4.2 VII-35. Series of Groups

To give insights into the structure of a group $G$, we study a series of embedding subgroups of $G$.

4.2.1 Subnormal and Normal Series

**Def 4.10.** A subnormal (or subinvariant) series of a group $G$ is a finite sequence of subgroups of $G$:

$$H_0 = \{ e \} < H_1 < H_2 < \cdots < H_n = G,$$

such that $H_i \triangleleft H_{i+1}$ (that is, $H_i$ is a normal subgroup of $H_{i+1}$).

**Def 4.11.** A normal (or invariant) series of a group $G$ is a finite sequence of subgroups of $G$:

$$H_0 = \{ e \} < H_1 < H_2 < \cdots < H_n = G,$$

such that $H_i \triangleleft G$ (that is, $H_i$ is a normal subgroup of $G$).

A normal series is always a subnormal series, but the inverse need not be true. If $G$ is an abelian group, then every finite sequence of subgroups

$$H_0 = \{ e \} < H_1 < H_2 < \cdots < H_n = G,$$

is both a subnormal and a normal series.

**Ex 4.12 (Ex 35.2, p.311).** Two examples of normal/subnormal series of $\mathbb{Z}$ under additions are

$$\{ 0 \} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z} \quad \text{and} \quad \{ 0 \} < 9\mathbb{Z} < \mathbb{Z}.$$

**Ex 4.13 (Ex 35.3, p.311).** The group $D_4$ in Example 8.10. The series

$$\{ \rho_0 \} < \{ \rho_0, \mu_1 \} < \{ \rho_0, \rho_2, \mu_1, \mu_2 \} < D_4$$

is a subnormal series, but not a normal series.

**Def 4.14.** A subnormal/normal series $\{ K_j \}$ is a refinement of a subnormal/normal series $\{ H_i \}$ of a group $G$, if $\{ H_i \} \subseteq \{ K_j \}$, that is, if each $H_i$ is one of the $K_j$.

---

1st HW: 1, 4, 22, 23
Ex 4.15 (Ex 35.5, p.311). The series \( \{0\} < 72\mathbb{Z} < 24\mathbb{Z} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z} \) is a refinement of the series \( \{0\} < 72\mathbb{Z} < 8\mathbb{Z} < \mathbb{Z} \).

If \( \{H_0 = \{e\} < H_1 < \cdots < H_n = G\} \) is a subnormal or normal series of \( G \), then \( H_i < H_{i+1} \) is always true. Many structural properties of \( G \) can be disclosed by studying the factor groups \( H_{i+1}/H_i \).

Def 4.16. Two subnormal/normal series \( \{H_i\} \) and \( \{K_j\} \) of the same group \( G \) are \textit{isomorphic} if there is a one-to-one correspondence between the collections of factor groups \( \{H_{i+1}/H_i\} \) and \( \{K_{j+1}/K_j\} \) such that the corresponding factor groups are isomorphic.

Ex 4.17. The series \( \{0\} < \langle 12 \rangle < \langle 4 \rangle < \mathbb{Z}_{24} \) and \( \{0\} < \langle 6 \rangle < \langle 3 \rangle < \mathbb{Z}_{24} \) of \( \mathbb{Z}_{24} \) are isomorphic series.

Ex 4.18. The series \( \{0\} < 24\mathbb{Z} < 12\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z} \) and \( \{0\} < 24\mathbb{Z} < 6\mathbb{Z} < 3\mathbb{Z} < \mathbb{Z} \) of \( \mathbb{Z} \) are isomorphic series.

4.2.2 The Schreier Theorem

Thm 4.19 (Schreier Theorem). Two subnormal (normal) series of a group \( G \) have isomorphic refinements.

Idea of the proof. Suppose \( G \) has two subnormal (normal) series
\[
\{e\} = H_0 < H_1 < \cdots < H_n = G,
\]
\[
\{e\} = K_0 < K_1 < \cdots < K_m = G.
\]
The subnormal (normal) series \( \{H_i \mid i = 1, \cdots, n\} \) can be refined as follow:
\[
\{e\} = H_{0,0} \leq H_{0,1} \leq \cdots \leq H_{0,m-1}
\]
\[
\leq H_{1,0} \leq H_{1,1} \leq \cdots \leq H_{1,m-1}
\]
\[
\leq \cdots
\]
\[
\leq H_{n-1,0} \leq H_{n-1,1} \leq \cdots \leq H_{n-1,m-1}
\]
\[
\leq H_{n-1,m} = G,
\]
in which \( H_{ij} := H_i(H_{i+1} \cap K_j) \). Likewise, we can make a refinement of \( \{K_j\} \). Then show that these two refinements are isomorphic.

