MATH 7610-7610 Preliminary Exam, Spring 2013

Problem One. Given a $n \times n$ matrix $A = (a_{ij})_{n\times n}$.

(a). Use A to give the definition of strictly diagonally dominant matrix.

(b). Assume that A is strictly diagonally dominant. Show that A is nonsingular.

(c). Show that if A is symmetric, strictly diagonally dominant, and $a_{ii} > 0$, then A is positive definite.

(d). Show that if A is diagonally dominant, then the Jacobi iterative method is convergent.

Problem Two. Consider the implicit finite difference scheme

$$\begin{cases} \frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{u_j^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\Delta x)^2} & 0 < n \le N,\ 1 < j \le J \\ u_j^0 = u_0(j\Delta x), \quad 1 < j \le J \\ u_0^n = u_J^n = 0 & 0 < n \le N, \end{cases}$$

where $\Delta t = 1/N$ and $\Delta x = 1/J$, for the initial boundary value problem of parabolic equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) & 0 < x < 1,\ 0 < t < 1, \\ u(x,0) = u_0(x), & 0 < x < 1, \\ u(0,t) = u(1,t) = 0 & 0 \le t \le 1, \end{cases}$$

(a). Write the finite difference scheme as a linear system of equation in the form of $AU^{n+1} = U^n$

(b). Use Problem One to show that A is positive definite, and the Jacobi iterative method is convergent.

(c). Show that the radius $\rho(A)$ of the spectrum of A satisfies $\rho(A) > 1$ and use it to prove the stability of the scheme.

Problem Three. Consider the Peaceman-Rachford ADI scheme

$$\frac{u_{r,s}^* - u_{r,s}^n}{\frac{1}{2}\Delta t} = \frac{\delta_x^2 u_{r,s}^*}{(\Delta x)^2} + \frac{\delta_y^2 u_{r,s}^n}{(\Delta y)^2}$$

$$\frac{u_{r,s}^{n+1} - u_{r,s}^{*}}{\frac{1}{2}\Delta t} = \frac{\delta_x^2 u_{r,s}^{*}}{(\Delta x)^2} + \frac{\delta_y^2 u_{r,s}^{n+1}}{(\Delta y)^2}$$

for solving the 2-D heat equation

$$u_t = u_{xx} + u_{yy}.$$

Show that the scheme is unconditionally stable, i.e., it is stable for all $\mu_x = \frac{\Delta t}{(\Delta x)^2}$ and $\mu_y = \frac{\Delta t}{(\Delta y)^2}$

Problem Four. Show that the implicit upper wind scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = a\frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x} \quad 0 < n \le N, \quad 1 < j \le J$$

for the hyperbolic PDE

$$\begin{cases} u_t(x,t) + au_x(x,t) = 0, \quad 0 < x < 1, \ 0 < t < t_F, \\ u(x,0) = u_0(x), \quad 0 < x < 1, \\ u(0,t) = 0 \quad 0 < t < t_F \end{cases}$$

is absolutely stable. Here $a > 0$ is a constant, $\Delta t = \frac{t_F}{N}$ and $\Delta x = \frac{1}{J}$.

Problem Five. For the mixed boundary value problem

$$\begin{cases} -u''(x) + u(x) = f(x), \quad 0 < x < 1, \\ u(0) = 1, \ \ u'(1) = 2 \end{cases}$$

(a). Derive the variational equation.

(b). Definite the piecewise continuous linear finite element spaces.

(c). Write the finite element equation for u_h using piecewise linear finite element spaces (no need to calculate the coefficient matrix).

(d). What is the convergence order of u_h to u and $\frac{du_h}{dx}$ to $\frac{du}{dx}$ in L^2 norm (proof is not needed).