Applied Stochastic Processes Prelim

8 a.m. – 12 p.m., 07/27/2018

Based on MATH 7820-7830 taught in Fall 2017 and Spring 2018

There are five problems which may contain several questions listed as (a), (b), (c), etc. The work on each problem will be assigned 10 points maximum, total 50. To pass the prelim 30 points are needed. The work for each problem should start on a separate page. Please enumerate all pages, put your initials on each page, and staple all. A stapler and blank paper are provided.

Previously prepared notes are allowed. They should be turned in, separately, with the exam. After the exams are graded they will be returned to the students. In addition, there is no restriction on what can be brought to the exam: calculators, laptops, tablets, etc., even books.

Final remarks:

- Some questions are "closed" and some questions are "open".
  - The closed questions are theorems to prove with the stated assumptions and theses.
  - The open questions appear after phrases such as "give examples", "discuss conditions", "describe applications", or "show a construction", etc. A brief mini-essay of mathematical nature, right to the point, is expected - yes, "brief" and "mini"! Please avoid unnecessary or unrelated divagations.
- Please make your writing legible and transparent.
- Use suitable verbal comments to explain your steps. Especially, refer to the known theorems that you apply in your reasoning. For example, while deriving a convergence result a phrase such as "by the Lebesgue Dominated Convergence Theorem" should appear.
- All sets and functions that appear here are assumed to be suitably measurable. For example, if you see $T \subset \mathbb{R}$, then you know that it is tacitly assumed that T is a Borel set.

Problem 1 - Poisson and related processes

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Introduction.

By $N(t) = N_t$ we denote the standard Poisson process on $[0, \infty)$ with unit intensity. A random Poisson measure (a.k.a. a generalized Poisson process) on a measure space $(T, \mathcal{T}, \Lambda)$ takes independent values on disjoint sets and $X(A)$ is Poisson with the intensity parameter$\Lambda(A)$, $A \in \mathcal{T}$. So Λ may be called the **intensity** or **control** measure. For a connected $T \subset \mathbb{R}$, $0 \in T$,, we write $X(t) = X_t = X[0, t]$.

Such processes are random point processes, i.e., they count random points, called **signals**, in particular sets.

Let $T \subset \mathbb{R}$ be a Borel set. A continuous nondecreasing function $C : T \to [0, \infty)$ is called a **clock**. Then $X(t) \overset{\text{def}}{=} N(C(t))$ is a generalized Poisson process on T with the control measure Λ.

Objectives.

(a) Given the intensity $\lambda(x) = \dfrac{1}{x}$, $x > 0$ (i.e., the density of Λ), find the clock $C(t)$.

(a) Given the clock $C(t) = \tan \pi t$, $-\frac{1}{2} < t < \frac{1}{2}$, find the intensity function $\lambda(x)$.

(c) In each case find the conditional probability of at least one signal in the interval $(\frac{1}{4}, 4]$ given there was no signal in the interval $(\frac{1}{2}, 2]$.

(d) Give some examples with explanations of possible physical (or financial, or natural, or human-made, etc.) phenomena that the above random Poisson measures in (a) and in (b) may serve as suitable models.

Problem 2 - Markov Chains

Introduction.

This is the classical Gambler's Ruin Problem. The whole game follows the Bernoulli process of iid two-outcome single games, played one at a time, where the win $+1\$$ happens with probability p and the loss $-1\$$ happens with probability $q = 1 - p$. Let X_n denote the gambler's fortune at game (or time) n. The whole game is over when $X_n = 0$ or $X_n = N$ for some predestined amount N. The gambler starts with $i\$$, $i \in \{0, 1, ..., N\}$, which is the state space.

Objectives.

(a) Find the transition matrix and transition probabilities.

(b) Classify all states as transient or recurrent, with proofs or explanations.

(c) Show that eventually, i.e., at a finite time, the game will be over with probability 1. Denote by f_i (that is also a function of N, $f_i = f_{iN}$) the probability that, starting at i, the gambler's fortune eventually will reach N. What is f_N?

(d) Find a recursive relation between f_{i-1}, f_i, f_{i+1}. Then solve for f_i as a function of f_1 (and of i and p, of course).

(e) Finally, let $N \to \infty$ and find the limit of f_{iN}. Give an interpretation of the obtained outcome. Why is this called "*The Gambler's Ruin Problem*"?

