z}{L^2}z, v = \frac{Z}{L^2}\rho, p = v^2, \omega = \frac{L^3}{Z^2}\Omega, f = \frac{L^4}{Z^3}F, \varepsilon_0 = \frac{L^2}{Z^2}E_0, \gamma = (\frac{Z}{cL})^2, h = \frac{L^2}{Z^2}H \quad (3)$$

we write the scaled Hamiltonian h:

$$h = \frac{1}{2p} - \frac{1}{\sqrt{w^2+p}} + \omega + \frac{p\omega^2}{2} + fw - \frac{\gamma}{2}((\frac{1}{\sqrt{w^2+p}} + \varepsilon_0)^2 + (\frac{1}{\sqrt{w^2+p}} + \varepsilon_0)p\omega^2 \\ + p\omega^3 + \frac{p^2\omega^4}{4} + \frac{\omega}{p} + \omega^2) \quad (4)$$

The unperturbed non-relativistic orbit has the eccentricity

$$\varepsilon_e = \sqrt{1 + \frac{2E_0L^2}{Z^2}} = \sqrt{1 + 2\varepsilon_0} \quad (5)$$

so the circular orbit, whose eccentricity is zero, corresponds to the scaled unperturbed energy $\varepsilon_0 = -1/2$, which we substitute into (4):

$$h = \frac{1}{2p} - \frac{1}{\sqrt{w^2+p}} + \omega + \frac{p\omega^2}{2} + fw - \frac{\gamma}{2}((\frac{1}{\sqrt{w^2+p}} - \frac{1}{2})^2 + (\frac{1}{\sqrt{w^2+p}} - \frac{1}{2})p\omega^2 \\ + p\omega^3 + \frac{p^2\omega^4}{4} + \frac{\omega}{p} + \omega^2) \quad (6)$$

For the non-relativistic case, corresponding to $\gamma = 0$, (6) coincides with Eq. (3) in paper [9].

To find the equilibrium points in the scaled (w, p) coordinate space, the derivatives of (6) with respect to both coordinates should vanish, which gives us the following two equations:

$$\frac{\partial h}{\partial w} = \frac{w}{(w^2+p)^{3/2}}(1 + \frac{(w^2+p)^{3/2}}{w}f + \gamma(\frac{1}{\sqrt{w^2+p}} + \frac{p\omega^2 - 1}{2})) = 0 \quad (7)$$

$$\frac{\partial h}{\partial p} = \frac{1}{2}(\frac{1}{(w^2+p)^{3/2}} - \frac{1}{p^2} + \omega^2 - \gamma((\frac{1}{\sqrt{w^2+p}} + \frac{p\omega^2 - 1}{2})(\omega^2 - \frac{1}{(w^2+p)^{3/2}}) \\ + \omega(\omega^2 - \frac{1}{p^2}))) = 0 \quad (8)$$

We solve (7) for f:

$$f(w,p,\omega,\gamma) = -\frac{w}{(w^2+p)^{3/2}}\left(1+\gamma\left(\frac{1}{\sqrt{w^2+p}}+\frac{p\omega^2-1}{2}\right)\right) \tag{9}$$

Then, if we define

$$s = \frac{1}{\sqrt{w^2+p}} \tag{10}$$

and substitute it into (8), we get an equation which is a 4th-degree polynomial with respect to s:

$$\gamma s^4 + \left(1+\frac{\gamma}{2}(p\omega^2-1)\right)s^3 - \gamma\omega^2 s + (1-\gamma\omega)\left(\omega^2-\frac{1}{p^2}\right) - \frac{\gamma}{2}\omega^2(p\omega^2-1) = 0 \tag{11}$$

Solving this equation for s and selecting the relevant roots we obtain the solution $s(p, \omega, \gamma)$, and we express the solution for w from (10):

$$w(p,\omega,\gamma) = -\sqrt{\frac{1}{s^2(p,\omega,\gamma)} - p} \tag{12}$$

(w is negative for $F_z > 0$). Then we substitute (12) into the equation for f given in (9) and obtain the expression for the scaled electric field depending on the squared scaled orbit radius p for the given values of ω and γ:

$$f(p,\omega,\gamma) = f(w(p,\omega,\gamma),p,\omega,\gamma) \tag{13}$$

Then we substitute the expression for w from (12) and the expression for f from (13) into (6) and obtain the energy of the electron depending on the squared scaled orbit radius p for the given values of ω and γ:

$$\varepsilon(p,\omega,\gamma) = h(w(p,\omega,\gamma),p,f(p,\omega,\gamma),\omega,\gamma) \tag{14}$$

Equations (13) and (14) represent a parametric dependence of the scaled energy ε on the scaled electric field f with the parameter p for the given values of ω and γ.

