Chapter 4

7. Advanced Group Theory

It is important to build up the correct visions about things in a group, a homomorphism, or so.

4.1 VII-34. Isomorphism Theory

Thm 4.1 (First Isomorphism Theorem). Let \( \phi : G \to G' \) be a group homomorphism with kernel \( K \). Let \( \gamma_K : G \to G/K \) be the canonical homomorphism defined by \( \gamma_K(g) := gK \). Then \( \phi \) is the composition of two homomorphisms:

\[
\phi : G \xrightarrow{\gamma_K} G/K \xrightarrow{\mu} \phi[G],
\]

that is, \( \phi = \mu \circ \gamma_K \), where \( \mu : G/K \to \phi[G] \) is the unique isomorphism defined by \( \mu(gK) := \phi(g) \). (cf. Figure 34.1 on p.307)

Ex 4.2. HW 1, p.310.

Below lemmas are helpful in understanding the upcoming theorems.

Lem 4.3. Let \( N \) be a normal subgroup of \( G \), and \( H \) be a subgroup of \( G \). Then \( HN = NH \) is the smallest subgroup that contains both \( H \) and \( N \). Moreover, if \( H \) is also normal, then \( HN \) is normal in \( G \).

Lem 4.4. Let \( N \) be a normal subgroup of \( G \). Let \( \gamma : G \to G/N \) be the canonical homomorphism. Then there is a bijection

\[
\phi : \{\text{all normal subgroups of } G \text{ containing } N\} \to \{\text{all normal subgroups of } G/N\}
\]

given by \( \phi(L) := \gamma(L) = LN \). (By figure)
Thm 4.5 (Second Isomorphism Theorem). Let $N$ be a normal subgroup of $G$, and $H$ be a subgroup of $G$. Then $(HN)/N \simeq H/(H \cap N)$.

Ex 4.6 (Ex 34.6). Let $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, $H = \mathbb{Z} \times \mathbb{Z} \times \{0\}$, what is $(HN)/N \simeq H/(H \cap N)$ means?

Ex 4.7. HW 3, p.310.

Thm 4.8 (Third Isomorphism Theorem). Let $H$ and $K$ be normal subgroups of a group $G$ with $K \leq H$. Then

$$G/H \simeq (G/K)/(H/K).$$

Ex 4.9. Hw 5, p.310.

4.1.1 Homework, VII-34, p.310-p.311

1. 1st HW: 2, 7.

2. 2nd HW: 4, 6, 8.