SOME EASY LOOKING SOLVED AND UNSOLVED PROBLEMS FOR POLYNOMIALS AND RELATED CLASSES OF FUNCTIONS

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Research Interests: Complex Variables, and Approximation Theory
By a real polynomial we will mean an expression of the form $\sum_{\nu=0}^{n} c_{\nu} x^{\nu}$, where $c_{\nu}$ is real and $x$ a real variable, and by a complex polynomial an expression of the form $\sum_{\nu=0}^{n} a_{\nu} z^{\nu}$, where $a_{\nu}$ are complex and $z$ a complex variable.

Polynomials have played a central role in Approximation Theory and Numerical Analysis for many years. To indicate why this is the case, we note that

1. The set of all polynomials of degree at most $n$ is a finite dimensional linear space with a convenient basis.
2. Polynomials are smooth functions
3. Polynomials are easy to store, manipulate, and evaluate on a computer.
4. The derivative and antiderivative of a polynomial are again polynomials whose coefficients can be found algebraically (even by a computer).
5. The number of zeros of a polynomial of degree $n$ is precisely $n$.
6. The sign structure and shape of a polynomial are intimately related to the sign structure of its set of coefficients.
7. Given any continuous function on an interval $[a, b]$, there exists a polynomial that is uniformly close to it.
8. Precise rates of convergence can be given for approximation of smooth functions by polynomials.
Although polynomials are among the nicest type of functions, but still there are many type of problems associated with polynomials and here we would be discussing mainly the following type of problems.

1. If a polynomial has zeros and if so where the zeros lie, that is, the problems concerning the **Location of the Zeros of a Polynomial**, **Eneström-Kakeya Theorem**, and **Sandov’s conjecture**.

2. How fast the maximum modulus of a polynomial can grow, that is, the problems concerning the **Growth of a Polynomial**.

3. Relationship between the growth of the derivative of a polynomial in terms of the growth of the polynomial, known as **Extremal Problems for Polynomials**.

4. If the time permits, we would discuss above problems for a larger class of functions, which includes polynomials, like **Extremal Problems for Rational Functions**, and **Entire Functions**.
1 Location of the Zeros of Polynomials

Let \( p(z) = \sum_{v=0}^{n} a_v z^v \) be a polynomial of degree \( n \). By Fundamental Theorem of Algebra, \( p(z) \) has exactly \( n \) zeros in the complex plane. But this theorem does not say anything regarding the location of the zeros of the polynomial, i.e., the region which contains some or all of the zeros of the polynomial. The problems of this kind were first studied by Gauss and Cauchy.

**THEOREM 1.1 (Gauss, 1799.)** Let \( p(z) = z^n + A z^{n-1} + \cdots + A_{n-1} z + A_n \) be a polynomial with real coefficients. Then \( p(z) \) has all its zeros in \( |z| \leq R \) where \( R = \max(1, S \sqrt{2}) \) where \( S \) is the sum of \( |A_k| \)'s.

In 1816 he showed that \( R = \max_{1 \leq k \leq n} (n \sqrt{2} |A_k|)^{1/2} \) whereas in 1849, he gave a bound for polynomials with arbitrary real or complex \( A_k \)'s, and showed that \( R \) may be taken as the positive root of the equation \( z^n - 2^{1/2} (|A_1| z^{n-1} + \cdots + |A_n|) = 0 \).

As a further indication of Gauss’s interest in the location of the complex zeros of a polynomial, we have his letter to Schumacher in which he tells of having written enough upon this topic to fill several volumes, but unfortunately the only results he subsequently published are those in his paper in 1850.

Cauchy obtained more exact bounds for the moduli of the zeros of a polynomial than those given by Gauss.

**THEOREM 1.2 (Cauchy, 1829)** All the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \) lie in

\[
|z| \leq 1 + A, \tag{1}
\]

where \( A = \max_{0 \leq j \leq n-1} |a_j| \).

**THEOREM 1.3 (Joyal, Labelle, and Rahman, Canadian Math. Bull. 1967)** All the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \) lie in the disk

\[
|z| \leq \frac{1}{2} \left( 1 + |a_{n-1}| + [ (1 - |a_{n-1}|)^2 + 4 \beta ]^{1/2} \right), \tag{2}
\]

where \( \beta = \max_{0 \leq j \leq n-2} |a_j| \).

Since \( \beta \leq A \), and \( |a_{n-1}| \leq A \), the Theorem 1.3 sharpens Theorem 1.2.

**THEOREM 1.4 (Datt and Govil, Jour. Approx. Theory 1978.)** All the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \), where \( a_v \) may be complex, lie in

\[
|z| \leq 1 + (1 - \frac{1}{(1+A)^n}) A. \tag{3}
\]

Theorem 1.3 clearly sharpens Theorem 1.2 due to Cauchy, because \( (1-\frac{1}{(1+A)^n}) < 1 \).
Let 

\[ p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0, \]

with \( a_i \neq 0 \), for at least one \( i \in I = \{0, 1, 2, \ldots, n-1\} \), be a polynomial of degree \( n \), with complex coefficients, and let \( Z[f(z)] \) denotes the set of all zeros of the polynomial \( f(z) \).

Then according to a classical result of Cauchy about the location of the zeros of polynomials,

**THEOREM 1.5 (Cauchy, 1829)** All the zeros of the polynomial 
\[ p(z) = z^n + \sum_{v=0}^{n-1} a_vz^v \] lie in the disk 
\[ \{ z : |z| \leq \eta \} \subset \{ z : |z| < 1 + A \}, \]

where \( \eta \) is the unique positive root of the equation
\[ z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_1|z - |a_0| = 0, \]
and \( A = \max_{0 \leq j \leq n-1} |a_j| \) is as in Theorem 1.2.

One has to assume that \( a_i \neq 0 \), for at least one \( i \in I = \{0, 1, 2, \ldots, n-1\} \), because if \( a_i = 0 \), for all \( i \in I = \{0, 1, 2, \ldots, n-1\} \), then the equation does not have any positive root, and so the above theorem does not hold. The above result of Cauchy has been refined among others by Zilovic et al [A bound for the zeros of polynomials, IEEE Trans. Circuits Syst. I 39 (1992), 476-478], and Sun and Hsieh [A note on circular bound of polynomial zeros, IEEE Trans. Circuits Syst. I 43 (1996), 476-478]. Following is the result due to Sun and Hsieh.

**THEOREM 1.6 (Sun and Hsieh, 1996)** All the zeros of the polynomial 
\[ p(z) = z^n + \sum_{v=0}^{n-1} a_vz^v \] lie in the disk 
\[ \{ z : |z| \leq \eta \} \subset \{ z : |z| < 1 + \delta_1 \} \subset \{ z : |z| < 1 + A \}, \]

where \( \eta \) is the unique positive root of the equation
\[ z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_1|z - |a_0| = 0, \]
and \( \delta_1 \) is the unique positive root of the equation
\[ z^3 + (2 - |a_{n-1}|)z^2 + (1 - |a_{n-1}| - |a_{n-2}|)z - A = 0. \]

Here \( A = \max_{0 \leq j \leq n-1} |a_j| \) is as in Theorem 1.2.

