Algebraic curves from function fields

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- But I am, at best an algebra-ist, with interest in the algebra that leads to algebraic-geometry codes.
- I have spent the last several decades running examples of various theoretical ideas using Computer Algebra Systems. I do not use any topology, geometry, or analysis when doing this.
- This leads me, not surprisingly, to an algebraic theory of the subjects with which I deal, unencumbered by any ideas of topology, geometry, or analysis.

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Divisors for the Klein quartic

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  are divisors describing that x/z and y/z are supposed to have 3 zeros and 3 poles each.
- This would seem to be a traditional approach in terms of Riemann-Roch spaces and the Riemann-Roch theorem in that these vector spaces are defined in terms of the numbers of zeros and poles of rational functions.

Riemann-Roch

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$$\mathbf{D} := \sum_P m_P(\mathbf{D}) \cdot P, \quad ((f)) = \sum_P \nu_P(f) \cdot P$$

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- and the Riemann-Roch theorem

$$1 + deg(\mathbf{D}) - dim(\mathbf{L}(\mathbf{D})) \leq g$$

with equality when $1 + deg(\mathbf{D}) \geq 2g$, g the genus.

Example spaces

$$\mathbf{L}(7P) = \left\langle 1, \frac{y}{z}, \frac{xy}{z^2}, \frac{y^2}{z^2}, \frac{x^2y}{z^3} \right\rangle$$

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$g = 3,\ 1 + deg(5P + 4Q) = 10,\ dim(\mathbf{L}(5P + 4Q)) = 10 - 3$

Laurent series expansions

For the Klein quartic example above,

$$\begin{array}{llll}
\frac{y}{z} = t_P^{-3} u_P^{-1} & \frac{x}{z} = t_P^{-2} u_P^{-1} & t_P := \frac{x}{y} & u_P := \frac{y^2 z}{x^3} \\
\frac{y}{z} = t_Q^{2} u_Q & \frac{x}{z} = t_Q^{-1} & t_Q := \frac{z}{x} & u_Q := \frac{x^2 y}{z^3} \\
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The induced equations used to define the units are gotten from:

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by ignoring the terms outside the parentheses; not by defining exceptional divisors and the like!

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Exceptional divisors are moot at the function field level

- $$\mathbf{K} = \mathbf{F}\left(\frac{x}{z}, \frac{y}{z}\right) / \left\langle \left(\frac{x}{z}\right)^3 \left(\frac{y}{z}\right) + \left(\frac{y}{z}\right)^3 + \left(\frac{x}{z}\right) \right\rangle$$

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A possible nonsingular model

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- These in turn describe x/z and y/z being regular functions except at P, Q where one or both have poles; x/y and z/y being regular functions except at Q, R where one or both have poles; and y/x and z/x being regular functions except at P, R where one or both have poles.

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- These three are consistent with each other if all three points P, Q, R are avoided.

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- And it generally rules out the choice of the functions that are given to define either a function field or curve, in that there may be multiple points, meaning points that aren't distinguished by their values relative to the defining functions.
- One big question is why affine or projective coordinates, but not rational coordinates?
- And why isn't choosing a set of coordinates that does distinguish points paramount?

Choices of coordinates

- $$\begin{aligned}(x:y:z)(P) &= (0:1:0)\\(x:y:z)(Q) &= (1:0:0)\\(x:y:z)(R) &= (0:0:1)\end{aligned}\ .$$

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- $$\begin{aligned}(f_7,f_5,f_3)(P) &= ((1:0),(1:0),(1:0)) = (\infty,\infty,\infty)\\(f_7,f_5,f_3)(Q) &= ((1:1),(0:1),(0:1)) = (1,0,0)\\(f_7,f_5,f_3)(R) &= ((0:1),(0:1),(0:1)) = (0,0,0)\end{aligned}.$$

Points as ring isomorphisms, Laurent series

Define ring homomorphisms

$$\pi \;:\; \mathbf{F}(\underline{x}) \to \mathbf{F}((t))$$

with $ker(\pi) = I$ to get induced ring isomorphisms

$$P \;:\; \mathbf{K} = \mathbf{F}(\underline{x})/I \to \mathbf{F}((t)).$$

$$P(f) := \sum_{j=\nu_P(f)} f_j t^j = t^{\nu_P(f)} u_{P,f}(t)$$

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Define equivalence classes of such, by

$$P_1 \equiv P_2 \text{ iff } \nu_{P_1}(f) = \nu_{P_2}(f) \text{ for all } f;$$

and call these equivalence classes of ring isomorphisms points.

