Name (readable): ______________________________

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Preliminary Examination in
ORDINARY DIFFERENTIAL EQUATIONS
January 2010

Instruction. 90 points are required for passing, 150 points for a "100% performance". Time: 3 hours.

1. Find the principal fundamental matrix solution of

$$\begin{cases} \dot{x} = x - y + z, \\ \dot{y} = -x + y - z, \\ \dot{z} = x - y + z. \end{cases} \tag{1}$$

20 pts.

2. Let $n \in \mathbb{N}$, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $V \in C^1(\mathbb{R}^n, \mathbb{R})$, $\lim\limits_{\|x\| \to \infty} V(x) = \infty$, and $f(x) \cdot \nabla V(x) \leq 0$ for $x \in \mathbb{R}^n$. Denote by φ the solution flow of

$$\dot{u} = f \circ u, \tag{2}$$

and set $Z := \{x \in \mathbb{R}^n : \nabla V(x) \cdot f(x) = 0\}$.

a. Show that $t^+(x) = \infty$ for $x \in \mathbb{R}^n$ and that $\gamma^+(x)$ is bounded for all $x \in \mathbb{R}^n$.

b. Show that the ω-limit set $\omega(x)$ of $\varphi(\cdot, x)$ is a subset of Z for $x \in \mathbb{R}^n$.

c. Show that (2) has no nontrivial periodic solution, if Z is an isolated set. (Z is called isolated, iff for $z \in Z$, there exists an $r \in (0, \infty)$ with $Z \cap B_z(r) = \{z\}$.)

30 pts.

3. a. State Gronwall's inequality.

b. State a variation of parameters formula.

Let $n \in \mathbb{N}$, $A, B \in C([0,\infty), \mathfrak{M}_{n,n})$, and Φ and Ψ be fundamental matrix solutions of $\dot{x}(t) = A(t)x(t)$ and $\dot{y}(t) = B(t)y(t)$, respectively. Assume $\sup\{\|\Phi(t)\|_{n,n} : t \geq 0\} < \infty$ and $\sup\{\|\Phi(t)^{-1}\|_{n,n} : t \geq 0\} < \infty$.

c. Prove that $u \equiv 0$ is the only solution $u \in C^1([0,\infty), \mathbb{R}^n)$ of $\dot{x}(t) = A(t)x(t)$ which satisfies $\lim\limits_{t \to \infty} u(t) = 0$.

d. Assume that $\int_0^\infty \|A(t) - B(t)\|_{n,n} \, dt < \infty$. Show that $\sup\{\|\Psi(t)\|_{n,n} : t \geq 0\} < \infty$.

30 pts.

4. Consider

$$\begin{cases} \dot{x} = -(x^2+y^2)y \\ \dot{y} = x + y - (x^2+y^2)y \end{cases} \tag{3}$$

a. Verify that $X : t \mapsto (\cos(t), \sin(t))$ is a (2π-)periodic solution of (3).

b. Prove that X is orbitally stable under the solution flow of (3).

25 pts.

5. a. Define the concept "center manifold" and state the center manifold theorem.

b. Let $f \in C^2(\mathbb{R}^3, \mathbb{R}^3)$, $f(\mathbf{0}) = \mathbf{0}$, $g \in C^2(\mathbb{R}^3, \mathbb{R})$, $S := \{\mathbf{x} : g(\mathbf{x}) = 0\}$, $\nabla g(\mathbf{x}) \neq \mathbf{0}$ for $\mathbf{x} \in S$, and $f(\mathbf{x}) \cdot \nabla g(\mathbf{x}) = 0$ for $\mathbf{x} \in S$. Denote by φ the solution flow of $\dot{u} = f \circ u$. Assume that

- $\mathbf{0}$ is an isolated zero of f;
- $0 \in \sigma((Df)(\mathbf{0}))$ with a 2-dimensional eigenspace E;
- $(Df)(\mathbf{0})$ has an eigenvalue with nonzero real part;
- $z \cdot \nabla g(\mathbf{0}) = 0$ for $z \in E$.

Prove that S is a center manifold of φ at $\mathbf{0}$.

25 pts.

6. Let $n \in \mathbb{N}$, $U \subseteq \mathbb{R}^n$ be open, $z \in U$, and $f \in C^1(U, \mathbb{R}^n)$ with $f(z) = 0$. Denote by φ the solution flow of $\dot{u} = f \circ u$, and assume that z is asymptotically stable under φ. Prove that $\{x \in U : z \in \omega(x)\}$ is open. Give an counterexample if the stability assumption is dropped.

20 pts.

7. Consider

$$\begin{cases} \dot{x} = -x^3 \\ \dot{y} = -y + x^2 \end{cases} \tag{4}$$

a. Show that (4) has no non-constant periodic solution.

b. Show that $(0,0)$ is the only rest point of the solution flow φ of (4) and determine the stable and center spaces of the linearization at $(0,0)$.

c. Prove that $(0,0)$ is a stable rest-point of φ.

30 pts.

8. a. Define the concept "global attractor" and state the existence theorem for a global attractor.

b. Denote by φ the solution flow of

$$\begin{cases} \dot{x} = x(1 - x^2 - y^2) \\ \dot{y} = y(1 - \frac{x^2}{4} - y^2) \end{cases} \tag{5}$$

(i) Find the nullclines for (5) and rest points of φ.

(ii) For each rest point, linearize (5) at the rest point and determine the stable, unstable and center spaces of the linearization.

(iii) Show that φ has a global attractor.

(iv) Determine positively invariant sets under φ and conclude that φ has no regular periodic orbit.

(v) Describe an under φ invariant compact set with $(0, 0)$ as interior point by means of unstable manifolds.

(vi) Use (v), possibly without proof, do describe the global attractor.

70 pts.