Problem 3 - Martingales and Lévy processes

Introduction

The characteristic function of a Lévy process $X_t, t \in T \subset \mathbb{R}$, has the form $\mathsf{E}\, e^{i\theta X_t} = \exp\{\, t\Psi(\theta)\,\}$, related to the fundamental Lévy-Khinchin formula.

Objectives.

(a) There are several forms of the function $\Psi(\theta)$. Show a few examples and explain the differences and similarities. Discuss the parameters and their uniqueness.

(b) Denote by $X(A)$ the generalized increment of the process, if well defined. Let $B \subset A$. If the process is integrable of mean 0, compute

$$\mathsf{E}\,[\,X(B)\,|\,X(A)\,]\ .$$

If you are not sure about the existence of $X(A)$ for arbitrary Borel sets, use at least usual intervals. What answer do you obtain when the assumption $B \subset A$ is suppressed?

(c) Define the "exponential process" $Y_t(z) = \exp\{\, zX_t - t\Phi(z)\,\}$. Describe conditions and the range of the parameter z under which this process is a martingale.

(d) Using major examples of Lévy processes (Poisson, stable, Gamma and symmetrized Gamma, and Wiener, with or without deterministic trend) give explicit formulas for the exponential martingales, if they exist. In particular, determine the cases when the partial derivatives $\frac{\partial^k Y}{\partial z^k}$ are also martingales.

Problem 4 - Brownian Motion

Suppose that the assets $A(t)$ at time $t \geq 0$ of some financial institution vary at random, proportionally to values of a standard Brownian motion, $A(t) \stackrel{\mathcal{D}}{=} aB_t$ (as stochastic processes). The institution files for bankruptcy when the assets reach the debt $-b$, where $b > 0$. Let $T = T_b$ denote the waiting time for that event. That is,

$$T_b > t \iff \min_{s \leq t} A(s) > -b.$$

(a) *Illustrate the above relation graphically.*

(b) **The main objective**: Find the probability distribution of T_b,

(c) *Show that $T_b < \infty$ with probability 1 but $\mathsf{E}\, T_b = \infty$.*

Remark. It may help to notice that this problem is equivalent to the situation when the institution pulls out of the market once its assets hit the positive level b. Also, it may help to strip the context of financial connotations and proceed in an abstract way.

Problem 5 - Stochastic integration

Using again at least three distinct examples of stochastic processes X_t on $T \subset \mathbb{R}$, show the construction of the integral $\int_T f(t)X(dt)$. Usually T is an interval, the closed or open positive half line, or $\mathbb{R}$. In particular:

(a) your work should exhibit the distinction between the Lebesgue-Stieltjes integral and truly stochastic integral (which does not exist in the former sense);

(b) examine the properties of the process $Y_t = \int_0^t f\, dX$ in regard to preservations of properties of the integrand X_t. For example, if X_t is a martingale (or its increments are independent, or they are uncorrelated, etc.), check if Y_t is also a martingale (preserves the original property);

(c) In your examples show a potential area of applicability of stochastic integration as a suitable model for real life phenomena.

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Problem 1 - Poisson and related processes

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By $N(t) = N_t$ we denote the standard Poisson process on $[0, \infty)$ with unit intensity. A random Poisson measure (a.k.a. a generalized Poisson process) on a measure space $(T, \mathcal{T}, \Lambda)$ takes independent values on disjoint sets and $X(A)$ is Poisson with the intensity parameter$\Lambda(A)$, $A \in \mathcal{T}$. So Λ may be called the **intensity** or **control** measure. For a connected $T \subset \mathbb{R}$, $0 \in T$,, we write $X(t) = X_t = X[0, t]$.

Such processes are random point processes, i.e., they count random points, called **signals**, in particular sets.

Let $T \subset \mathbb{R}$ be a Borel set. A continuous nondecreasing function $C : T \to [0, \infty)$ is called a **clock**. Then $X(t) \stackrel{\text{def}}{=} N(C(t))$ is a generalized Poisson process on T with the control measure Λ.

Objectives.

(a) Given the intensity $\lambda(x) = \dfrac{1}{x}$, $x > 0$ (i.e., the density of Λ), find the clock $C(t)$.

(b)[1] Given the clock $C(t) = \tan \pi t$, $-\frac{1}{2} < t < \frac{1}{2}$, find the intensity function $\lambda(x)$.