$$P(f) := \sum_{j=\nu_P(f)} f_j t^j = t^{\nu_P(f)} u_{P,f}(t)$$

Points as ring isomorphisms, Laurent series

Define ring homomorphisms

$$\pi \ : \ \mathbf{F}(\underline{x}) \to \mathbf{F}((t))$$

with $ker(\pi) = I$ to get induced ring isomorphisms

$$P \ : \ \mathbf{K} = \mathbf{F}(\underline{x})/I \to \mathbf{F}((t)).$$

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$$P(f) := \sum_{j=\nu_P(f)} f_j t^j = t^{\nu_P(f)} u_{P,f}(t)$$

The trailing exponent $\nu_P(f)$ is called a valuation. The other object independent of the representative, P, is the coordinate value defined by f_0 if $\nu_P(f) \geq 0$ and ∞ if $\nu_P(f) < 0$.

Harris Lecture 7 quote

- Let $X \subset \mathbf{A}^n$ be an irreducible affine variety.

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- but despite its name, f itself is not a function on X;
- even if we allow ∞ as a value at points where $h = 0$, we cannot in general make sense of f at points where both g and h vanish.
- We will see shortly in what sense we can deal with these objects as maps.

Another quote from Harris lecture 7

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- ...a rational map, despite its name, is not a map, since it may not be defined at some points of X.
- But if a rational map is not a map, what sort of object is it?
- **Definition 7.3** Let X be an irreducible variety and Y any variety. A rational map

$$\phi \ : \ X - - \to Y$$

is defined to be an equivalence class of pairs (U, γ) with $U \subset X$ a dense Zariski open subset and $\gamma \ : \ U \to Y$ a regular map, where two such pairs (U, γ) and (V, η) are said to be equivalent if $\gamma|_{U \cap V} = \eta|_{U \cap V}$.

regular function versus poles

- $$\left(\frac{x}{z}\right)^3\left(\frac{y}{z}\right)+\left(\frac{y}{z}\right)^3+\left(\frac{x}{z}\right)=0$$

- $$\left(\frac{x_1}{h_1}\right)^3\left(\frac{x_2}{h_2}\right)+\left(\frac{x_2}{h_2}\right)^3+\left(\frac{x_1}{h_1}\right)=0$$

regular function versus poles

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- $$\frac{x}{z}=\frac{y^3+z^2x}{x^2y}=\frac{y^3}{x^2y+z^3}$$
  is a rational function, regular except when $z=0=xy$.

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- $\frac{x_i}{h_i}$ is a rational function with a pole when $h_i=0$ (and $x_i=1$).

The Wikipedia page for singularity theory is:
https://en.wikipedia.org/wiki/Singularity_theory

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either the example of a cusp or the example of a multiple point is an algebraic curve.

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Neither the example of a cusp nor the example of a multiple point is an algebraic curve.

At best, each is a graph of part of a curve projected relative to the functions x and y.

Quiz

Which of the following doesn't belong?

1. $$y^2 = x^3$$
2. $$y^2 = x^3 + x^2$$
3. $$y^2 = x^3 + x^2 + x$$
4. $$y^2 = 1 - x^2$$
5. $$\mathbf{P}^1(\mathbf{F}), \text{ the projective line}$$

Answers to quiz

1. $$y^2 = x^3,\ y = t^3,\ x = t^2, t := y/x$$

   as elements of $\mathbf{F}(t)$.

2. $$y^2 = x^3 + x^2,\ y = t^3 - t,\ x = t^2 - 1,\ t := y/x$$

   as elements of $\mathbf{F}(t)$.

3. $y^2 = x^3 + x^2 + x$ is an elliptic curve, so of genus 1, not genus 0, at least in characteristic not 3.

4. $$y^2 = 1 - x^2,\ x = \frac{2t}{1+t^2},\ y = \frac{1-t^2}{1+t^2},\ t := \frac{x}{y+1}$$

   as elements of $\mathbf{F}(t)$ in characteristic not 2.