We use some examples to illustrate the theorem.

Ex 4.20 (Ex 35.8, p.312). Find isomorphic refinements of the series
\[
\{0\} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z} \quad \text{and} \quad \{0\} < 9\mathbb{Z} < \mathbb{Z}.
\]

Ex 4.21. Find isomorphic refinements of the series
\[
\{0\} < \langle 30 \rangle < \langle 10 \rangle < \mathbb{Z}_{120} \quad \text{and} \quad \{0\} < \langle 24 \rangle < \langle 3 \rangle < \mathbb{Z}_{120}
\]
4.2.3 The Jordan-Hölder Theorem

**Def 4.22.** Let \( \{ H_i \} \) be a subnormal/normal series of a group \( G \), such that all the factor groups \( H_{i+1}/H_i \) are simple.

1. If \( \{ H_i \} \) is a subnormal series, then \( \{ H_i \} \) is called a **composition series**.
2. If \( \{ H_i \} \) is a normal series, then \( \{ H_i \} \) is called a **principal or chief series**.

**Thm 4.23 (Jordan-Hölder Theorem).** *Any two composition (principal) series of a group \( G \) are isomorphic.*

It can be proved by Schreier Theorem. The Jordan-Hölder Theorem tells us that the factor groups \( \{ H_{i+1}/H_i \} \) in any composition (principal) series \( \{ H_i \} \) of a group \( G \) is uniquely determined by \( G \).

**Ex 4.24.** The additive group \( \mathbb{Z} \) has no composition (principal) series.

**Ex 4.25.** The additive group \( \mathbb{Z}_{24} \) has the following isomorphic composition (principal) series:

\[
\begin{align*}
\{0\} &< \langle 8 \rangle < \langle 4 \rangle < \langle 2 \rangle < \mathbb{Z}_{24}, \\
\{0\} &< \langle 12 \rangle < \langle 4 \rangle < \langle 2 \rangle < \mathbb{Z}_{24}, \\
\{0\} &< \langle 12 \rangle < \langle 6 \rangle < \langle 2 \rangle < \mathbb{Z}_{24}, \\
\{0\} &< \langle 12 \rangle < \langle 6 \rangle < \langle 3 \rangle < \mathbb{Z}_{24}.
\end{align*}
\]

**Ex 4.26.** When \( n \geq 5 \), \( S_n \) has the only composition (principal) series

\[
\{e\} < A_n < S_n.
\]

**Prop 4.27.** If \( G \) has a composition (principal) series, and if \( N \) is a proper normal subgroup of \( G \), then there exists a composition (principal) series containing \( N \).

4.2.4 Examples of Series

1. **Def 4.28.** A group \( G \) is solvable if it has a composition series \( \{ H_i \} \) such that all factor groups \( H_{i+1}/H_i \) are abelian.

Recall that we constructed “the descending commutator subgroup series”

\[
G^{(0)} := G \geq G^{(1)} \geq G^{(2)} \geq \cdots
\]

where \( G^{(i+1)} := [G^{(i)}, G^{(i)}] \) is the commutator subgroup of \( G^{(i)} \). Then \( G \) is solvable if and only if \( G^{(n)} = \{e\} \) for certain integer \( n \).
**Ex 4.29 (Ex 35.19, p.318).** The group $S_3$ is solvable since it has a composition series

$$\{e\} < A_3 < S_3$$

where $A_3/\{e\} \simeq Z_3$ and $S_3/A_3 \simeq Z_2$ are abelian.

2. (opt: The Ascending Central Series) Let

$$Z(G) := \{z \in G \mid zg = gz \text{ for all } g \in G\}$$

be the center of a group $G$. Then $Z(G)$ is a normal subgroup of $G$, so that $G/Z(G)$ is a factor group. Likewise, $Z(G/Z(G))$ is a normal subgroup of $G/Z(G)$. By Lemma 34.3, there is a normal subgroup $Z_1(G)$ of $G$ such that $Z_1(G)/Z(G) = Z(G/Z(G))$. Repeating the process, we obtain a series

$$\{e\} \leq Z(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots$$

where $Z_{i+1}(G)/Z_i(G)$ is the center of $G/Z_i(G)$. This is call the *ascending central series* of the group $G$.

**4.2.5 Homework, Section 35, p.319-p.321**

1st: 1, 4, 22

2nd: Find all composition series of $Z_5 \times Z_5$ and $Z_{25}$, 10, 17, 23