(c) In each case find the conditional probability of at least one signal in the interval $(\frac{1}{4}, 4]$ given there was no signal in the interval $(\frac{1}{2}, 2]$.

(d) Give some examples with explanations of possible physical (or financial, or natural, or human-made, etc.) phenomena that the above random Poisson measures in (a) and in (b) may serve as suitable models.

[1] originally there was a typo: "(a)"

Problem 1 - Answers

This problem is conceptual and not computational. It's focused on the distinction between the classical concept of a stochastic process X_t versus stochastic measure $X(A)$, even on the real line, even for usual intervals $A = (a, b]$. The mathematical apparatus must be adapted to conform to the requirements dictated by reality and potential applications. A student is expected to at least address the forthcoming issue when the introduced concept of the "clock" is insufficient and thus must be re-interpreted.

The question (c) is closed.

One should distinguish between two meanings of the term "intensity". The confusion originates from the intersection of the probabilistic and analytic (or physical) terminology. First, it may (and often does) denote the parameter of a Poisson random variable. Secondly, it may denote the density (e.g., the derivative if it exists):

$$\Lambda(a, b] = \int_a^b \lambda(x)\, dx.$$

So, $\Lambda(A)$ appears in the former sense while $\lambda(x)$ appears in the latter sense. Therefore, it is better to distinguish the objects, calling the former the "control measure" while keeping the term "intensity" for $\lambda(x)$.

In addition, the "clock" function in (b) is negative for $t < 0$, contradicting the requirement of positivity. The quick integration in (a) yields a function that is either negative or decreasing on some interval.

The point is that the given definition of the "clock" depends on the linear order of the real line while for the measure Λ the order is irrelevant. The order is reflected only in intervals $(a, b]$, where by convention $a \le b$. The issue disappears away from the real line, e.g., while considering a Poisson measure on the plane, which has no linear order that is compatible with the control measure. On the plane the concept of a "clock" is meaningless.

While often $X(t) = X_t = X[0, t]$ is used but this notation should be used cautiously. The existence of random measure $X(a, b]$ does not imply the existence (as a finite number) of $X(0, t]$.

Such processes are random point processes, i.e., they count random points, called **signals**, in particular sets.

A student should observe that the process $X(t) = N(C(t))$ has also independent increments since $C(t)$ is nondecreasing. Therefore the one-dimensional distributions determine the joint distributions of all finite sequence of increments. It may help to write down the formula for the Poisson measure $X(A)$, a.k.a. generalized Poisson process, using the Laplace transform:

$$\mathsf{E}\, e^{-\theta(X(b)-X(a))} = e^{-\Lambda(a,b]\left(1-e^{-\theta}\right)}. \tag{0.1}$$

(Alternatively, the Poisson integral or just the pmf may be used.)

Again, "X_t" should be used carefully for reasons explained, that also transpire in examples (a) and (b).

The relation between a non-decreasing $C(t)$ and the control measure Λ should be established. For example, using the Laplace transform,

$$\mathsf{E}\, e^{-\theta X(t)} = \mathsf{E}\, e^{-\theta N(C(t))} = \exp\left\{ -C(t)\left(1 - e^{-\theta}\right) \right\} .$$

Thus, from the first formula in (0.1) it follows that the "clock" $C(t)$ is the cumulative function of the intensity $\lambda(x)$, subject to subtle differences to be analyzed in the forthcoming (a).

In order to interpret intensities or measures it would be helpful to express the generated process in terms of transformation of the standard signals S_n that posses the n-Gamma distribution (or n-Erlang's). It turns out that $T_n = \phi(S_n)$, where $\phi = C^{-1}$ (using the generalized inverse when $C(t)$ is piecewise constant). Indeed,

$$\mathsf{P}(T_n \geq t) = \mathsf{P}(X(t) \leq n) = \mathsf{P}(N(C(t)) \leq n) = \mathsf{P}(S_n \geq C(t) = \mathsf{P}(C^{-1}(S_n) \geq t). \qquad (0.2)$$

Question (a).

The domain is the open set $T = (0, \infty)$. Here $X(t)$ does not exists, since the count of signals in $(0, t)$ is infinite, so $C(t)$ or $\Lambda(0, t]$ do not exist (or are infinite). Still, the intensity x^{-1}, $x > 0$, yields the control measure

$$\Lambda(a, b] = \ln(b/a).$$

Therefore, the clock, if it exists, should use another reference point, distinct from 0.