In the absence of the electric field, $w = 0$ from (9). Substituting $w = 0$ into (8), we obtain an implicit dependence $p(\omega)$ for the given γ in the absence of the electric field. Figure 1 presents this dependence for $\gamma = 0.1$, corresponding to the nuclear charge $Z = 43$ and angular momentum $L = 1$.

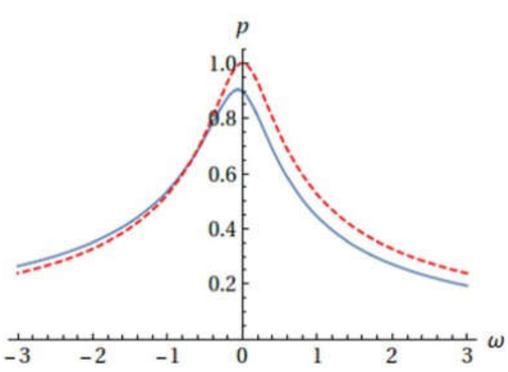

Fig. 1. Dependence of the squared scaled orbit radius of the electron on the scaled magnetic field in the absence of the electric field for the case of γ = 0.1, in the relativistic case (blue, solid curve) and in the non-relativistic case (red, dashed curve).

It is seen that the relativistic effect compresses the orbit in the case of positive ω and, starting from some negative value of ω, expands the orbit as the absolute value of ω increases from that point. By equating the values of (8) at $\gamma = 0$ and a non-zero γ (both for the case of $f = 0$, i.e., $w = 0$), this point is numerically found to be at $\omega_c = -0.6753$ and $p_c = 0.6509$. It is also seen that the magnetic field shifts the maximum of the curve in Fig. 1 to some negative value of ω, corresponding to the antiparallel configuration of vectors **B** and **L**.

Solving numerically (8) at $w = 0$ for p, we find the minimum value of p corresponding to given ω and γ, and we take the maximum value to be $1/|\omega|$, as in the non-relativistic case. We plot the parametric dependence $\varepsilon(f)$ of the scaled energy of the electron on the scaled electric field with the parameter p varying in the range determined by the above-mentioned limits, for various values of the magnetic field in the case of $Z = 14$ and $L = 1$ ($\gamma = 0.01$). We compare the relativistic plots with the non-relativistic plots in Figure 2.

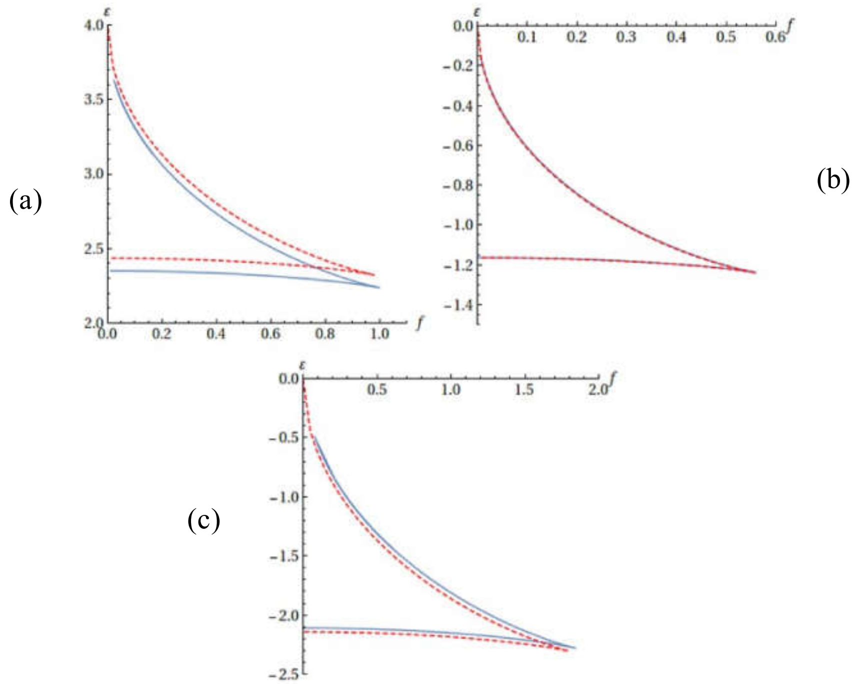

Fig. 2. Dependence of the scaled electron energy ε on the scaled electric field *f* for the scaled magnetic field (a) ω = 2, (b) ω = –1, and (c) ω = –4 in the case of *Z* = 14 and *L* = 1, in the relativistic case (blue, solid curve) and the non-relativistic case (red, dashed curve).

