Graph Theory Prelim 2008

1) Let G be a bipartite graph with bipartition (A, B), and at least one edge.

a) State Hall's theorem on matchings saturating A.

b) Suppose (for this part b only) that for all $a \varepsilon A$, $b \varepsilon B$, that $1 \le \deg(b) \le \deg(a)$. Prove that G has a matching saturating A.

c) State König's theorem on the chromatic index (edge chromatic number) of G.

d) Prove that G has a matching saturating all its vertices of maximum degree.

2) Let G be a simple graph with minimum degree $\delta \ge 2$.

a) Prove that G contains cycles of at least $\delta - 1$ different lengths. (Hint – consider a longest path.)

b) Prove that G contains a cycle of length at least $\delta + 1$.

c) Prove that G contains a path of length exactly δ.

d) For each $\delta \ge 2$, find an example of such a connected G with no paths of length $\delta + 1$.

e) Prove or disprove: your example in part d is unique (up to isomorphism).

3) A *kernel* of a directed graph D is a set of independent vertices K such that for each vertex *z* in V(D)\K, there exists an arc directed from *z* to a vertex in K. The kernel number *K(D)* is the size of a smallest kernel in D, if D has a kernel. (If D has no kernel, define *K(D)* to be ∞.) For an undirected graph G, the kernel number *K(G)* is the size of a smallest kernel among all orientations of G (an orientation of G is formed by replacing each edge of G with an arc).

a) Find a directed graph D = (V, A) on 5 vertices, with a kernel K of size 3, and with V\K is another kernel.

b) Show that if K is a kernel of a directed graph D, then K is a maximal independent set in D.

c) Let $i(G)$ be the size of a smallest maximal independent set in the G. Show that $i(G)=K(G)$ (Hint: Show that $i(G) \le K(G)$, and show that $i(G) \ge K(G)$.)