The number 1 seems to be a natural choice that may yield the definition $X_t = X(1, t]$ for $t \geq 1$ and $X_t = X(t, 1]$ for $t \in (0, 1]$. Consequently, the requirement of the non-decrease of $C(t)$ is no longer reasonable. To resolve the issue, one may consider $C(t) = |\ln t|$, which is decreasing for $t < 1$. The answer "$|\ln t|$" alone is insufficient, because it contradicts the definition of a "clock".

One of consequences of the extended definition is the significant change of enumeration of signals. No longer T_1 is the first signal after time 0 but rather the first signal in the vicinity of time $t = 1$. Then the definition of the second signal T_2, and then of consecutive signals, becomes ambiguous. To resolve this issue one may consider transformations of two independent Poisson process, the standard $N_t \leftrightarrow (S_n)$ and a generalized $\tilde{N}_t \leftrightarrow (\tilde{S}_n)$ on $[0, 1]$ with time reversed $t \mapsto 1 - t$. In the latter case $\phi(t)$ may and should be replaced by $\tilde{\phi} = 1/(1 - t)$. Hence, $T_n = \exp\{ S_n \} \in [1, \infty)$ and $\tilde{T}_n = 1 - \exp\left\{ -\tilde{S}_n \right\} \in [0, 1)$.

Question (b).

$C(t)$ is negative for $t < 0$, contradicting the given definition of a clock, so the problem is ill-posed. However, the intensity, hence the control measure Λ, is well defined: $\lambda(t) = \pi \sec^2 t$. Hence, the same issue as in (a) arises, which can be resolved in the same manner. Alternatively, the definition of the clock may be extended, admitting negative $C(t)$, interpreted as time running backward.

However, the specific computational question in (c) allows to postpone this discussion, since only $t > 0$ is involved.

Question (c).

Case (a): Denote $A = (\frac{1}{4}, 4]$, $B = (\frac{1}{2}, 2]$, $B \subset A$.

$$p = \mathsf{P}(X(A) \geq 1 | X(B) = 0) = 1 - \mathsf{P}(X(A) = 0 | X(B) = 0) = 1 - e^{\Lambda(B) - \Lambda(A)} = 1 - e^{\Lambda(A \setminus B)}.$$

Therefore,

$$\Lambda(A \setminus B) = \ln \frac{1/2}{1/4} + \ln \frac{4}{2} = 2 \ln 2,$$

so $p = 1 - e^{-2 \ln 2} = 1 - 2^{-2} = \frac{3}{4}$.

Case (b): $\Lambda[1/2, \infty] = 0$. Hence $\mathsf{P}(X(B) = 0) = 1$. Also, $\Lambda(A) = \Lambda(1/4, 1/2) = \infty$. Thus $\mathsf{P}(X(A) \geq 1) = 1$. Hence $\mathsf{P}(X(A) \geq 1 | X(B) = 0) = \mathsf{P}(X(A) \geq 1) = 1$.

Question (d).

In the case (a) an "explosion at time 0" model applies, eventually signals become more and more sparse. In contrast, in the case (b) the initial signals after $t = 0$ are rare but then their intensity rapidly increases to infinity on a bounded interval, yielding a "catastrophe" at the end.

A student taking the prelim may elaborate to use more specific examples illustrating the described phenomena. In the current prelim a student used the example of a star's photon emission. One may add that the case (a) might describe the process after s star is becoming supernova while (b) (for $t > 0$) may describe the process before the supernova happens.

Problem 2 - Markov Chains

Introduction.

This is the classical Gambler's Ruin Problem. The whole game follows the Bernoulli process of iid two-outcome single games, played one at a time, where the win $+1$\$ happens with probability p and the loss -1\$ happens with probability $q = 1 - p$. Let X_n denote the gambler's fortune at game (or time) n. The whole game is over when $X_n = 0$ or $X_n = N$ for some predestined amount N. The gambler starts with i\$, $i \in \{0, 1, ..., N\}$, which is the state space.

Objectives.

(a) Find the transition matrix and transition probabilities.

(b) Classify all states as transient or recurrent, with proofs or explanations.

(c) Show that eventually, i.e., at a finite time, the game will be over with probability 1. Denote by f_i (that is also a function of N, $f_i = f_{iN}$) the probability that, starting at i, the gambler's fortune eventually will reach N. What is f_N?