5. $$\mathbf{F}(t)$$

Quiz

Which of the following function fields doesn't belong?

1. $$\mathbf{F}(y,x)/\langle y^3 + yx^3 + x\rangle$$

2. $$\mathbf{F}(f_5,f_3)/\langle f_5^3 + f_5f_3 + f_3^5\rangle$$

3. $$\mathbf{F}(f_7,f_5,f_3)/\langle f_7^2 + f_7 + f_5f_3^3,\ f_7f_5 + f_5 + f_3^4,\ f_5^2 - f_7f_3\rangle$$

4. $$\mathbf{F}(f_7,f_5,f_3)/\langle f_7^2+f_7+f_5f_3^3,\ f_7f_5+f_5+f_3^4,\ f_7f_3-f_5^2,\ f_5^3+f_5f_3+f_3^5\rangle$$

Answers to quiz

A trick question. $f_3 = y$, $f_5 = yx$, $f_7 = yx^2$, $x = f_5/y$.

Answers to quiz

A trick question. $f_3 = y$, $f_5 = yx$, $f_7 = yx^2$, $x = f_5/y$. Which of the following quotient rings doesn't belong?

1. $$\mathbf{F}[y,x]/\langle y^3 + yx^3 + x\rangle$$

2. $$\mathbf{F}[f_5,f_3]/\langle f_5^3 + f_5f_3 + f_3^5\rangle$$

3. $$\mathbf{F}[f_7,f_5,f_3]/\langle f_7^2 + f_7 + f_5f_3^3,\ f_7f_5 + f_5 + f_3^4,\ f_5^2 - f_7f_3\rangle$$

4. $$\mathbf{F}[f_7,f_5,f_3]/\langle f_7^2+f_7+f_5f_3^3,\ f_7f_5+f_5+f_3^4,\ f_7f_3-f_5^2,\ f_5^3+f_5f_3+f_3^5\rangle$$

Birational equivalence

The quotient ring

$$A_1 := \mathbf{F}[x, y]/\langle x^3 y + y^3 + x \rangle$$

is supposedly birationally equivalent to

$$A_2 := \mathbf{F}[f_7, f_5]/\langle f_7^5 + f_7^4 + f_5^7 \rangle.$$

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That is, there are ring homomorphisms $\phi : A_1 \to A_2$ and $\psi : A_2 \to A_1$ defined by $\phi(x) := f_7/f_5$, $\phi(y) := f_5^2/f_7$ and $\psi(f_7) := x^2 y$, $\psi(f_5) := xy$, which should be inverses of each other.

$$\phi(x^3y + y^3 + x) = (f_7/f_5)^3(f_5^2/f_7) + (f_5^2/f_7)^3 + (f_7/f_5)$$

$$= (f_7^5 + f_7^4 + f_5^7)/(f_7^3 f_5)$$

$$\phi(x^3y + y^3 + x) = (f_7/f_5)^3(f_5^2/f_7) + (f_5^2/f_7)^3 + (f_7/f_5)$$
$$= (f_7^5 + f_7^4 + f_5^7)/(f_7^3 f_5)$$

$$\psi(f_7^5 + f_7^4 + f_5^7) = (x^2y)^5 + (x^2y)^4 + (xy)^7 = x^7y^4(x^3y + y^3 + x)$$

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$$\psi(f_7^5 + f_7^4 + f_5^7) = (x^2y)^5 + (x^2y)^4 + (xy)^7 = x^7y^4(x^3y + y^3 + x)$$

Should we worry about the extra factors x^7y^4 and $f_7^3 f_5$ produced in this process? There are things called exceptional divisors, normal crossings, and on and on, in the theory of desingularizing curves and surfaces that suggest the answer is yes; but I say no.

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$$K := Q(A_1) = Q(A_2).$$

That is $x = f_7/f_5$, $y = f_5^2/f_7$, $f_7 = x^2y$, and $f_5 = xy$, if they are all viewed as elements of the same function field. So not only are ϕ and ψ inverses of each other, they are both the identity map on the common function field.