**THEOREM 1.7 (V.K. Jain, 2006)** All the zeros of the polynomial 
\[ p(z) = z^n + \sum_{v=0}^{n-1} a_vz^v \] lie in the disk 
\[ \{ z : |z| \leq \eta \} \subset \{ z : |z| < 1 + \delta_0 \} \subset \{ z : |z| < 1 + \delta_1 \} \subset \{ z : |z| < 1 + A \}, \]

where \( \eta, \delta_1, \) and \( A \) are as in the above Theorem, and \( \delta_0 \) is the unique positive root of the equation
\[ z^4 + (3 - |a_{n-1}|)z^3 + (3 - 2|a_{n-1}| - |a_{n-2}|)z^2 + (1 - |a_{n-1}| - |a_{n-2}| - |a_{n-3}|)z - A = 0. \]
2 Eneström-Kakeya Theorem

THEOREM 2.1 (Eneström-Kakeya) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying
\[
0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n,
\]
then \( p(z) \) has all its zeros in \( |z| \leq 1 \).

Joyal, Labelle and Rahman dropped the hypothesis that coefficients be all positive and proved

THEOREM 2.2 (Joyal, Labelle and Rahman, Canad. Math. Bull. 1967) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying
\[
a_0 \leq a_n \leq a_2 \leq \cdots \leq a_n,
\]
then \( p(z) \) has all its zeros in
\[
|z| \leq \frac{a_n - a_0 + a_0}{a_n}.
\]

THEOREM 2.3 (Dewan and Govil, Jour. Approx. Theory, 1984) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients such that
\[
a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq a_n
\]
then \( p(z) \) has all its zeros in the annulus
\[
R_1 \leq |z| \leq R_2
\]
where \( R_1 \) and \( R_2 \) are constants depending on the coefficients \( a_0, a_1, a_{n-1} \) and \( a_n \). Moreover
\[
0 \leq R_1 \leq 1 \leq R_2 \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.
\]

THEOREM 2.4 (Govil and Jain, Jour. Approx. Theory, 1978) Let \( p(z) = \sum_{v=0}^{n} a_v z^v(\neq 0) \) be a polynomial with complex coefficients such that
\[
|\arg a_v - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad v = 0, 1, 2, \ldots, n.
\]
for some real \( \beta \), and
\[
|a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_n|
\]
then \( p(z) \) has all its zeros in \( R_1 \leq |z| \leq R_2 \), where \( R_1 \) and \( R_2 \) are constants depending on \( a_0, a_{n-1}, a_n \) and \( \alpha \).
3 Some Further Extensions of Eneström-Kakeya Theorem.

If instead of having information about the moduli of the coefficients we have the information about their real and imaginary parts, the following results could be of interest.

THEOREM 3.1 Let \( p(z) = \sum_{v=0}^{n} a_v z^v \), \( a_n \neq 0 \), \( \text{Re} \ a_j = \alpha_j, \text{Im} \ a_j = \beta_j \) for \( j = 0, 1, 2, \ldots, n \).

If
\[
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \quad \text{and} \quad \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.
\]

In particular if all the coefficients are real, it gives an improvement of the result of Joyal, Labelle and Rahman. Further if the coefficients are as well positive it gives the Eneström-Kakeya Theorem.

THEOREM 3.2 If
\[
\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \quad \text{and} \quad \beta_0 \geq \beta_1 \geq \cdots \geq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)}{|a_n|}.
\]

THEOREM 3.3 If
\[
\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \quad \text{and} \quad \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| + \alpha_0 - \beta_0 - \alpha_n + \beta_n} \leq |z| \leq \frac{|a_0| + \alpha_0 - \beta_0 - \alpha_n + \beta_n}{|a_n|}.
\]

THEOREM 3.4 If
\[
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \quad \text{and} \quad \beta_0 \geq \beta_1 \geq \cdots \geq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| - \alpha_0 + \beta_0 + \alpha_n - \beta_n} \leq |z| \leq \frac{|a_0| - \alpha_0 + \beta_0 + \alpha_n - \beta_n}{|a_n|}.
\]

All the above four results are in fact corollaries of a more general theorem due to Gardner & Govil, Journal of Approximation Theory 78 (1994), 286-292.
THEOREM 3.5 (Gardner & Govil, Acta Math. Hungar. 1997) Let $p(z) = \sum_{v=0}^{n} a_v z^v$, $\Re(a_j) = \alpha_j$ and $\Im(a_j) = \beta_j$ for $j = 0, 1, \ldots, n$, $a_n \neq 0$ and for some $k,$

$$
\beta_0 \leq t \beta_1 \leq t^2 \beta_2 \leq \cdots \leq t^k \beta_k \geq t^{k+1} \beta_{k+1} \geq t^{k+2} \beta_{k+2} \geq \cdots \geq t^n \beta_n
$$

for some positive $t$. Then $p(z)$ has all its zeros in $R_1 \leq |z| \leq R_2,$ where

$$
R_1 = t|a_0| \left(2t^k \beta_k - \beta_0 - t^n \beta_n + t^n \alpha_0 + |\alpha_0| + |\alpha_n| t^n + 2 \sum_{j=1}^{n-1} |\alpha_j| t^j\right),
$$

and

$$
R_2 = \max \left\{ \left| a_0 \right| t^{n+1} + \left(2t^2 + 1\right) t^{n-k} \beta_k - t^{n-1} \beta_0 - t \beta_n + (t^2 - 1) \sum_{j=1}^{k-1} t^{n-j-1} \beta_j 
+ \left(1 - t^2\right) \sum_{j=k+1}^{n-1} t^{n-j-1} \beta_j + \sum_{j=1}^{n} \left( |\alpha_{j-1}| + t|\alpha_j| \right) t^{n-j} \right\} / \left| a_n \right| , \frac{1}{t} \right\}.
$$

With the suitable choices of $t$ and $k$ in the above theorems, one can also obtain the following corollaries which appear to be interesting and useful.