(d) Find a recursive relation between f_{i-1}, f_i, f_{i+1}. Then solve for f_i as a function of f_1 (and of i and p, of course).

(e) Finally, let $N \to \infty$ and find the limit of f_{iN}. Give an interpretation of the obtained outcome. Why is this called "*The Gambler's Ruin Problem*"?

Problem 2 - Answers

This classical problem is well described in many standard textbooks, e.g., in *Stochastic Processes* by Sheldon M. Ross, Example 4.4(A). No further details will be provided.

A student taking the prelim may use the Ross' elementary method by mathematical induction of solving the recurrence equation in (d), or may use his or hers alternative approach, e.g., using the method of generating functions:

$$F(t) = \sum_{n=0}^{\infty} f_n \, t^n,$$

and rewriting the recurrence equation as a second order linear differential equation and solving it.

This approach should earn extra points because it is more elegant, and also more powerful in other, more complicated problems.

In the Ross' presentation, literally the probability of "getting rich" rather than probability of ruin is presented. Of course, by symmetry, both interpretations are equivalent - as should be noted in the introduction:

"the win $+1$\$ happens with probability p and the loss -1\$ happens with probability $q = 1 - p$."

by adding "or conversely".

Problem 3 - Martingales and Lévy processes

Introduction

The characteristic function of a Lévy process $X_t, t \in T \subset \mathbb{R}$, has the form $\mathsf{E}\, e^{i\theta X_t} = \exp\{\, t\Psi(\theta)\,\}$, related to the fundamental Lévy-Khinchin formula.

Objectives.

(a) There are several forms of the function $\Psi(\theta)$. Show a few examples and explain the differences and similarities. Discuss the parameters and their uniqueness.

(b) Denote by $X(A)$ the generalized increment of the process, if well defined. Let $B \subset A$. If the process is integrable of mean 0, compute

$$\mathsf{E}\,[\, X(B) \,|\, X(A)\,]\ .$$

If you are not sure about the existence of $X(A)$ for arbitrary Borel sets, use at least usual intervals. What answer do you obtain when the assumption $B \subset A$ is suppressed?

(c) Define the "exponential process" $Y_t(z) = \exp\{\, zX_t - t\Phi(z)\,\}$. Describe conditions and the range of the parameter z under which this process is a martingale.

(d) Using major examples of Lévy processes (Poisson, stable, Gamma and symmetrized Gamma, and Wiener, with or without deterministic trend) give explicit formulas for the exponential martingales, if they exist. In particular, determine the cases when the partial derivatives $\frac{\partial^k Y}{\partial z^k}$ are also martingales.

Problem 3 - Answers

Problem (b) is closed, (a) and (c) are partially open and partially closed, (d) is an open problem.

A student should give the original Lévy-Khinchin formula for a single infinitely divisible distribution, with or without extracting the atom of the Lévy measure at 0:

$$\begin{aligned}\ln \mathsf{E}\, e^{i\theta X_t} &= iat + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - \frac{i\theta x}{1+x^2} \right) \nu(dx) \\ &= iat - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\theta x} - 1 - \frac{i\theta x}{1+x^2} \right) \nu_0(dx),\end{aligned}$$

where $\nu\{0\} = \sigma^2$ while $\nu_0\{0\} = 0$ (one may still use ν as long as the domain $\mathbb{R}\setminus\{0\}$ is stated). Then the triple $[a, \sigma^2, \nu_0]$ is unique.

(a) The function $\dfrac{x}{1+x^2}$ may be replaced by an equivalent function $[\![x]\!]$. One of the reasons for such replacement is to provide a clear analog of the LC-formula in higher dimensions. Also, another choice may alleviate cumbersome computations.

The equivalence is defined simply by the integrability of the difference between the new and the original function. A replacement causes the change in the shift parameter $a \mapsto a'$. The new triple $[a', \sigma^2, \nu]$ is also unique for a fixed specific function $[\![x]\!]$. For example, one may choose $[\![x]\!] \approx x$ near 0 and bounded away from 0. A popular choices are $[\![x]\!] = x\mathbb{1}_{\{|x|\le 1\}}$ (conveniently extendable to $\mathbb{R}^n$ or even Banach spaces) or $[\![x]\!] = x\mathbb{1}_{\{|x|\le 1\}} + \mathrm{sign}(x)\mathbb{1}_{\{|x|>1\}}$ (only for $\mathbb{R}$). Zolotarev (after Lévy) used $[\![x]\!] = \sin x$ in derivation of the ch.f. of a stable distribution that is commonly used as a default.