Divisors for the Klein quartic

$$a^3c^2 + abc^3 + b^5 = 0$$

also defines the Klein quartic.

$$\left(\frac{a}{c}\right)^3 + \left(\frac{a}{c}\right)\left(\frac{b}{c}\right) + \left(\frac{b}{c}\right)^5 = 0$$

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$$\left(\left(\frac{a}{c}\right)\right) = -5 \cdot P + 1 \cdot Q + 4 \cdot R$$

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describes that the homogeneous, rational functions a/c and b/c with a/c having 5 poles and zeros; b/c, 3 each.

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describes that the homogeneous, rational functions a/c and b/c with a/c having 5 poles and zeros; b/c, 3 each.
But now all the poles are at P with $(a : b : c)(P) = (1 : 0; 0)$, while $(a : b : c)(Q) = (0 : 0 : 1) = (a : b : c)(R)$ is a double point.

Laurent series expansions

For the Klein quartic example above,

$$\begin{array}{ll} b/c = t_P^{-3} u_P^{-2} & a/c = t_P^{-5} u_P^{-3} \\ b/c = t_Q^{2} u_Q & a/c = t_Q^{1} \\ b/c = t_R^{1} & a/c = t_R^{4} u_R \end{array}$$

Towers

$$x_8^2 + x_4^2 x_8 + x_4 = 0,\ \ x_4^2 + x_2^2 x_4 + x_2 = 0,\ \ x_2^2 + x_1^2 x_2 + x_1 = 0$$

Towers

$$x_8^2 + x_4^2 x_8 + x_4 = 0,\ x_4^2 + x_2^2 x_4 + x_2 = 0,\ x_2^2 + x_1^2 x_2 + x_1 = 0$$

$$\begin{array}{llllllll}
((x_8)) = & -8 \cdot P_1 & +4 \cdot P_2 & +1 \cdot P_3 & +1 \cdot P_4 & +1 \cdot P_5 & +1 \cdot P_6 \\
((x_4)) = & -4 \cdot P_1 & -4 \cdot P_2 & +2 \cdot P_3 & +2 \cdot P_4 & +2 \cdot P_5 & +2 \cdot P_6 \\
((x_2)) = & -2 \cdot P_1 & -2 \cdot P_2 & -2 \cdot P_3 & -2 \cdot P_4 & +4 \cdot P_5 & +4 \cdot P_6 \\
((x_1)) = & -1 \cdot P_1 & -1 \cdot P_2 & -1 \cdot P_3 & -1 \cdot P_4 & -4 \cdot P_5 & +8 \cdot P_6
\end{array}$$

Towers

$$x_8^2 + x_4^2 x_8 + x_4 = 0,\ x_4^2 + x_2^2 x_4 + x_2 = 0,\ x_2^2 + x_1^2 x_2 + x_1 = 0$$

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$$P_1 = (1:0:0:0:0),\ P_2 = (0:1:0:0:0),$$

