Algebra I Homework One

Name:___________________

Instruction: In the following questions, you are requested to work out the solutions in a clear and concise manner. You may discuss the ideas with peers, but you should write down the solutions independently. Three questions will be randomly selected and checked for correctness; they count 50% grades of this homework set. The other questions will be checked for completeness; they count the rest 50% grades of the homework set. You should self-check the answers with those posted on the course website. Staple this sheet of paper as the cover page of your homework set.

1. (Section 0.7) Let \( S \) be a set. A choice function for \( S \) is a function from the set of all nonempty subsets of \( S \) to \( S \) such that \( f(A) \in A \) for all \( A \neq \emptyset, A \subset S \). Show that the Axiom of Choice is equivalent to the statement that every set \( S \) has a choice function.

2. (Section 0.8) Let \( A, B \) and \( C \) be sets such that \( B \) and \( C \) are disjoint and the cardinality numbers \( |A| = \alpha, |B| = \beta \) and \( |C| = \gamma \). Define \( \alpha^\beta \) to be the cardinal number of the set of all functions \( B \to A \). Recall that \( \beta + \gamma := |B \cup C| \) and \( \beta\gamma := |B \times C| \).
   (a) Show that \( \alpha^{\beta+\gamma} = (\alpha^\beta)(\alpha^\gamma); \ (\alpha\beta)^\gamma = (\alpha^\gamma)(\beta^\gamma); \ \alpha^{\beta\gamma} = (\alpha^\beta)^\gamma \).
   (b) If \( \alpha \leq \beta \), then \( \alpha^\gamma \leq \beta^\gamma \).
   (c) If \( P(A) \) is the power set of a set \( A \), then \( |P(A)| = 2^{|A|} \).

3. (Section 1.1) Write down a group table for the group \( D_4^* \), the group of symmetries of a square.

4. (Section 1.1) Prove that the symmetric group on \( n \) letters, \( S_n \), has order \( n! \).

5. (Section 1.2) If \( f : G \to H \) is a homomorphism of groups, then \( f(e_G) = e_H \) and \( f(a^{-1}) = f(a)^{-1} \) for all \( a \in G \). Show by example that the first conclusion may be false if \( G, H \) are monoids that are not groups.

6. (Section 1.2) Let \( f : G \to H \) be a homomorphism of groups, \( A \) a subgroup of \( G \), and \( B \) a subgroup of \( H \).
   (a) \( \text{Ker} \ f \) and \( f^{-1}(B) \) are subgroups of \( G \).
   (b) \( f(A) \) is a subgroup of \( H \).

7. (Section 1.3) If \( f : G \to H \) is a homomorphism, \( a \in G \), and \( f(a) \) has finite order in \( H \), then \( |a| \) is infinite or \( |f(a)| \) divides \( |a| \).

8. (Section 1.4) If \( H, K \) and \( N \) are subgroups of a group \( G \) such that \( H < N \), then \( HK \cap N = H(K \cap N) \).

9. (Section 1.5) If \( H \) is a cyclic subgroup of a group \( G \) and \( H \) is normal in \( G \), then every subgroup of \( H \) is normal in \( G \).

10. (Section 1.5) If \( f : G \to H \) is a homomorphism with kernel \( N \) and \( K < G \), then prove that \( f^{-1}(f(K)) = KN \). Hence \( f^{-1}(f(K)) = K \) if and only if \( N < K \).