(b) It suffices (as it was written) to deal with common intervals. By stationarity, one may choose $A = [0, a]$ and $B = [0, b]$. If b is of the form $b_k = a + \dfrac{k(b-a)}{n}$, $k = 1, \ldots, n$, i.e., it is a point of uniform partition of $[0, a]$ into n intervals, then, denoting by $B_1 = [0, b_1]$, $B_k = (b_k, b_{k+1}]$, $k = 2, \ldots, n$, by stationarity it follows that

$$X(A) = \mathsf{E}\,[\,X(A)\,|\,X(A)\,] = \sum_k \mathsf{E}\,[\,X(B_k)\,|\,X(A)\,] = n\,\mathsf{E}\,[\,X(B_{k_0})\,|\,X(A)\,]$$

for any k_0. Hence,

$$\mathsf{E}\,[\,X(B_k)\,|\,X(A)\,] = \frac{1}{n}\,X(A).$$

By additivity, if B is a union of some B_k,

$$\mathsf{E}\,[\,X(B)\,|\,X(A)\,] = \frac{|B|}{|A|}\,X(A).$$

For an arbitrary $b < a$ one may use a decreasing sequence of b_k's, or B_k's from the above uniform partitions, The limit $|B| = b$ on the right hand side is obvious. On the left hand side $\mathsf{E}\,[\,X(B_k)\,|\,X(A)\,]$ form a martingale with respect to (reversed) filtration $\mathcal{F}_t = \sigma\{\,X_s : s \ge t\,\}$. Then by the Doob's convergence theorem, the left hand sides converge to $X(B_k)$ a.s. and in L^1.

Note that the slightly more complicated downward approximation of a point from the right is applied to obtain $X(0, b]$, while an upward approximation would yield $X(0, b)$. That is, the right continuity of trajectories of a Lévy process is used.

The case of an arbitrary Borel set B (or A) requires a more advanced measure-theoretical argument. First of all, the quantity $X(B)$ would have to be defined. In the course of 2017/208 the concept of the integral $Xf = \int f\, dX$ was introduced first, which for Lévy processes is particularly easy. Then simply $X(B) = X(\mathbb{1}_B)$. This part is not required for the prelim but would give a student some additional leverage.

When B is not a subset of A, the decomposition $B = (B \cap A) \cup (B \setminus A)$ helps. Hence

$$\begin{aligned} \mathsf{E}\left[X(B) \,|\, X(A) \right] &= \mathsf{E}\left[X(B \cap A) \,|\, X(A) \right] + \mathsf{E}\left[X(B \setminus A) \,|\, X(A) \right] \\ &= \frac{|B \cap A|}{|A|} X(A) + \mathsf{E}\, X(A \setminus B) = \frac{|B \cap A|}{|A|} X(A), \end{aligned}$$

since $X(B \setminus A)$ is independent of $X(A)$ and the zero mean was assumed.

(c) First of all, the process $Y_t(z) = \exp\{ zX_t - t\Phi(z) \}$ must be integrable, i.e., at least some exponential moments of the process X_t must exist. This determines the domain of z's. The natural filtration $\mathcal{F}_t$ generated by the process may be used. Let $s < t$. Then

$$\mathsf{E}\left[Y_t \,|\, \mathcal{F}_s \right] = \mathsf{E}\left[e^{zX_s + z(X_t - X_s) - t\Phi(z)} \,|\, \mathcal{F}_s \right] = e^{zX_s - t\Phi(z)} \mathsf{E}\, e^{z(X_t - X_s)} = e^{zX_s - t\Phi(z)} e^{(t-s)\Phi(z)} = Y_s.$$

(d) In the first part it suffices simply to list the forms of the function $\Psi(z)$:

a zero mean Brownian motion with variance σ^2: $\Psi(z) = \dfrac{\sigma^2 z^2}{2}$, $z \in \mathbb{R}$.
(a.k. the Geometric Brownian Motion)
a standard Poisson process with intensity λ: $\Psi(z) = \lambda\left(e^{-\theta} - 1\right)$, $z = -\theta < 0$.
a Gamma process: $\Psi(z) = \ln(1+z)$, $z > -1$.