Corollary 1. If $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ then all the zeros of $p(z)$ lie in $R_1 \leq |z| \leq R_2$

where $R_1 = |a_0| \left( \alpha_n - \alpha_0 + |\alpha_n| + |\beta_0| + |\beta_n| + 2 \sum_{j=1}^{n-1} |\beta_j| \right)$

and $R_2 = \left( |a_0| - \alpha_0 + \alpha_n + |\beta_0| + |\beta_n| + 2 \sum_{j=1}^{n-1} |\beta_j| \right) / |a_n|.$

Corollary 2. If $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n$ then all the zeros of $p(z)$ lie in $R_1 \leq |z| \leq R_2$

where $R_1 = |a_0| \left( \alpha_0 - \alpha_n + |\alpha_n| + |\beta_0| + |\beta_n| + 2 \sum_{j=1}^{n-1} |\beta_j| \right)$

and $R_2 = \left( |a_0| + \alpha_0 - \alpha_n + |\beta_0| + |\beta_n| + 2 \sum_{j=1}^{n-1} |\beta_j| \right) / |a_n|.$

Corollary 3. If $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_n$ then all the zeros of $p(z)$ lie in $R_1 \leq |z| \leq R_2$

where $R_1 = |a_0| \left( \beta_n - \beta_0 + |a_n| + |\alpha_0| + |\alpha_n| + 2 \sum_{j=1}^{n-1} |\alpha_j| \right)$

and $R_2 = \left( \beta_n - \beta_0 + |a_0| + |\alpha_0| + |\alpha_n| + 2 \sum_{j=1}^{n-1} |\alpha_j| \right) / |a_n|.$

Corollary 4. If $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ then all the zeros of $p(z)$ lie in $R_1 \leq |z| \leq R_2$

where $R_1 = |a_0| \left( \beta_0 - \beta_n + |a_n| + |\alpha_0| + |\alpha_n| + 2 \sum_{j=1}^{n-1} |\alpha_j| \right)$ and

$$
R_2 = \left( \beta_0 - \beta_n + |a_0| + |\alpha_0| + |\alpha_n| + 2 \sum_{j=1}^{n-1} |\alpha_j| \right) / |a_n|.
$$
4 Some Recent Extensions of Eneström Kakeya Theorem.

THEOREM 4.1 (Eneström-Kakeya) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying
\[
0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n,
\]
then \( p(z) \) has all its zeros in \( |z| \leq 1 \).

Joyal, Labelle and Rahman dropped the hypothesis that coefficients be all positive and proved

THEOREM 4.2 (Joyal, Labelle and Rahman, Canad. Math. Bull. 1967) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying
\[
a_0 \leq a_n \leq a_2 \leq \cdots \leq a_n,
\]
then \( p(z) \) has all its zeros in
\[
|z| \leq \frac{a_n - a_0 + a_0}{a_n} \quad (6)
\]

THEOREM 4.3 (Aziz and Zarga, Glasnik Matematicki (1996)) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) such that for some \( K \geq 1 \),
\[
K a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0, \quad (7)
\]
then all zeros of \( p(z) \) lie in
\[
|z + (K - 1)| \leq \frac{K a_n + |a_0| - a_0}{|a_n|} \quad (8)
\]

Theorem 4.3. Let \( p(z) \) be a polynomial of degree \( n \) with \( \text{Re}(a_v) = \alpha_v \) and \( \text{Im}(a_v) = \beta_v \). If for some \( K \geq 1 \)
\[
K \alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0, \quad (9)
\]
then \( p(z) \) has all its zeros in
\[
|z + (K - 1)| \leq \frac{K \alpha_n - \alpha_0 + |a_0| + 2 \sum_{v=0}^{n} |\beta_v|}{|\alpha_n|} \quad (10)
\]

Theorem 4.4. Let \( p(z) \) be a polynomial of degree \( n \) with \( \text{Re}(a_v) = \alpha_v \) and \( \text{Im}(a_v) = \beta_v \). If for some \( K \geq 1 \)
\[
K \beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0, \quad (11)
\]
then \( p(z) \) has all its zeros in
\[
|z + (K - 1)| \leq \frac{K \beta_n - \beta_0 + |\beta_0| + 2 \sum_{v=0}^{n} |\alpha_v|}{|\beta_n|} \quad (12)
\]
5 The Conjecture of Sendov.

Blagovest Sendov, a Bulgarian mathematician has enriched Approximation Theory by numerous important results. He is best known for his work on Hausdorff metric, positive and monotonic operators, splines, segment analysis, Whitney constants and several other topics. Although amongst his more than 150 publications one can hardly find any result on zeros and critical points of polynomials, his name has become famous in the Analytic Theory of Polynomials since it has been Sendov to whom this discipline owes its most challenging conjecture, which has resisted a proof for more than forty years.

CONJECTURE 5.1 (Sendov) Let \( p(z) = \prod_{v=0}^{n} (z - z_v) \) be a polynomial of degree \( n \geq 2 \) with all its zeros in the closed unit disk. Then each of the disks \( \{ z : |z - z_v| \leq 1 \} \) contains at least one critical point, that is at least one zero of the derivative \( p'(z) \).

For a long time, the origins of Sendov’s conjecture remained in obscurity. Neither a date for the statement, nor a motivation and not even the true author were known. According to the several testimonies, it now appears that Sendov made his conjecture already in 1959 to Obreschkoff (who ignored in his book), in 1962 at the International Congress of mathematicians in Stockholm to Marden, and independently to other mathematicians including Illief. In 1965, at the conference on Complex Analysis and Applications in Yerevan (Armenia), Illief spoke about this conjecture at an informal meeting between participants. This way Walter K. Hayman learnt about the problem, but he misunderstood its origin. In 1967, when Hayman published his book on open problems in Complex Analysis, he included Sendov’s problem as a conjecture of Illief. Since then this conjecture has been known to the public, but for than 10 years it has erroneously attributed to Illief. Often a conjecture arises from a missing step in an approach towards a desired result, and so when some mathematicians met Sendov and tried to learn about the possible backgrounds of Sendov’s conjecture, it turned out that there was nothing that was a motivation, and it seems that Sendov just wanted to puzzle the mathematical community with something that might look quite easy at a first glance. In fact, except for the trivial case when \( z_v = 0 \), the disks \( \{ z : |z - z_v| \leq 1 \} \) contain lot of points that cannot be critical points of the polynomial \( p \). As a consequence of Gauss Gauss-Lucas Theorem, Sendov’s conjecture would imply that already each of the lens-shaped domains

\[
\{ z : |z - z_v| \leq 1 \} \cap \{ z : |z| \leq 1 \} \quad (v = 1, 2, \ldots, n)
\]

contains a critical point. Several non-trivial refinements of Sendov’s conjecture have been proposed, and some disproved with the help of computational methods. The most promising refinement is due to Phelps and Rodriguez.