It is seen from Fig. 2 that the allowance for the relativism decreases the energy for $\omega > \omega_c$ and increases the energy for $\omega < \omega_c$.

It is important to clarify the following. In each of the plots in Fig. 2, there are two branches of the energy. The lower branch corresponds to the stable motion (bound state), while the upper branch corresponds to the unstable motion leading to the ionization. The value of the electric field where the two branches meet represents the ionization threshold. Thus, from Fig. 2 it is also seen that the relativistic effects increase the critical electric field required for the ionization.

Figure 3 shows the dependence of the electric field at the ionization threshold on the scaled magnetic field ω. It is seen that the relativistic effect increases the critical electric field corresponding to the ionization threshold.

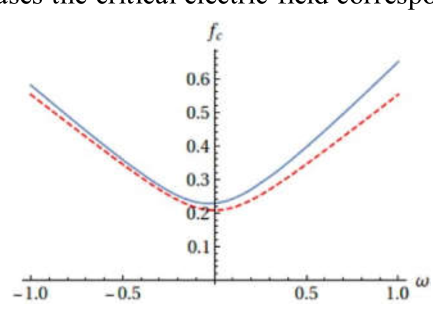

Fig. 3. Dependence of the electric field f_c at the classical ionization threshold on the scaled magnetic field ω for γ = 0.1 (blue, solid curve) and in the non-relativistic case (red, dashed curve).

Figure 4 shows the dependence of the energy at the ionization threshold on the scaled magnetic field ω. It is seen that the relativism decreases the energy at the ionization threshold for the scaled magnetic field above the critical value, and increases it for the scaled magnetic field below the critical value.

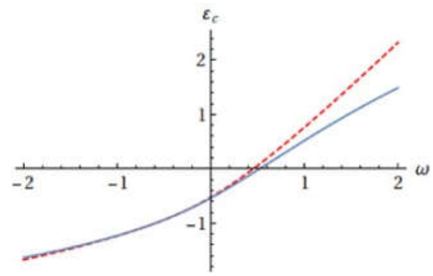

Fig. 4. Dependence of the energy ε_c at the classical ionization threshold on the scaled magnetic field ω for γ = 0.1 (blue, solid curve) and in the non-relativistic case (red, dashed curve).

3. CONCLUSIONS

We studied the role of relativistic effects for hydrogenic Rydberg ions under collinear electric and magnetic fields of arbitrary strengths. We found that the primary outcome of the allowance for the relativism is the increase of the critical value of the electric field, at which the ionization occurs. In other words, relativistic effects work as the stabilizing factor. Physically, this is because with the allowance for the relativism, the mass of the bound electron increases, so that it becomes more difficult to push it away from the nucleus.

References

1. E. Lee, D. Farrelly, T. Uzer, Optics Express **1**, 221 (1997)
2. T.C. Germann, D.R. Herschbach, M. Dunn, and D.K. Watson, Phys. Rev. Lett. **74**, 658 (1995)
3. C.H. Cheng, C.Y. Lee, and T.F. Gallagher, Phys. Rev. Lett. **73**, 3078 (1994)
4. L. Chen, M. Cheret, F. Roussel, and G. Spiess, J. Phys. B **26**, L437 (1993)
5. S.K. Dutta, D. Feldbaum, A. Walz-Flannigan, J.R. Guest, and G. Raithel, Phys. Rev. Lett. **86**, 3993 (2001)
6. R.G. Hulet, E.S. Hilfer, and D. Kleppner, Phys. Rev. Lett. **55**, 2137 (1985)
7. K.B. MacAdam and E. Horsdal-Petersen, J. Phys. B **36**, R167 (2003)
8. V.M. Vainberg, V.S. Popov, and A.V. Sergeev, Sov. Phys. JETP **71**, 470 (1990)
9. E. Oks, E., Eur. Phys. J. D **28**, 171 (2004).