Graph Theory Prelim 2013

1. (a) Define the terms *path* and *connected*.

   (b) Prove that if G is a connected graph, then any two paths of maximum length in G have a common vertex.

   (c) What simple graphs have the property that every two edges have exactly one vertex in common? Prove your answer to be true.

2. All graphs in this question are simple. If H is a *spanning subgraph* of G (a subgraph of G with $V(H) = V(G)$), the *relative complement of H in G* is defined to be the spanning subgraph of G whose edge set is $E(G) \setminus E(H)$. If G is a complete graph, then this graph is called the *complement* of H.

   (a) If k is a positive integer, and H is a subgraph of the complete bipartite graph $K_{2,k}$, prove that H is isomorphic to its relative complement in $K_{2,k}$ if and only if H has exactly k edges.

   (b) Consider the following graph H on nine vertices: its vertices are the nine cells of a three by three checkerboard, and two different vertices are adjacent if and only if they lie in either the same row, or the same column. Prove that H is isomorphic to its complement.

3. (a) Let G be an n-vertex graph. Denote by $\alpha(G)$ the maximum size of an independent set in G, and denote by $\tau(G)$ the minimum size of a *vertex cover* in G (a set of vertices S in G such that every edge in G is incident to at least one vertex in S).

      i. Prove that $\tau(G) + \alpha(G) = n$.

      ii. Prove that if G is simple and triangle-free, then

      $$|E(G)| \leq \alpha(G) \cdot \tau(G)$$

   (b) Use part (a) to prove that an n-vertex triangle-free simple graph has at most $\frac{n^2}{4}$ edges. (This result is part of *Mantel's Theorem*, which asserts, in addition, that if G is a triangle-free n-vertex simple graph, then $|E(G)| = \lfloor \frac{n^2}{4} \rfloor$ if and only if $G \approx K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$).

   (c) State *Turán's Theorem* and explain briefly how it generalizes Mantel's Theorem.

4. (a) Suppose that $n > 1$ and $0 < d_1 \leq d_2 \leq \cdots \leq d_n$ are integers. Prove that there is a loopless multigraph on vertices $v_1, \ldots, v_n$ in which v_i has degree d_i, $i = 1, \ldots, n$, if and only if

      i. $\sum_{j=1}^{n} d_j$ is even, and

      ii. $\sum_{j=1}^{n-1} d_j \geq d_n$.

   (b) Suppose that $n > 1$ and $0 < p_1 \leq \cdots \leq p_n$ are integers. Give, with proof, necessary and sufficient conditions for $K_{p_1,\ldots,p_n}$ to have a perfect matching ($K_{p_1,\ldots,p_n}$ is the *complete multipartite* graph consisting of n independent pairwise disjoint sets of vertices with sizes $p_1, \ldots, p_n$, respectively, where every pair of vertices in different sets is joined by an edge).

   (c) Suppose that $0 < p_1 \leq p_2 \leq p_3 \leq p_4 \leq p_5$ are integers. Find, in terms of $p_1, \ldots, p_5$, the number of edges in a *smallest maximal matching* in K_{p_1,p_2,p_3,p_4,p_5}.