$$P_3 = P_4 = (0:0:1:0:0),$$

$$P_5 = (0:0:0:1:0),\ P_6 = (0:0:0:0:1)$$

Special one-point position

For $y23 := x_8^2 x_4 x_2 x_1$ and $y8 := x_8$, both with poles only at one point, there is a special one-point description of the curve.

```
loadPackage "QthPower";
wtr=matrix{{23,8}};
R=ZZ/2[y23,y8,Weights=>entries weightGrevlex(wtr)];
GB={y23^8+y8^23+y23^4*y8^10+y23^2*y8^15+y23^5*y8^6+y23^6*y8^2+y23*y8^16+y23^4*y8^7+y23^5*y8^3
+y23^3*y8^8+y23*y8^13+y23^4*y8^4+y23^2*y8^9+y23^3*y8^5+y23*y8^10+y23^4*y8+y23*y8^7+y23^2*y8^3}
time ic2=qthIntegralClosure(wtr,R,GB);
toString ic2
```

```
(ZZ/2)[f33, f21, f19, f18, f15, f14, f12, f8],     <-- weighted ring
matrix{{33,  21,  19,  18,  15,  14,  12,  8}})

delta=y8^40+y8^37+y8^34+y8^31+y8^16+y8^13+y8^10+y8^7, <-- denominator
n12=y23^4*y8^30+y23^5*y8^26+y23*y8^36+y23^4*y8^27+y23^7*y8^18+y23^2*y8^32
   +y8^37+y23^2*y8^29+y8^34+y23^3*y8^25+y23^4*y8^21+y23^6*y8^13+y23^4*y8^18
   +y23^6*y8^10+y23*y8^24+y23^4*y8^15+y23^2*y8^20+y8^25+y23^6*y8^7+y23^2*y8^17
   +y23^5*y8^8+y23*y8^18+y23^2*y8^14+y8^19+y23^3*y8^10+y23*y8^15+y23^4*y8^6
   +y8^16+y23*y8^12+y23^2*y8^8+y8^13+y8^7,
n14=y23^2*y8^36+y23^6*y8^23+y23^7*y8^19+y23^5*y8^24+y8^38+y23^3*y8^29
   +y23^6*y8^20+y23*y8^34+y23^4*y8^25+y23^7*y8^16+y23^2*y8^30+y23^5*y8^21
   +y8^35+y23^3*y8^26+y23^4*y8^22+y23^2*y8^27+y23^6*y8^14+y23^5*y8^15
   +y23*y8^25+y23^4*y8^16+y23^7*y8^7+y23^2*y8^21+y8^26+y23*y8^22+y23^4*y8^13
   +y23^2*y8^18+y23^5*y8^9+y23^3*y8^14+y23^6*y8^5+y23*y8^19+y23^4*y8^10
   +y23^5*y8^6+y23^3*y8^11+y23^6*y8^2+y23^4*y8^7+y23^5*y8^3+y23^3*y8^8
   +y23^2*y8^9+y8^14+y23^3*y8^5+y23*y8^10+y23^2*y8^6+y23*y8^7+y23^2*y8^3
   +y8^8+y23*y8^4,
n15=y23*y8^39+y23^2*y8^35+y23^6*y8^22+y23*y8^36+y23^7*y8^18+y23^5*y8^23
   +y8^37+y23^3*y8^28+y23^6*y8^19+y23^4*y8^24+y23^7*y8^15+y23^2*y8^29
   +y23^5*y8^20+y8^34+y23^3*y8^25+y23*y8^30+y23^4*y8^21+y23^2*y8^26
   +y23^6*y8^13+y23^5*y8^14+y23*y8^24+y23^4*y8^15+y23^7*y8^6+y23^2*y8^20
   +y8^25+y23*y8^21+y23^4*y8^12+y23^2*y8^17+y23^5*y8^8+y23^3*y8^13
   +y23^6*y8^4+y23*y8^18+y23^4*y8^9+y23^5*y8^5+y23^3*y8^10+y23^6*y8
   +y23*y8^15+y23^4*y8^6+y23^5*y8^2+y23^3*y8^7+y23*y8^12+y23^2*y8^8+y8^13
   +y23^3*y8^4+y23^2*y8^5+y23^2*y8^2+y8^7+y23*y8^3,
n18=y23^6*y8^25+y23^7*y8^21+y23^2*y8^35+y23^3*y8^31+y23^6*y8^22+y23^2*y8^32
   +y23^5*y8^23+y23^3*y8^28+y23*y8^33+y23^4*y8^24+y23^2*y8^29+y23^5*y8^20
   +y23^3*y8^25+y23^6*y8^16+y23*y8^30+y23^7*y8^12+y23^2*y8^26+y23^5*y8^17
   +y23^6*y8^13+y23^7*y8^9+y23^3*y8^19+y23^6*y8^10+y23^4*y8^15+y23^5*y8^11
   +y23^3*y8^16+y23^6*y8^7+y23*y8^21+y23^2*y8^17+y23^3*y8^13+y23^2*y8^14