CONJECTURE 5.2 (Sendov’s conjecture, Stengthened Form of Phelps-Rodriguez)) Let \( p(z) = \prod_{v=0}^{n} (z - z_v) \) be a polynomial of degree \( n \geq 2 \) with all its zeros in the closed unit disk. Then each of the open disks \( \{ z : |z - z_v| < 1 \} \), for \( 1 \leq v \leq n \) contains at least one critical point of the polynomial \( p(z) \), unless \( p(z) = z^n - c \), \( |c| = 1 \).
6 The Classes for Which the Conjectures Were Proved

6.1 Polynomials of Small Degree: A natural way of approaching a conjecture on polynomials is to try a verification for small degrees. While Sendov’s conjecture is trivial for polynomials of degree \( n = 2 \), it is not so obvious for \( n = 3 \). In 1968, Branan presented a proof for \( n = 3 \), and in the same year Rubinstein verified the conjecture for \( 3 \leq n \leq 4 \). In 1969, Joyal, and Schmeisser obtained the stronger form of the conjecture for \( 3 \leq n \leq 4 \), and in the same year Meir and Sharma verified Sendov’s conjecture for \( n = 5 \). In 1971, Gacs extended the stronger conclusion by Schmeisser to \( n = 5 \). The case \( n = 6 \) had to wait for more than twenty years till Brown (P.A.M.S. 1991) made some progress. In the meantime, the rumors had been spread, saying that in 1986, the conjecture was proved for all degrees \( n \). This had blocked further research for some years. In 1992, Katsoprinakis (Bull. London Math. Soc. 1992) published a proof for \( n = 6 \), but he used a lemma that was incorrectly stated in a book, and so his proof contained a gap. In 1996, Borcea, and Katsoprinakis who filled the gap in his former proof, both gave correct proofs for \( n = 6 \). In the same year, Borcea Analysis, 1996) obtained a proof for \( n = 7 \), and in 1999, Brown and Xiang settled the case for \( n = 8 \). This seems to be all that one knows, and in all the cited cases, it turns out that Sendov’s conjecture holds in the strengthened form of Phelps-Rodriguez. We may summarize these results in the following theorem.

**THEOREM 6.1** Let \( p(z) = \prod_{v=0}^{n} (z - z_v) \) be a polynomial of degree \( 2 \leq n \leq 8 \) with all its zeros in the closed unit disk. Then each of the open disks \( \{ z : |z - z_\nu| < 1 \} \) contains at least one critical point, that is at least one zero of the derivative \( p'(z) \), unless \( p(z) = z^n - c, |c| = 1 \).

In fact, for \( 2 \leq n \leq 5 \), each of the disks \( \{ z : |z - z_\nu| < 1 \} \) can be replaced by the largest open disk contained in the lens-shaped domain. Hopes that this statement may extend to \( n > 5 \) were destroyed by Miller (Trans.A.M.S. 1990). Employing computational methods, he constructed polynomials of degrees 6, 8, 10, and 12, for which the above result does not hold. Kumar and Shenoy (1991) added counterexamples for the degrees 7, 9, and 11.

6.2 Polynomials with Real Zeros: For polynomials with real zeros, the location of the critical points can be well described with the help of Rolle’s theorem. It is not difficult to verify the Sendov’s conjecture for such polynomials, and Phelps and Rodriguez did it. In fact, one can easily obtain the following refined statement.

**THEOREM 6.2** Let \( p(z) = \prod_{v=0}^{n} (z - z_v) \) be a polynomial of degree \( n \geq 2 \) with real zeros ordered as \(-1 \leq x_1 \leq \ldots \leq x_n \leq 1\). Then each of the intervals

\[
\left[ x_1, x_1 + \frac{2}{n} \right], \left[ x_n - \frac{2}{n}, x_n \right], \text{ and } \left[ -1 + x_\nu + |x_\nu|, 1 + x_\nu - |x_\nu| \right],
\]

for \( 2 \leq \nu \leq (n - 1) \), contains a critical point of \( p(z) \).
6.3 Polynomials with Real Coefficients:

The class of polynomials with real coefficients covers the monic polynomials with only real zeros. So far, no one has succeeded in verifying Sendov’s conjecture for polynomials with real coefficients. However for the subclass of the so-called Cauchy polynomials, the conjecture was proved in the strengthened form of Phelps and Rodriguez.

THEOREM 6.3 (Analytic Theory of Polynomials, by Rahman and Schmeisser, p.408)

Let \( p(z) = \prod_{\nu=0}^{n} (z - z_{\nu}) = z^n - \sum_{\nu=0}^{n-1} a_{\nu} z^{\nu} \) (\( n \geq 2 \)) have all its zeros in the closed unit disk, and suppose that \( a_0, \ldots, a_{n-1} \) are all nonnegative. Then each of the open disks \( \{ z : |z - z_{\nu}| < 1 \} \), for \( 1 \leq \nu \leq n \) contains a critical point of \( p \), unless \( p(z) = z^n - 1 \).

6.4 Polynomials Having Zeros on a Circle:

If in the situation of Sendov’s conjecture, the polynomial \( p \) has a zeros on the unit circle then the statement on the location of a critical point relative to this zeros not only holds but can be refined, and this was done by Rubinstein.

THEOREM 6.4 (Rubinstein, Pacific J. Math. 1968) Let \( p(z) = \prod_{\nu=0}^{n} (z - z_{\nu}) \) be a polynomial of degree \( n \geq 2 \) having all its zeros in the closed unit disk. Suppose that \( |z_1| = 1 \). Then \( p \) has a critical point in the open disk \( \{ z : |z - z_1| < 1 \} \), unless \( p(z) = z^n - z_1^n \).

As an immediate consequence of the above theorem, one gets

THEOREM 6.5 (Rubinstein, Pacific J. Math. 1968) Let \( p(z) = \prod_{\nu=0}^{n} (z - z_{\nu}) \) be a polynomial of degree \( n \geq 2 \) having all its zeros on the unit circle. Then for each \( \nu \in \{1, \ldots, n\} \), \( p \) has a critical point in the open disk \( \{ z : |z - z_{\nu}| < 1 \} \), unless all the critical points lie on the unit circle. In fact one can prove that each disk \( \{ z : |z - z_{\nu}/2| < 1/2 \} \) contains a critical point.

6.5 Polynomials Having a Zero at the Origin:

It is curious phenomenon that Sendov’s conjecture becomes an easy problem as soon as the polynomial has a zero at the origin, while its other zeros may be arbitrarily located in the unit disk. This was first observed by Schmeisser [Math. Z. 1969]. An alternative proof of this was given by Gacs [JMAA, 1971].

THEOREM 6.6 Let \( p(z) = \prod_{\nu=0}^{n} (z - z_{\nu}) \) be a polynomial of degree \( n \geq 2 \) having all its zeros in the closed unit disk, and suppose that \( f(0) = 0 \). Then each of the open disks \( \{ z : |z - z_{\nu}| < 1 \} \), for \( \nu = 1, \ldots, n \} \) contains at least one critical point of the polynomial \( p \).
The Conjecture of Sendov.

