Graph Theory Prelim **Name (Print):** ______________________

08/24/24

Time Limit: 4 hours

This exam contains 3 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

Instructions:

- You are required to **show your working** and justify your answers for all questions *except where explicitly stated.*
- **Organize your work clearly**. Dedicate a separate, labeled page for the solution to each question.
- Materials, such as calculators, cell phones, books, and notes, are prohibited.

Academic integrity is expected of all Auburn University students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.
Please sign below to indicate that you have read and agree to these instructions.

Signature of student

Graphs on this exam are simple, but a multigraph may contain parallel edges or loops.

1. Let n and $d_1, \dots, d_n$ be non-negative integers and suppose that $d_1 \geq d_2 \geq \cdots \geq d_n$.
   (a) Prove that $d_1, \dots, d_n$ are the vertex degrees of some multigraph iff $\sum_{i=1}^n d_i$ is even.
   (b) Consider the following theorem.
   Theorem. (Hakimi, 1960s) *The values $d_1, \dots, d_n$ are the vertex degrees of some loopless multigraph iff $\sum_{i=1}^n d_i$ is even and $d_1 \leq d_2 + d_3 + \cdots + d_n$*
   State clearly which direction of this theorem is the *easy direction* (i.e. either the forwards or backwards direction), and then prove this easy direction.
   (c) The theorem of Hakimi stated in part (b) becomes false if the term "loopless multigraph" is replaced by "simple graph". State precisely which direction(s) of the theorem fail after this replacement, providing explicit counterexample(s).

2. This question is about connectivity.
   (a) Carefully define the *vertex-connectivity* $\kappa(G)$ of a graph G.
   (b) State the local version of Menger's Theorem.
   (c) Consider the 5×5 grid graph G whose vertex set is all pairs of integers between 1 and 5 inclusive, and with an edge between two vertices (x, y) and (x', y') if and only if
   - $x = x'$ and y, y' differ by 1, or
   - $y = y'$ and x, x' differ by 1.

   Determine, with proof, the minimum size of a vertex cut separating the vertices $(2, 2)$ and $(4, 4)$.
   (d) Suppose that G is a graph on at least $k + 1$ vertices, and that $\kappa(G) \geq k$. Prove that G has k internally disjoint paths between any pair of vertices (you can apply Menger's Theorem).

3. Let G be a graph and let the deficiency of G be defined as $\text{def}(G) = \max\{o(G - S) - |S| : S \subseteq V(G)\}$, where $o(G - S)$ is the number of components of $G - S$ that are of odd order. Suppose that $S \subseteq V(G)$ is maximal with $o(G - S) - |S| = \text{def}(G)$. Prove the following statements.
   (a) $G - S$ has no even component.
   (b) Every odd component D of $G - S$ is factor critical, i.e., D has no 1-factor but $D - v$ has a 1-factor for any $v \in V(D)$.
   (c) Let H be the bipartite multigraph obtained from G by contracting each component of $G - S$ into a single vertex and deleting edges within S. Then H has a matching saturating S.
   (d) Deduce, by applying Parts (a)-(c), that G has a matching of size $\frac{1}{2}(|V(G)| - \text{def}(G))$.

4. This question is about Hamilton cycles.
   (a) Prove that every cycle on at least three vertices is 1-tough.

(b) Let G be a 1-tough $(P_2 \cup P_1)$-free graph on at least three vertices. Show that G is Hamiltonian by applying Dirac's Theorem (1952) on Hamilton cycles.

(c) Prove that a $(P_3 \cup P_1)$-free graph G on at least three vertices is Hamiltonian if and only if G is 1-tough.

5. Let $s, t \in \mathbb{Z}^+$. Let $R(s,t)$ be the minimum n such that every simple n-vertex graph contains either K_t or $\overline{K_s}$, if such an n exists (if not, set $R(s,t) = \infty$).

   (a) Determine $R(s,t)$ for $s \in \{1, 2\}$.

   (b) Prove that $R(s,t) \leq R(s-1,t) + R(s,t-1)$ for all $s, t \in \mathbb{Z}^+$ with $s, t \geq 2$.
   Hint: Let $N = R(s-1,t) + R(s,t-1)$, and let G be any simple graph on N vertices.

   (c) State *Ramsey's Theorem* and prove it using *induction on $s + t$* (and using (a) and (b)).