   +y23^5*y8^5+y23^6*y8+y23*y8^15+y23^2*y8^11+y23^5*y8^2+y8^16+y23^3*y8^7
   +y23*y8^12+y23^3*y8^4+y23*y8^9+y8^10+y23^2*y8^2+y8^7+y23*y8^3,
```

```
n19=y23^5*y8^28+y23*y8^38+y23^4*y8^29+y23^7*y8^20+y23^2*y8^34+y23*y8^35
   +y23^2*y8^31+y23^5*y8^22+y8^36+y23^3*y8^27+y23*y8^32+y23^7*y8^14
   +y23^2*y8^28+y8^33+y23^6*y8^15+y23*y8^29+y23^4*y8^20+y23^2*y8^25+y8^30
   +y23^3*y8^21+y23^6*y8^12+y23*y8^26+y23^2*y8^22+y23^4*y8^14+y23^2*y8^19
   +y23^5*y8^10+y23^6*y8^6+y23^4*y8^11+y23^3*y8^12+y23^6*y8^3+y23*y8^17
   +y23^2*y8^13+y23^5*y8^4+y8^18+y8^15+y23^3*y8^6+y8^12+y23*y8^8+y23^2*y8^4
   +y23*y8^5,
n21=y23^3*y8^34+y23^7*y8^21+y23^2*y8^35+y23^5*y8^26+y23^4*y8^27+y23^7*y8^18
   +y23^2*y8^32+y23^5*y8^23+y8^37+y23^6*y8^19+y23^7*y8^15+y23^2*y8^29
   +y23^5*y8^20+y8^34+y23^6*y8^16+y23*y8^30+y23^4*y8^21+y23^2*y8^26
   +y23^5*y8^17+y23^6*y8^13+y23*y8^27+y23^4*y8^18+y23^2*y8^23+y23^3*y8^19
   +y23^6*y8^10+y23*y8^24+y23^4*y8^15+y23^7*y8^6+y23^2*y8^20+y8^25
   +y23^3*y8^16+y23*y8^21+y23^7*y8^3+y23^6*y8^4+y23^7+y23^2*y8^14+y23^3*y8^10
   +y23^6*y8+y23^4*y8^6+y23^2*y8^11+y23^4*y8^3+y23^2*y8^8+y8^13+y23^3*y8^4
   +y23*y8^9+y23^3*y8+y23*y8^6+y23^2*y8^2,
n33=y23^7*y8^24+y23^3*y8^31+y23*y8^36+y23^4*y8^27+y23^7*y8^18+y23^5*y8^23
   +y8^37+y23*y8^33+y23^2*y8^29+y23^5*y8^20+y23^4*y8^21+y23^7*y8^12+y23*y8^27
   +y23^7*y8^9+y23^2*y8^23+y23^5*y8^14+y23^6*y8^10+y23*y8^24+y23^2*y8^20
   +y23^5*y8^11+y8^25+y23*y8^21+y23^4*y8^12+y23^7*y8^3+y23^2*y8^17
   +y23^4*y8^9+y23^2*y8^14+y8^19+y23^2*y8^11+y8^13+y23^3*y8^4+y8^7
```

```
f33^2+f33*f8^3+f18*f8^6+f18*f8^3+f18+f15+f14*f8^2+f12*f8^3+f8^6+f8^3+1,
f33*f21+f33+f18*f8^3+f18+f15*f8^3+f15+f14*f8^5+1,
f33*f19+f21*f8^2+f19*f8^3+f14*f8+f12*f8^5+f12*f8^2,
f33*f18+f33+f21+f19*f8^4+f19*f8+f18*f8^3+f18+f15*f8^3+f12*f8^3+f8^3+1,
f33*f15+f21+f15*f8^3+f15+f12*f8^3+f8^6+f8^3,
f33*f14+f21*f8+f15*f8^4+f15*f8+f14*f8^3+f12*f8+f8,
f33*f12+f21*f8^3+f19*f8+f18+f15+f14*f8^2+f12*f8^3+f8^3+1,

f21^2+f33+f21+f18*f8^3+f14*f8^2+f12,
f21*f19+f19+f14*f8+f12*f8^2+f8^5,
f21*f18+f19*f8+f18+f15*f8^3+f15+f12+f8^3+1,
f21*f15+f21+f18+f15+f12*f8^3+f12+f8^3+1,
f21*f14+f19*f8^2+f15*f8+f14,
f21*f12+f33+f21+f18+f15+f12+f8^3,

f19^2+f14*f8^3+f12*f8+f8,
f19*f18+f21*f8^2+f12*f8^2+f8^2,
f19*f15+f19+f18*f8^2+f8^2,
f19*f14+f33+f18+f15+1,           <-- strict affine F_2[f8] algebra presentation
f19*f12+f15*f8^2+f8^2,

f18^2+f19*f8+f15+f12*f8^3+f12+f8^3+1,
f18*f15+f33+f21+f15+f12,
f18*f14+f15*f8+f12*f8+f8^4+f8,
f18*f12+f18+f14*f8^2,

f15^2+f18+f14*f8^2+f12+1,
f15*f14+f21*f8+f14+f12*f8+f8,
f15*f12+f19*f8+f15+f12+1,

f14^2+f19+f12*f8^2,
f14*f12+f18*f8,

f12^2+f12+f8^3
```

