Name: ____________________________________

Mathematical Statistics Preliminary Examination

August 11, 2021, 9:00am - 1:00pm, Room 224, Parker Hall

Directions:

1. This is a closed-book in-class exam.
2. Your proctor will determine your seat in the room where you take the exam.
3. You may not use a calculator.
4. The proctor will provide as many blank sheets of paper as you need.
5. Work any five out of the eight problems. You may submit solutions for at most five problems.
6. You need to start each problem on a new page. Clearly label each problem and number each page and write your name on top right of each page.
7. To get full credit you need to properly document your solutions.
8. Each problem is worth 10 points.
9. You need to turn in the typeset pages that were given to you along with your solutions.

Please mark the five problems you are submitting for grading in the table below.

Problem	1	2	3	4	5	6	7	8
Submit for grading								
Score								

1. Let $X_1, X_2, \ldots$ be a sequence of iid random variables from distribution F.

(a) If $EX_i = \mu$ and $\text{Var}X_i = \sigma^2$ with $0 < \sigma^2 < \infty$, then does X_i converge in probability? If so, find the limiting rv, if not provide a counterexample.

(b) If $EX_i = \mu$ and $\text{Var}X_i = \sigma^2$ with $0 < \sigma^2 < \infty$, then does X_i converge a.s.? If so, find the limiting rv, if not provide a counterexample.

(c) Does X_i converge in distribution? If so, find the limiting distribution, if not provide a counterexample.

(d) For $EX_i = \mu$ and $\text{Var}X_i = \sigma^2 = 0$, repeat parts (a)-(c).

(e) Let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Assume $EX_i = \mu$ and $\text{Var}X_i = \sigma^2$ and that MGFs of X_i exist in a neighborhood of 0 for all i. Repeat parts (a)-(c) for Y_n.

Hints:

(I) You may use the Cauchy criteria for convergence of a random sequence:

  (i) X_n converges a.s. $\iff P(\lim_{m,n\to\infty} |X_m - X_n| = 0) = 1$.

  (ii) X_n converges in probability $\iff \lim_{m,n\to\infty} P(|X_m - X_n| < \epsilon) = 1$ for all $\epsilon > 0$.

(II) You may also use the (well-known) named results concerning the sample mean in part (e).

2. Suppose $X_1, X_2, \ldots, X_n$ are iid random variables from Uniform$(\theta, \theta+1)$ distribution with $-\infty < \theta < \infty$.

(a) Show that $T(\mathbf{X}) = \left(X_{(1)}, X_{(n)}\right)$ is a minimal sufficient statistic for θ.

(b) Is $X_{(n)} - X_{(1)}$ also a minimal sufficient statistic?

(c) Show that T is not complete.

(d) Find an MLE of θ.

(e) Can you find a complete sufficient statistic for θ?

3. Let $X_1, X_2, \ldots, X_n$ are iid random variables from the Exponential(θ) distribution with pdf $f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}$ for $x > 0$ and $\theta > 0$. This density has mean θ and variance θ^2 (no need to verify these).
Hints:
I- The pdf of the Gamma(α, β) distribution is

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \quad \text{for} \quad x \geq 0, \alpha, \beta > 0.$$

II- Gamma$(\alpha,\beta) \stackrel{d}{=} \frac{\beta}{2}\chi^2_{2\alpha}$ (That is, if $X \sim$ Gamma(n,β), then $2X/\beta \sim \chi^2_{2\alpha}$.)

(a) Use Neyman-Pearson Lemma to find a uniformly most powerful (UMP) size α test of $H_0 : \theta = 2$ vs $H_1 : \theta = 1$. Express your answer in terms of a chi-square (χ^2) distribution.

(b) Construct a UMP level α test of $H_0 : \theta \geq 2$ vs $H_1 : \theta < 2$. (Justify your answer.)

(c) Express the power function $\beta(\theta)$ for your test in part (b) in terms of a chi-square distribution. Use this to show that your test in part (b) actually is a size α unbiased test.

(d) Derive the likelihood ratio test (LRT) for testing $H_0 : \theta = 2$ vs $H_1 : \theta \neq 2$. State an approximation to the distribution of the LRT statistic under H_0 when n is large (no proof necessary for this approximation).

4. Suppose X is a random variable with $E(X) = \mu \neq 0$ and $\text{Var}(X) = \sigma^2 < \infty$.

(a) Let $g(x)$ be a differentiable function of x in its domain. Estimate the mean and variance of the random variable $g(X)$ in terms of μ and σ^2 (i.e., what would be the mean and variance of $g(X)$ approximately?).

(b) Find the approximations in part (a) for $g(X) = X^3$.

(c) Suppose now that we have the mean of a random sample of size n, i.e., $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ where $X_i \stackrel{iid}{\sim} F$ with $\text{Var}X_i = \sigma^2$ with $0 < \sigma^2 < \infty$, then find the asymptotic distribution of $\bar{X}^3$.

