Name: ______________________________

Mathematical Statistics Preliminary Examination

August 14, 2020, 9:00am - 1:00pm

Directions:

1. This is a closed-book online exam which will be administered via Zoom. Please arrange your video options under Zoom so that you and your work are visible and keep your sound on at all times during the exam. If you want to ask a question, use the private chat option in Zoom (to communicate with me). The exam will be recorded at our end in case needed for later reference. At the end of the exam, you need to scan or take images of your work and send the scanned file or image file to me via email (keep yourself visible during this scanning/imaging process as well and do this process as fast as you can).
2. It is your responsibility to choose where you take this exam, but we recommend a quiet place with minimal disturbance possible during the entire exam period.
3. You may not use a calculator.
4. It is your responsibility to prepare spare blank sheets of paper as you need before the exam.
5. Work any five out of the eight problems. You may submit solutions for at most five problems.
6. You need to start each problem on a new page. Clearly label each problem and write your name at top right of each page.
7. To get full credit you need to properly document and explain/justify your solutions.
8. Each problem is worth 10 points.

Please mark the five problems you are submitting for grading in the table below.

Problem	1	2	3	4	5	6	7	8
Submit for grading								
Score								

1. Let $X_1, X_2, \ldots, X_n$ be random sample (i.e. X_i are independent, identically distributed (iid) random variables) from a distribution with probability density function

$$f(x|\theta) = \theta e^{-\theta x} \quad \text{for } x \geq 0 \text{ and } \theta > 0.$$

(a) Find the maximum likelihood estimator (MLE) of θ (denote this MLE as $\hat{\theta}$).

(b) Show that the method of moments estimator (MoM) of θ (denoted $\tilde{\theta}$) is the same as the MLE in part (a).

(c) Let $\tau(\theta) = 1/\theta$. Find the MLE of $\tau(\theta)$ (denote this MLE as $\hat{\tau}(\theta)$. State any property/result you are using when answering this. Is $\hat{\tau}(\theta)$ unbiased for $\tau(\theta)$?

(d) Find the variance of $\hat{\tau}(\theta)$.

(e) Suppose $n = 36$ and $\theta = 1/2$. Estimate $P(|\hat{\theta}| > 2/5)$ using the Central Limit Theorem. Express your answer in terms of the cumulative distribution function Φ of the standard normal distribution (i.e., you do not need to evaluate your answer).

2. Let random vectors $\mathbf{U}_i$ be iid from a bivariate normal (BVN) distribution, that is,

$$\mathbf{U}_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix} \overset{iid}{\sim} \text{BVN}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \right) \quad \text{for } i = 1, 2, \ldots, n$$

where $\theta > 0$. The length of random vector $\mathbf{U}_i$ is $Z_i = \sqrt{X_i^2 + Y_i^2}$ has pdf

$$f_Z(z) = \frac{z}{\theta} e^{-z^2/(2\theta)} \quad \text{for } z > 0$$

(You do not need to derive this.)

(a) Find $E(Z_i)$ and $\text{Var}(Z_i)$. (Hint: To find the variance, finding the distribution of Z_i^2 first might help.)

(b) Derive a level α uniformly most powerful test of $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$ based on $Z_1, \ldots, Z_n$. Find also the critical value for the test.

3. Let $(X_1, X_2, X_3) \sim$ Multinomial$(n; p_1, p_2, p_1 + p_2)$ where $X_1 + X_2 + X_3 = n$.
(If $(X_1, X_2, \ldots, X_k) \sim$ Multinomial$(n; p_1, p_2, \ldots p_k)$, then its pmf is
$f(x_1, x_2, \ldots, x_k | p_1, p_2, \ldots, p_k) = \frac{n!}{x_1! x_2! \cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$.)

(a) Find the range of the parameter p_1?

(b) What is a minimal sufficient statistic T for p_1 (if exists). (Justify your answer.)

(c) If T in part (b) is complete, find the UMVUE of p_1, denoted as $\tilde{p}_1$. If T is not complete, can you find a (nontrivial) ancillary statistic U? Also, what is the conditional distribution $T|U$? What would be the (conditional) UMVUE $\tilde{p}_1$ based on this conditional distribution?

(d) Find the MLE $\hat{p}_1$ of p_1 (and determine when it exists). Also derive the asymptotic distributions of $\sqrt{n}(\hat{p}_1 - p_1)$ and $\sqrt{n}(\tilde{p}_1 - p_1)$? (You can assume that the regularity conditions for the asymptotic distribution of MLEs are satisfied, hence you do not need to check them.)

(e) Notice that $X_1 \sim$ Binomial(n, p_1), and so $p_1^u = X_1/n$ is clearly an unbiased estimator of p_1 (you do not need to verify these). Find the (asymptotic) relative efficiency of p_1^u with respect to $\hat{p}_1$ and $\tilde{p}_1$. Which of the estimators is preferable?