genus

0	1	2	3	4	5	6	7
8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
32	33						

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The genus is easily computed as

$$g = |\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 17, 25\}| = 13$$

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4. Does it help to write things homogeneously as $xy = h^2$ or $\frac{y}{h} = \frac{h}{x}$, and think in projectively?
5. How about $xy = hk$ or $\frac{y}{k} = \frac{h}{x}$ instead?

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5. $\frac{x}{h}\frac{y}{k} = 1$ (or $xk = yh$) is.

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- By Theorem A2.27 the map g is an isomorphism over the part of X that is smooth, or even normal.
- The map g is a finite morphism in the sense that the coordinate ring of $\overline{X}$ is finitely generated *as a module* over the coordinate ring of X; this is a strong form of the condition that each fiber $g^{-1}(x)$ is a finite set.

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- and this means the singular locus of Y is of codimenion at least 2.
- Desingularization in codimension 1 is the most that can be hoped, in general, from a finite morphism.

- For example, the quadric cone $X \subset \mathbf{K}^3$ defined by the equation $x^2 + y^2 + z^2 = 0$ is normal, and it follows that any finite map $Y \to X$ that is isomorphic outside the singular point must be an isomorphism.

- For example, the quadric cone $X \subset \mathbf{K}^3$ defined by the equation $x^2 + y^2 + z^2 = 0$ is normal, and it follows that any finite map $Y \to X$ that is isomorphic outside the singular point must be an isomorphism.
- However, for any affine or projective variety X over a field it is conjectured that there is actually a *resolution of singularities*: that is, a *projective* map $\pi : Y \to X$ (this means that Y can be represented as a closed subset of $X \times \mathbf{P}^n$ for some projective space $\mathbf{P}^n$) where Y is a smooth variety, and the map π is an isomorphism over the part of X that is already smooth.

- In the example above, there is a desingularization (the *blowup* of the origin in X) that may be described as the subset of $X \times \mathbf{P}^2$, with coordinates x, y, z for X and u, v, w for $\mathbf{P}^2$, defined by the vanishing of the 2×2 minors of the matrix $\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}$ together with the equations $xu + yv + zw = 0$ and $u^2 + v^2 + w^2 = 0$.

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- It is described algebraically by the Rees algebra $R \oplus I \oplus I^2 \cdots$
- where $R = \mathbf{K}[x, y, z]/(x^2 + y^2 + z^2)$ is the coordinate ring of X
- and $I = (x, y, z) \subset R$.

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- But then one would need to find a way to explain that there were singularities when $h = 0$ as well as the one at $x = 0, y = 0, z = 0, h = 1$.

- Looking at things in rational terms is a better alternative, as the rational function equation:

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- with singularities at $h_1 = 0 = h_2$, $h_1 = 0 = h_3$, and $h_2 = 0 = h_3$, in addition to the one at $x_1 = x_2 = x_3 = 0$.

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to get a non-singular surface coordinatized over $(\mathbf{P}^1(\mathbf{F}))^6$; with additional induced relations

$$1 + \left(\frac{h_4}{x_4}\right)^2 + \left(\frac{x_6}{h_6}\right)^2 = 0;$$

$$\left(\frac{x_4}{h_4}\right)^2 + 1 + \left(\frac{h_5}{x_5}\right)^2 = 0;$$

$$\left(\frac{h_6}{x_6}\right)^2 + \left(\frac{x_5}{h_5}\right)^2 + 1 = 0.$$

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- Clearly $\phi(xu + yv + zw) = 0t$ and $\phi(u^2 + v^2 + w^2) = 0t^2$.

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- Clearly $\psi(x^2 + y^2 + z^2) \mapsto (xu + yv + zw)t \mapsto (u^2 + v^2 + w^2)t^2$.