It appears to have been in 1958 that Sendov stated the conjecture on critical points of a polynomial. After slow communication among some specialists, it rapidly spread by a publication in 1967, where it was erroneously attributed to Ilieff. Since then the conjecture has been studied vigorously in numerous papers but it still seems to be unsolved.

CONJECTURE 7.1 (Sendov) Let \( p(z) = \prod_{v=0}^{n} (z - z_v) \) be a polynomial of degree \( n \geq 2 \) with all its zeros in the closed unit disk. Then each of the disks \( \{ z : |z - z_v| \leq 1 \} \) contains at least one critical point, that is at least one zero of the derivative \( p'(z) \).

It appears that Sendov made his conjecture in 1959 to Obreschkoff, in 1962 at the International Congress of Mathematicians in Stockholm to M. Marden, and independently to other mathematicians including L. Ilieff. In 1965, at the conference on Complex Analysis and Applications in Armenia, Ilieff spoke about this conjecture at an informal meeting between participants and due to some misunderstanding W.K. Hayman stated the problem in his book as Ilieff’s conjecture. Since then the conjecture has been known to the public, but for about 10 years it was erroneously attributed to Ilieff. Despite the remarkable number of contributions, the conjecture still seems to be far from complete solution. Let \( p(z) = \prod_{v=0}^{n} (z - z_v) \) be a polynomial of degree \( n \geq 2 \) with all its zeros in the closed unit disk. Then each of the open disks \( \{ z : |z - z_v| < 1 \} \) contains at least one critical point, that is at least one zero of the derivative \( p'(z) \), unless \( p(z) = z^n - c, |c| = 1 \).

When Sendov’s conjecture became known to the public, it was natural to try a proof for small degree \( n \). While the case \( n = 2 \) is trivial, the conjecture for \( n = 3 \) was proved by Branan (1968), for \( 3 \leq n \leq 4 \) by Rubinstein (1968), Joyal (1969), Schmeisser (1969), and for for \( n = 5 \) by Meir and Sharma (1969). Also in 1969, Goodman, Rahman and Ratti proved it with the additional hypothesis that \( p(z) \) has all its zeros on \( |z| = 1 \), rather than in \( |z| \leq 1 \). The case where \( n = 6 \) had to wait for twenty years until Brown (1991) made some progress. Katsoprinankis (1992) thought to have a complete proof but he got caught in a trap by employing a lemma which was incorrectly stated in a book. For \( n = 7 \), this conjecture was verified by Borcea (1996), and for \( n = 8 \) by Brown and Xiang (1999). The conjecture has recently been verified by Schmeisser in the case when the polynomial \( p(z) \) has a zero at the origin.

After more than thirty years of research on Sendov’s conjecture, it seems that the standard methods from the theory of polynomials have been exhausted and new approaches are needed. For references, I suggest the book, Analytic Theory of Polynomials by Q. I. Rahman and G. Schmeisser, Oxford University Press, 2003, and a recent survey paper The Conjectures of Sendov and Smale by Gerhard Schmeisser, Approximation Theory, DARBA, Sofia, 2002, pp. 353-369.
8 Extremal Problems for Polynomials

THEOREM 8.1 (D. I. Mendeleev) *If* \( p(x) = \sum_{v=0}^{2} c_v x^v \) *is a real polynomial of degree 2*, then

\[
\max_{-1 \leq x \leq 1} |p'(x)| \leq 4 \max_{-1 \leq x \leq 1} |p(x)|
\]

THEOREM 8.2 (A. A. Markov, Zapiski Imp. Akad. Nauk 62 (1889), 1-24) *If* \( p(x) = \sum_{v=0}^{n} c_v x^v \) *is a real polynomial of degree* \( n \), then

\[
\max_{-1 \leq x \leq 1} |p'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |p(x)|.
\]

The result is best possible and the equality holds for Tchebychev’s polynomial of first kind \( T_n(x) = \cos(n \cos^{-1} x) \).

The above result was generalized by A. A. Markov’s brother, W. W. Markov, who proved

THEOREM 8.3 (W. Markov, Math. Annalen 77 (1916), 213-258) *If* \( p(x) \) *is a real polynomial of degree* \( n \), then

\[
\max_{-1 \leq x \leq 1} |p^{(s)}(x)| \leq \frac{n^2(n^2-1^2)(n^2-2^2)\ldots(n^2-s^2)}{1 \cdot 3 \cdot 5 \ldots (2s-1)} \max_{-1 \leq x \leq 1} |p(x)|.
\]

Again the equality holds for the polynomial \( T_n(x) = \cos(n \cos^{-1} x) \).

THEOREM 8.4 (S. Bernstein, Memoire de l’Académie Royale de Belgique (1912), 1-103) *If* \( p(x) \) *is a real trigonometric polynomial of degree* \( n \), then

\[
|p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \max_{-1 \leq x \leq 1} |p(x)|.
\]

Remark: In the neighborhood of origin, Theorem 8.4 gives a better bound than Theorem 6.2 while in the neighborhood of \( x = \pm 1 \), the bound obtained by Theorem 8.2 is better than that from Theorem 8.4.
9 Bernstein’s Inequality

THEOREM 9.1 (S. Bernstein, Memoire de l’Académie Royale de Belgique (1912), 1-103)
If \( p(x) \) is a real polynomial of degree at most \( n \), then
\[
|p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \max_{-1 \leq x \leq 1} |p(x)|.
\]

Since \( p(\cos \theta) = \sum_{v=0}^{n} c_v \cos^v \theta \) can be written as a trigonometric polynomial of degree \( n \), the following result is a generalization of Theorem 7.1.

THEOREM 9.2 (S. Bernstein). If \( t(\theta) = \sum_{v=0}^{n} (a_v \cos v\theta + b_v \sin v\theta) \) is a trigonometric polynomial of degree \( n \), then for real \( \theta \),
\[
|t'(\theta)| \leq n \max_{-\pi \leq \theta \leq \pi} |t(\theta)|.
\]

The following inequality which follows readily from Theorem 7.2 is also known as Bernstein’s inequality.

THEOREM 9.3 (Bernstein’s Inequality) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \), then
\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.
\]
The result is best possible and the equality holds for \( p(z) = \lambda z^n \), \( \lambda \) being a complex number.

Bernstein in fact proved Theorem 7.2 with \( 2n \) in place of \( n \). Theorem 7.2 in the present form appeared in print for the first time in a paper of Feketé who attributes the proof to Fejer. Alternate proofs of this theorem were given by Rogosinski, de la Vallee Poisson and others.

