1. (III.3.5) Let \( R \) be a principal ideal domain.
   (a) Every proper ideal is a product \( P_1 P_2 \cdots P_n \) of maximal ideals, which are uniquely determined up to order.
   
   (b) An ideal \( P \) in \( R \) is said to be primary if \( ab \in P \) and \( a \notin P \) imply \( b^n \in P \) for some \( n \). Show that \( P \) is primary if and only if for some \( n \), \( P = (p^n) \), where \( p \in R \) is prime (=irreducible) or \( p = 0 \).
   
   (c) If \( P_1, \cdots, P_n \) are primary ideals such that \( P_i = (p^n_i) \) and the \( p_i \) are distinct primes (up to associate), then \( P_1 P_2 \cdots P_n = P_1 \cap P_2 \cap \cdots \cap P_n \).
   
   (d) Every proper ideal in \( R \) can be expressed (uniquely up to order) as the intersection of a finite number of primary ideals.

2. (III.3.11) Let \( R \) be a Euclidean ring and \( a \in R \). Then \( a \) is a unit in \( R \) if and only if \( \varphi(a) = \varphi(1_R) \).

3. If \( R \) is a principal ideal domain, then a greatest common divisor of \( X \subset R \) exists and is of the form \( r_1a_1 + \cdots + r_na_n \) for some \( a_i \in X \) and \( r_i \in R \).

4. Let \( S \) be a multiplicative subset of \( R \).
   (a) If \( I \) is an ideal of \( R \), then \( S^{-1}I \) is an ideal of \( S^{-1}R \). Conversely, every ideal of \( S^{-1}R \) is of the form \( S^{-1}I \) for \( I \subset R \).
   
   (b) If \( J \) is another ideal of \( R \), then the following equalities hold for ideals of \( S^{-1}R \):
   
   \[
   S^{-1}(I + J) = S^{-1}I + S^{-1}J, \quad S^{-1}(IJ) = (S^{-1}I)(S^{-1}J), \quad S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J
   \]

   You are required to prove only one of the equalities.

5. (III.4.16) Every nonzero homomorphic image of a local ring is local.

6. (III.6.11) If \( c_0, c_1, \cdots, c_n \) are distinct elements of an integral domain \( D \) and \( d_0, \cdots, d_n \) are any elements of \( D \), then there is at most one polynomial \( f \) of degree \( \leq n \) in \( D[x] \) such that \( f(c_i) = d_i \) for \( i = 0, 1, \cdots, n \). [For the existence of \( f \), see Exercise III.6.12]

7. (III.6.14) Let \( R \) be an integral domain and \( c, b \in R \) with \( c \) a unit.
   (a) Show that the assignment \( x \mapsto cx + b \) induces a unique automorphism of \( R[x] \) that is the identity on \( R \). What is its inverse?
   
   (b) Show that every automorphism of \( R[x] \) that is the identity on \( R \) is of the type above.

8. (IV.1.6) A finitely generated \( R \)-module need not be finitely generated as an abelian group.