A stable random variable does not have higher moments, hence the exponential martingale doesn't exist for real z. However, for $z = i\theta$, choosing for simplicity a symmetric α-stable process,

$$Y_t = e^{i\theta X_t - i^\alpha\, t\theta^\alpha}$$

is a complex valued martingale.

The differentiation is possible under stronger integrability assumptions. Then the martingale property is preserved (essentially, one relies on the Lebesgue Dominated Convergence Theorem). For the k^{th} derivative,

$$\mathsf{E}\, |X_t|^k\, e^{zX_t} < \infty$$

is required. It is satisfied for a Brownian motion, Poisson process, and Gamma process for any natural number k for z listed. In the stable case, only $k = 1$ is possible for $\alpha > 1$.

Problem 4 - Brownian Motion

Suppose that the assets $A(t)$ at time $t \geq 0$ of some financial institution vary at random, proportionally to values of a standard Brownian motion, $A(t) \stackrel{\mathcal{D}}{=} aB_t$ (as stochastic processes). The institution files for bankruptcy when the assets reach the debt $-b$, where $b > 0$. Let $T = T_b$ denote the waiting time for that event. That is,

$$T_b > t \iff \min_{s \leq t} A(s) > -b.$$

(a) *Illustrate the above relation graphically.*

(b) **The main objective**: Find the probability distribution of T_b,

(c) *Show that* $T_b < \infty$ *with probability 1 but* $\mathsf{E}\, T_b = \infty$.

Remark. It may help to notice that this problem is equivalent to the situation when the institution pulls out of the market once its assets hit the positive level b. Also, it may help to strip the context of financial connotations and proceed in an abstract way.

Problem 4 - Answers

All questions are closed. The derivation of (c) from (b) is a routine probabilistic procedure, reducing the work to Calculus. It is expected.

This is (after the switch to maximum) the classical Maximum Principle, resulting from the Reflection Principle for a Brownian Motion. Most of standard textbooks contain it. For example, in Ross' *Stochastic Processes*, 2nd. ed., Section 8.2.

Therefore, no further details will be given.

Problem 5 - Stochastic integration

Using again at least three distinct examples of stochastic processes X_t on $T \subset \mathbb{R}$, show the construction of the integral $\int_T f(t)X(dt)$. Usually T is an interval, the closed or open positive half line, or $\mathbb{R}$. In particular:

(a) your work should exhibit the distinction between the Lebesgue-Stieltjes integral and truly stochastic integral (which does not exist in the former sense);

(b) examine the properties of the process $Y_t = \int_0^t f\, dX$ in regard to preservations of properties of the integrand X_t. For example, if X_t is a martingale (or its increments are independent, or they are uncorrelated, etc.), check if Y_t is also a martingale (preserves the original property);

(c) In your examples show a potential area of applicability of stochastic integration as a suitable model for real life phenomena.

Problem 5 - Answers

All questions are open.

In (a), essentially, a Poisson integral and Wiener (or Brownian) integral should be described. The constructions differ significantly yet the first step - the integral of a step function, is the same, as is the same virtually in stochastic integration with respect to any numerical process, random or not, even vector-valued.

The significant differences appear when the integral is extended beyond simple integrands and the issue of a limit arises.

In the Poisson case the best tool is the Laplace transform of the joint distribution of Poisson increments. It takes exactly the form of the Laplace transform of a Poisson integral of a step function. The a.s. limit then exists provided $\int(|f| \wedge 1) < \infty$. The resulted integral is a path Lebesgue-Stieltjes integral. This can be quickly generalized to abstract Poisson measures.

In contrast, the Wiener integral is based on the L^2 isometry, and utilizes the completeness of L^2. The path integrals do no exist because of unbounded variation, in contrast to the finiteness of the quadratic variation.

Other integrators may use one of these principal approaches. For example, for the stable integrator X_t, if $\alpha < 1$ and $X_t \geq 0$, the Poissonian approach works perfectly. For a symmetric α-stable process, $\alpha < 2$, the Wiener-like approach also works, but then instead of isometry one relies on an isomorphism in L^α.

(b) Examples of properties are listed. Arguments, including computations, are straightforward for most of the chosen properties (and such should be chosen).

(c) This is a very rich topic and a good brief essay may be written.