To obtain Theorem 7.3 from Theorem 7.2, simply apply Theorem 7.2 to \( t(\theta) = p(e^{i\theta}) = \sum_{v=0}^{n} a_v e^{i v \theta} \), which is a trigonometric polynomial of degree \( n \).
Theorem 10.1 (A. Markov, 1889) If \( p(x) = \sum_{v=0}^{n} c_v x^v \) is a real polynomial of degree \( n \), then
\[
\max_{-1 \leq x \leq 1} |p'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |p(x)|.
\] (13)

Theorem 10.2 If \( t(\theta) = \sum_{v=0}^{n} c_v e^{iv\theta} \) is a trigonometric polynomials of degree \( n \), then for real \( \theta \), we have
\[
|t'(\theta)| \leq n \max_{-\pi \leq \theta \leq \pi} |t(\theta)|.
\] (14)

Theorem 10.3 (S. Bernstein, 1926) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree at most \( n \), then
\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|
\] (15)

Theorem 10.4 (S. Bernstein) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree at most \( n \), then
\[
\max_{|z|=R \geq 1} |p(z)| \leq R^n \max_{|z|=1} |p(z)|.
\] (16)

Theorem 10.5 (R. S. Varga, J. Soc. Indust. Appl. Math. 1957, page 44) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \), then
\[
\max_{|z|=r \leq 1} |p(z)| \geq r^n \max_{|z|=1} |p(z)|
\] (17)

Theorem 8.4 is a simple deduction from the Maximum Modulus Principle. Theorem 8.3, which can be deduced from Theorem 8.2, is better known as S. Bernstein’s Inequality, although, it appeared in print for the first time in a paper of Feketé who attributes the proof to Fejer. Alternative proofs of this inequality were given by M. Riesz, Rogosinski, de la Vallee Poussin, and others. Bernstein had proved Theorem 8.2 with \( 2n \) in place of \( n \). Bernstein in 1930 observed that Theorem 8.3 can be obtained from Theorem 8.4 as well by making use of Gauss-Lucas Theorem.

It was not known that Theorem 8.4 can also be deduced from Theorem 8.3. Recently, Govil, Qazi and Rahman (2003) have shown that Theorem 8.4 can also be deduced from Theorem 8.3, and thus all the three, Theorems 8.3, 8.4 and 8.5 are all equivalent in the sense that they can be deduced from each other.
Theorem 7.3 [S. Bernstein, 1926] If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree at most \( n \), then
\[
|p'(z)| \leq n \max_{|z|=1} |p(z)|
\]

Theorem 7.4 [S. Bernstein] If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree at most \( n \), then
\[
\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|.
\]

In order to deduce Theorem 7.4 from Theorem 7.3, let \( p(z) \not\equiv M(p; 1) e^{i\gamma} z^n \) for all \( \gamma \in \mathbb{R} \). Applying Theorem 7.3 to the polynomial \( p(\rho z) \), we get on \( |z|=1 \)
\[
\rho |p'(\rho z)| < M(p; \rho) n,
\]
which is equivalent to
\[
|p'(\rho z)| < \frac{M(p; \rho)}{\rho} n, \quad |z|=1.
\]

For any given \( R > 1 \), let \( M(p; R) = |p(R e^{i\varphi})| \). Then
\[
M(p; R) = \left| p(e^{i\varphi}) + \int_{1}^{R} p'(\rho e^{i\varphi}) e^{i\varphi} d\rho \right| < M(p; 1) + \int_{1}^{R} \frac{n}{\rho} M(p; \rho) d\rho. \tag{18}
\]

Denoting the right-hand side of the preceding inequality by \( \Phi(R) \) we see that
\[
\frac{d}{dR} \{R^{-n} \Phi(R)\} = R^{-n} \Phi'(R) - n R^{-n-1} \Phi(R)
\]
\[
= R^{-n-1} n M(p; R) - n R^{-n-1} \Phi(R)
\]
\[
= n R^{-n-1} (M(p; R) - \Phi(R)) < 0 \quad \text{for} \quad R > 1
\]

Thus, \( R^{-n} \Phi(R) \) is a decreasing function of \( R \) for \( R > 1 \) and hence,
\[
R^{-n} \Phi(R) < \Phi(1) = M(p; 1) \tag{19}
\]

implying that \( \Phi(R) < M(p; 1) R^n \), which in view of (6) gives
\[
M(p; R), \quad \text{which is} \quad < \Phi(R) < \Phi(1) R^n = M(p; 1) R^n.
\]

Thus Theorems 7.3 and 7.4 are equivalent.

Also, Theorems 7.4 and 7.5 are equivalent. For this, we have just to observe that \( p \) is a polynomial of degree at most \( n \) if and only if \( q(z) = z^n p\left(\frac{1}{z}\right) \) is.

Thus all the three Theorems 7.3, 7.4 and 7.5 are all equivalent, in the sense that they can be obtained from each other.
THEOREM 12.1 If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \), then
\[
\begin{align*}
\max_{|z|=1} |p'(z)| & \leq n \max_{|z|=1} |p(z)|, \quad (20) \\
\max_{|z|=R \geq 1} |p(z)| & \leq R^n \max_{|z|=1} |p(z)|, \quad (21) \\
\max_{|z|=r \leq 1} |p(z)| & \geq r^n \max_{|z|=1} |p(z)|. \quad (22)
\end{align*}
\]

In all above inequalities, the equality holds for \( p(z) = \lambda z^n \).

THEOREM 12.2 (Frappier, Rahman and Ruscheweyh, Trans. A. M. S. 1985) If \( p(z) \) is a polynomial of degree at most \( n \), then
\[
\begin{align*}
\max_{|z|=1} |p'(z)| & \leq \begin{cases} 
 n \max_{|z|=1} |p(z)| - \frac{2n}{n+2} |p(0)| & \text{if } n \geq 2 \\
 \max_{|z|=1} |p(z)| - |p(0)| & \text{if } n = 1 \end{cases} \quad (23) \\
\max_{|z|=R} |p(z)| & \leq R^n \max_{|z|=1} |p(z)| - (R^n - R^{n-2})|p(0)| \quad \text{if } n \geq 2. \quad (24)
\end{align*}
\]

THEOREM 12.3 If \( p(z) \) is a polynomial of degree \( n \), \( p(z) \neq 0 \) in \( |z| < 1 \), then
\[
\begin{align*}
\max_{|z|=1} |p'(z)| & \leq \frac{n}{2} \max_{|z|=1} |p(z)|, \quad (25) \\
\max_{|z|=R \geq 1} |p(z)| & \leq \frac{(R^n + 1)}{2} \max_{|z|=1} |p(z)|, \quad (26) \\
\max_{|z|=r \leq 1} |p(z)| & \geq \left(\frac{r+1}{2}\right)^n \max_{|z|=1} |p(z)|. \quad (27)
\end{align*}
\]

Inequality (24), which is known as Erdős-Lax Theorem, is due to P. D. Lax [Bull A.M.S. (1944)], inequality (25) is due to Ankeney & Rivlin [Pacific J. Math (1955)] and inequality (26) is due to Rivlin [ Amer. Math. Monthly (1960)].
13

THEOREM 13.1 (P. D. Lax, Bull. A.M.S. 1944) If $p(z)$ is a polynomial of degree $n$, $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.$$  \hfill (28)

R. P. Boas raised the following question

“How large can the bound in (27) be if $p(z)$ has $k$ zeros on or outside the unit circle?”

