Design Theory Prelim

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1. Let v, k and λ be positive integers, with $1 < k < v - 1$. We say that (V, B) is a *(v, k, λ) design* if V is a set of *v points*, B is a collection of k-element subsets of V called *blocks*, and every pair of points is contained in exactly λ blocks.
   a. How many blocks are there?
   b. Let x be a point. How many blocks contain x?
   c. Let y be a point other than x. How many blocks contain x, but not y?
   d. We define the *complementary design* (V, B') as follows: $B' = \{V \setminus b \mid b \in B\}$. Prove that (V, B') is a $(v, v-k, \lambda')$ design, where $\lambda' = \lambda(v-k)(v-k-1)/k(k-1)$.
   e. List the blocks of a (7, 4, 2) design, which you may find by applying the construction in (1d) to the appropriate Steiner triple system.

2. A quasigroup $(V,\circ)$ is said to be *antisymmetric* if $u \circ w = w \circ u$ implies that $u = w$. A quasigroup of order n, $(Z_n = \{0, 1, \ldots n\text{-}1\}, \circ)$, is said to be a *shift-right* quasigroup if $u \circ w = (u+1) \circ (w+1)$ for all u,w in Z_n (reducing the sums modulo n).
   a. Find an antisymmetric quasigroup of order 4.
   b. Find all the values of n for which shift-right quasigroups are antisymmetric. Give reasons for your answer (both for the values that do produce quasigroups that are antisymmetric, and those that do not). (Hint: it suffices to consider the first row!)
   c. Suppose that $(V_1,\circ_1)$ and $(V_2,\circ_2)$ are quasigroups. Show that the direct product of $(V_1,\circ_1)$ and $(V_2,\circ_2)$ is antisymmetric if and only if $(V_1,\circ_1)$ and $(V_2,\circ_2)$ are antisymmetric.

3. Let $(V,\circ)$ be a quasigroup of order n.
   a. Describe a variation of the Bose Construction that uses $(V,\circ)$ to produce a $(v, 3, 2)$ design of order $v \equiv 1 \pmod 3$, $v \geq 10$.
      i. What additional property does $(V,\circ)$ need to satisfy if your construction is to work?
      ii. What is the value of n in your construction?
   b. What additional properties could you require in your construction to ensure that your $(v, 3, 2)$ design contains no repeated triples? Describe why such properties would have the desired effect, but do not prove that the ingredients exist. (Hint; Question 2 may be of use.)
   c. Construct a (7, 3, 2) design that has no repeated triples.

4. In the following, it may help to know that $1+x+x^3$ is an irreducible polynomial over GF(2).
   a. Find the first two rows of a pair of orthogonal latin squares of order 8, then simply describe how you would complete the latin squares.
   b. Using just the finite field and direct product constructions for sets of MOLS, how many pairwise orthogonal latin squares of order 400 could you make? Give a reason for your answer.
   c. $\{i, i+1, i+4, i+6 \mid 0 \leq i \leq 13\}$ is a set of blocks of a GDD of order 14 on the symbols in Z_{14} (reducing the sums modulo 14).
      i. What are the groups in this GDD?
      ii. Describe how it can be used to make a PBD of order 43 with 7 blocks of size 7 and the rest of size 4.