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on $(0, \theta)$. That is, $X_i \overset{iid}{\sim} U(0, \theta)$ for $i = 1, \ldots, n$ where $\theta > 0$ is an unknown parameter.

(a) Find an expression for the k^{th} (noncentral) moment of X_i's. Recall that k^{th} (noncentral) moment of a random variable X is $\mu_k = E(X^k)$ where $k \geq 1$ is an integer.

(b) Find an expression for $\text{Var}(X_i^k)$.

(c) Find a method of moments (MoM) estimator $\tilde{\theta}_k$ of θ based on the k^{th} (noncentral) sample moment $m_k = \frac{\sum_{i=1}^n X_i^k}{n}$ of the sample $X_1, X_2, \ldots, X_n$. Derive the asymptotic distribution of $\tilde{\theta}_k$. Hint: First, find the asymptotic distribution of m_k and then apply the Delta method to an appropriately chosen function of m_k.

(d) Show that for any MoM estimator of μ_k, there exists another MoM estimator (possibly with a different k) having an asymptotic distribution with smaller variance.

5. Let $X \sim$ Binomial(n, p) where $p \in (0, 1)$.

(a) Find $E(X/(n - X))$.

(b) Find $E(X/(n + 1 - X))$ and show that it converges to $p/(1 - p)$ as $n \to \infty$.

(c) Does a bounded unbiased estimator of $p/(1 - p)$ based on X exist? Explain your answer. (Hint: Try finding $g(x)$ with $Eg(X) = p/(1 - p)$ and check whether $g(x)$ is bounded or not over the support of X.) Does an asymptotically unbiased estimator of $p/(1 - p)$ exist? Explain your answer.

(d) Can you find the asymptotic distribution of $X/(n - X)$? If so, derive it. If not, explain why it does not exist.

6. The number of calls a customer service department of a company receives has a Poisson distribution with rate λ calls per 10 minutes. That is, if X is the number of calls received in (any) 10 minutes, then $X \sim \text{Poisson}(\lambda)$ with pmf $f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0, 1, \ldots$ and $\lambda \geq 0$. Based on a single observation X, we want to find the UMVUE of $\theta = P(\text{no calls in the next 20 minutes})$.

(a) What is the distribution of Y, the number of calls in (any) 20 minutes?

(b) Write θ in terms of λ.

(c) Is X a complete sufficient statistic for λ? (Justify your answer.)

(d) Find an unbiased estimate of θ based on X. [Hint: Rather than the mechanism provided in Casella & Berger, it might help to employ the Taylor expansion $e^{-\lambda} = \sum_{x=0}^{\infty} \frac{(-1)^x \lambda^x}{x!}$ and find $g(x)$ so that $Eg(X) = \theta$.)

(e) Why is the estimate you found in part (d) the UMVUE (i.e., the best unbiased estimate) of θ?

(f) Is the UMVUE in part (d) a reasonable estimator of θ? (i.e., Would you use it in practice to estimate θ?) Explain your answer.

7. Let $X_i \overset{iid}{\sim} U(0, \theta)$ for $i = 1, 2$. For the simple hypothesis tests of $H_0 : \theta = 1$ vs $H_1 : \theta = 2$, consider the tests

$$\phi_1 : \text{Reject } H_0 \text{ if } x_1 > c_1$$
$$\text{and}$$
$$\phi_2 : \text{Reject } H_0 \text{ if } x_1 + x_2 > c_2.$$

(a) Find the values of c_1 and c_2 that make each of the above tests has size 0.02.

(b) Compute the power of each of the two size 0.02 tests ϕ_1 and ϕ_2 above.

(c) Are either of the two tests ϕ_1 and ϕ_2 most powerful size 0.02 for the composite hypothesis tests of $H_0 : \theta \leq 1$ vs $H_1 : \theta \geq 1$? If so, prove it. If not, obtain the most powerful size 0.02 test and compute its power at $\theta = 2$.

8. Let $X_1, \ldots, X_n$ be independent random variables where $X_i \sim \text{Poisson}(\theta\kappa_i)$, $\theta > 0$ is unknown and κ_i is a known positive constant for $i = 1, \ldots, n$. (For the Poisson pmf, see Question 6).

(a) Provide a minimal sufficient statistic for θ. Is it also complete?

(b) Determine the Cramér-Rao lower bound for the variance of unbiased estimators of θ.

(c) Find the UMVUE of θ. Also, find the variance of this estimator.

(d) Let the prior density for θ be exponential distribution with mean 1 (i.e., $\pi(\theta) = e^{-\theta}$, $\theta \geq 0$). Find an explicit expression for a Bayes estimate of θ. (Hint: If $X \sim \text{Gamma}(\alpha, \beta)$ then its pdf is $f_X(x) \propto x^{\alpha-1}e^{-x/\beta}$ for $x \geq 0$, $\alpha, \beta > 0$ and $E(X) = \alpha\beta$.)