(d) The asymptotic variance in part (c) involves σ^2 and μ which may both be unknown. Approximate the asymptotic variance (so that it will be computable from the sample) and restate the asymptotic distribution in part (c) with this approximate variance, and justify the convergence in distribution in this setting.

5. Let X_i be independent random variables from the Poisson($c_i\lambda$) distribution with c_i a known positive constant for $i = 1, \ldots, n$, and $\lambda > 0$. (Recall that $X \sim$ Poisson(λ) has pmf $f(x|\lambda) = \dfrac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0, 1, \ldots$ and $\lambda \geq 0$.)

(a) Find the MLE of λ.

(b) Is the MLE in part (a) a minimal sufficient statistic for λ. Is it also complete?

(c) Let $\tau(\lambda) = P(\sum_{i=1}^n X_i = 0)$. Compute the Cramér-Rao lower bound for the variance of unbiased estimators of $\tau(\lambda)$.

(d) Find also the UMVUE of $\tau(\lambda)$ in part (c). Express this UMVUE in terms of sample statistics (so that it is computable given a sample).

(e) Let the prior density of λ be $\pi(\lambda) \equiv$ Exponential(1/10). Compute a Bayes estimate of λ. (For the pdfs of Exponential(θ) and Gamma(α, β) distributions, see Problem 3. Also, for $X \sim$ Gamma(α, β), $EX = \alpha\beta$.)

6. Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ and $\bar{X}$ and S^2 be the sample mean and sample variance, respectively.

(a) Find two pivotal quantities: one based on $\bar{X}$, μ and σ^2 and the other based on $\bar{X}$, μ and S^2. Also find the distribution of each pivotal quantity.

(b) Construct a $1 - \alpha$ pivotal confidence interval for μ (based on each pivot in part (a)). Make sure you use the correct quantiles of the distributions of the pivots.

(c) Consider the two cases: case (1) σ^2 is known, and case (2) σ^2 is unknown. Compute the expected length of each CI and comment on which of the CI's in part(b) is better to use in each case.

(d) When σ^2 is unknown, find a pivotal quantity based on σ^2 and S^2 and construct a $1 - \alpha$ confidence interval for σ^2 based on this pivot.

7. Let $X_1, X_2, \ldots, X_n$ be iid rv's from the Uniform$(-\theta, \theta)$ distribution with $0 < \theta < \infty$.

(a) Find the MLE of θ. (Hint: Consider $Y_i = |X_i|$ for $i = 1, 2, \ldots, n$)

(b) Find a pivotal quantity for θ based on the MLE in part (a)? Explain your answer.

(c) Find the shortest $1 - \alpha$ pivotal CI for θ based on the pivot in part (b).

(d) Find the corresponding $1 - \alpha$ test which will give you the CI in part (c) after inversion.

8. Drs. Xavier, Serene, and King want to generate samples of size n uniformly (i.e., n uniform data points) in the unit circle (i.e., in the circle centered at the origin with unit radius) in $\mathbb{R}^2$. That is, they want to generate X_i, Y_i iid with joint pdf

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi}, & \text{if } x^2 + y^2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Hints:
(I) The point (x, y) in Cartesian coordinates corresponds to (r, θ) in polar coordinates with $x = r\cos\theta$ and $y = r\sin\theta$.
(II) $\frac{d}{dx}\arctan x = \frac{1}{1 + x^2}$
(III) Let $A, B \subset \mathbb{R}^2$. Then, for $(X, Y) \sim$ Uniform(A), $P((X, Y) \in B) = \frac{Area(B \cap A)}{Area(A)}$.

(a) Dr. Xavier suggests the following approach: generate R_i iid from Uniform$(0, 1)$ (the uniform distribution on $(0, 1)$) and θ_i iid from Uniform$(0, 2\pi)$ and set $(X_i, Y_i) = (R_i \cos\theta_i, R_i \sin\theta_i)$ for $i = 1, 2, \ldots, n$. Is this strategy correct? Justify your answer.

(b) Dr. Serene believes Dr. Xavier's method is flawed, but cannot figure out a correct strategy. If Dr. Serene is right, can you help her out (in finding the correct transformation of the (R_i, θ_i) in part (a) to (X_i, Y_i) iid uniform on the unit circle)?

(c) On the other hand, Dr. King is a practical fellow and claims a version of rejection sampling (if correctly done) should work. So, he suggests the following: Generate X_i iid from Uniform$(-1, 1)$ and Y_i iid from Uniform$(-1, 1)$ and keep the only (X_i, Y_i) pairs satisfying $X_i^2 + Y_i^2 \leq 1$ for $i = 1, 2, \ldots, n$. Do you think he is correct? Justify your answer.