One would expect the answer to be $(n - \frac{k}{2})$.

THEOREM 13.2 (Giroux and Rahman, Trans. A. M. S. 1974) For every positive integer $n$, there exists a polynomial $p(z)$ of degree $n$ having a zero on $|z| = 1$, such that

$$\max_{|z|=1} |p'(z)| \geq (n - c/n) \max_{|z|=1} |p(z)|.$$  

On the other hand for an arbitrary polynomial $p(z)$ of degree $n$ having a zero on $|z| = 1$, we have

$$\max_{|z|=1} |p'(z)| \leq (n - \frac{1 - \sin \frac{1}{4\pi n}}{4\pi n}) \max_{|z|=1} |p(z)|.$$  

THEOREM 13.3 (St. A. Ruscheweyh, Complex Variables 6 (1986), 11-21) There exist polynomials $p(z)$ of degree $n$ having all but one zero on $|z| = 1$, such that

$$\max_{|z|=1} |p'(z)| = [An + o(n)] \max_{|z|=1} |p(z)|,$$

where $A \simeq 0.884$.

The result of Ruscheweyh thus shows that even if we assume that all but one zeros lie on $|z| = 1$, the bound in the Bernstein’s inequality cannot really be very significantly improved.
THEOREM 14.1 (P. D. Lax, Bull. A. M. S. 1944) If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then
\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{29}
\]

Here is another question raised by R. P. Boas.

"How large can the bound in (28) be if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, K > 0 \)?"

THEOREM 14.2 (M. A. Malik, Jour. Lond. Math. Soc. 1969) If \( p(z) \) is a polynomial of degree \( n \) having no zero in \( |z| < K, K \geq 1 \), then
\[
\max_{|z|=1} |p'(z)| \leq \left( \frac{n}{1 + K} \right) \max_{|z|=1} |p(z)|. \tag{30}
\]
Equality holds for \( p(z) = (z + K)^n \).

THEOREM 14.3 (N. K. Govil and Q. I. Rahman, Trans. Amer. Math. Soc. 1969) If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, K \geq 1 \), then
\[
\max_{|z|=1} |p^{(s)}(z)| \leq \frac{n(n - 1)\ldots(n - s + 1)}{1 + K^s} \max_{|z|=1} |p(z)|. \tag{31}
\]

Remark 1. The bound in Theorem 12.2 depends on the moduli of the zeros of smallest modulus. For example, for both polynomials \((z + K)^n\) and \((z + K)(z + K + l)^{n-1}\), the inequality (22) gives the same bound, \( \frac{n}{1 + K} \).

THEOREM 14.4 (N. K. Govil & G. Labelle, Jour. Math. Analysis & Applications 1987) Let \( p(z) = a_n \prod_{v=1}^n (z - z_v), a_n \neq 0 \). If \( |z_v| \geq K_v \geq 1, 1 \leq v \leq n \), then
\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left\{ 1 - \frac{1}{1 + \frac{2}{n} \sum_{v=1}^n \frac{1}{K_v - 1}} \right\} \max_{|z|=1} |p(z)| \tag{32}
\]
If \( K_v = 1 \) for some \( v \), it reduces to Lax’s result. If \( K_v \geq K \geq 1 \), it reduces to Malik’s result, Theorem 12.2.

Remark 2. One need not know the location of all the zeros of the polynomial in order to apply this theorem. No doubt the usefulness of the theorem will be heightened if we have information about all the zeros.
THEOREM 15.1 (P. D. Lax, Bull. Amer. Math. Soc. 1944.) If \( p(z) \) is a polynomial of degree \( n \), having no zeros in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.
\] (33)

The result is best possible and the equality holds for \( p(z) = (z + 1)^n \) which has all its zeros on \( |z| = 1 \).

If a polynomial \( p(z) \) has zeros on \( |z| = 1 \), then \( \min_{|z|=1} |p(z)| = 0 \). Hence if we exclude polynomials having zeros on \( |z| = 1 \), it should be possible to improve upon the bound in (25). This fact was observed by Aziz & Dawood, who proved

THEOREM 15.2 (A. Aziz and Q. M. Dawood, Jour. Approx. Theory 1988.) If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left( \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right).
\] (34)

The result is best possible with equality holding for \( p(z) = (z + 1)^n \) for which \( \min_{|z|=1} |p(z)| = 0 \).

THEOREM 15.3 (N. K. Govil, Jour. Approx. Theory 66 (1991).) If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, K \geq 1 \), then

\[
\max_{|z|=1} |p^{(s)}(z)| \leq \frac{n(n-1)\ldots(n-s+1)}{1 + K^s} \left( \max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)| \right).
\] (35)

For \( s = 1 \), the above theorem gives

COROLLARY 15.1 If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, K \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + K} \left( \max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)| \right).
\] (36)

We do not know if the bound in Theorem 13.3 is best possible.
THEOREM 16.1 (P. D. Lax, Bull. Amer. Math. Soc. 1944.) If $p(z)$ is a polynomial of degree $n$, having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|$$ \hspace{1cm} (37)

The result is best possible and the equality holds for $p(z) = (z + 1)^n$ which has all its zeros on $|z| = 1$.

If a polynomial $p(z)$ has zeros on $|z| = 1$, then $\min_{|z|=1} |p(z)| = 0$. Hence if we exclude polynomials having zeros on $|z| = 1$, it should be possible to improve upon the bound in (29). This fact was observed by Aziz & Dawood, who proved

THEOREM 16.2 (A. Aziz and Q. M. Dawood, Jour. Approx. Theory 1988.) If $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} (\max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)|).$$ \hspace{1cm} (38)

The result is best possible with equality holding for $p(z) = (z + 1)^n$, for which $\min_{|z|=1} |p(z)| = 0$.

THEOREM 16.3 (T. N. Chan and M. A. Malik, Proc. Indian Acad. Sci.1983.) If $p(z) = a_0 + \sum_{\nu=m}^n a_\nu z^\nu$ is a polynomial of degree $n$ having no zeros in $|z| < K$, $K \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + K^m} \max_{|z|=1} |p(z)|.$$ \hspace{1cm} (39)

The result is best possible and equality here holds for $p(z) = (z^m + K^m)^{n/m}$, $n$ being a multiple of $m$.

THEOREM 16.4 (N. K. Govil, Jour. Approx. Theory 66 (1991).) If $p(z)$ is a polynomial of degree $n$ having no zeros in $|z| < K$, $K \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + K} (\max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)|).$$ \hspace{1cm} (40)

THEOREM 16.5 (N. K. Govil, Journal of Inequalities & Applications, 2002) If $p(z) = a_0 + \sum_{\nu=m}^n a_\nu z^\nu$ is a polynomial of degree $n$ having no zeros in $|z| < K$, $K \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + K^m} (\max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)|).$$ \hspace{1cm} (41)

For $m = 1$, it reduces to Theorem 14.4 and for $m = 1$ and $K = 1$, it reduces to Theorem 14.2 due to Aziz and Dawood. In general, for all polynomials having no zeros on $|z| = 1$, it gives a bound sharper than obtainable from Theorem 14.3 of Chan and Malik.
Theorem 10.1 [N. K. Govil, to appear.] If \( p(z) = a_0 + \sum_{m=v}^{n} a_m z^m \) is a polynomial of degree \( n \) having no zeros in \( |z| < K \), \( K \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + K^m} \left( \max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)| \right). \tag{42}
\]

The proof is based on the use of the following lemmas.

**Lemma 1.** If \( p(z) = a_0 + \sum_{m=v}^{n} a_m z^m \) is a polynomial of degree \( n \), having no zeros in \( |z| < K \), \( K \geq 1 \), then on \( |z| = 1 \),

\[
K^m |p'(z)| \leq |q'(z)|,
\]

where \( q(z) = z^n p_n(1/z) \).

**Lemma 2.** If \( p(z) = a_0 + \sum_{m=v}^{n} a_m z^m \) is a polynomial of degree \( n \), having no zeros in \( |z| < K \), \( K \geq 1 \), then on \( |z| = 1 \),

\[
|q'(z)| \geq n \min_{|z|=K} |p(z)|,
\]

where \( q(z) = z^n p_n(1/z) \) is as in Lemma 1.

**Lemma 3.** If \( p(z) = a_0 + \sum_{m=v}^{n} a_m z^m \) is a polynomial of degree \( n \), having no zeros in \( |z| < K \), \( K \geq 1 \), then on \( |z| = 1 \),

\[
|q'(z)| \geq K^m |p'(z)| + n \min_{|z|=K} |p(z)|,
\]

where \( q(z) = z^n p_n(1/z) \) is as in Lemma 2.

**Lemma 4.** Let \( p(z) \) be a polynomial of degree \( n \). Then on \( |z| = 1 \),

\[
|p'(z)| + |q'(z)| \leq n \|p\|,
\]

where \( q(z) = z^n p_n(1/z) \) is as defined in Lemma 2.

Lemma 1 is due to Chan and Malik, and Lemma 4 is a special case of a more general result due to N. K. Govil and Q. I. Rahman [Trans. Amer. Math. Soc. 1969].

Theorem 10.1 now follows on combining Lemmas 3 and 4.
THEOREM 18.1 (Bernstein Inequality) For any polynomial \( p(z) \) of degree \( n \), we have

\[
M(p, R) \leq R^n \|p\|. \tag{43}
\]

THEOREM 18.2 (Frappier, Rahman and Ruscheweyh, Trans. A. M. S. 1985) If \( p(z) \) is a polynomial of degree at most \( n \), then

\[
\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - (R^n - R^{n-2})|p(0)| \quad \text{if} \ n \geq 2. \tag{44}
\]

THEOREM 18.3 (N. K. Govil, Complex Variables, 2000) If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^\nu \) is a polynomial of degree \( n \), then for \( R \geq 1 \) we have

\[
M(p, R) \leq R^n \left\{ 1 - \frac{(\|p\| - |a_n|)^2(R^2 - 1)}{|a_n| (\|p\| - |a_n|) + \|p\| |a_{n-1}| R + \|p\| (\|p\| - |a_n|) R^2} \right\}\|p\| \tag{45}
\]

The above inequality is best possible and the equality holds for \( p(z) = \lambda z^n \), \( \lambda \) being a complex number.

THEOREM 18.4 (N. C. Ankeny, T. J. Rivlin, Pacific Jour. Math., 1955) If \( p(z) \) has no zeros in \( |z| < 1 \), then

\[
M(p, R) \leq \frac{(R^n + 1)}{2} \|p\|. \tag{46}
\]

THEOREM 18.5 (N. K. Govil and Q. I. Rahman, 1969) If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, \ K \geq 1 \), then for \( 1 \leq R \leq K^2 \)

\[
M(p, R) \leq \left( \frac{R + K}{1 + K} \right)^n \|p\|. \tag{47}
\]

The above result is sharp, with equality holding for \( p(z) = (z + K)^n \).

Although the inequality (36) is sharp, but it holds only in the range \( 1 \leq R \leq K^2 \). The following result, although is not best possible, but holds for \( R \geq K^2 \).

THEOREM 2 [N. K. Govil, M. A. Qazi and Q. I. Rahman, 2003]. Let \( p(z) := \sum_{\nu=0}^{n} a_{\nu} z^\nu \neq 0 \) for \( |z| < K, \) where \( K > 1 \). Then,

\[
M(p; R) \leq \frac{R^n}{K^n} \left( \frac{K^n}{K^n + 1} \right)^{(R - K^2)/(R + K^2)} M(p; 1) \quad (R \geq K^2). \tag{48}
\]

If \( p(z) = a_0 + \sum_{v=m}^{n} a_v z^v \), where \( m \geq 1 \), is a polynomial of degree \( n \geq 2 \), having no zeros in \( |z| < K, K \geq 1 \), then for \( R \geq 1 \),

\[
M(p, R) \leq \left( \frac{R^n + K^m}{1 + K^m} \right) \|p\| - \left( \frac{R^n - 1}{1 + K^m} \right) \min_{|z|=K} |p(z)| \\
- |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right)
\]

(49)

if \( n > 2 \), and

\[
M(p, R) \leq \left( \frac{R^n + K^m}{1 + K^m} \right) \|p\| - \left( \frac{R^n - 1}{1 + K^m} \right) \min_{|z|=K} |p(z)| - |a_1| \frac{(R - 1)^n}{2},
\]

(50)

if \( n = 2 \).

For the proof of this theorem, we will need the following lemmas.

LEMMA 19.1 If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \), \( n \geq 2 \), then for all \( R \geq 1 \),

\[
\max_{|z|=R} |p(z)| \leq R^n \|p\| - \left( R^n - R^{n-2} \right) |p(0)|, \text{ if } n \geq 2
\]

(51)

and

\[
\max_{|z|=R} |p(z)| \leq R^n \|p\| - \frac{(R - 1)^n}{2} |p(0)|, \text{ if } n = 1.
\]

(52)


LEMMA 19.2 If \( p(z) = a_0 + \sum_{v=m}^{n} a_m z^m \) is a polynomial of degree \( n \), having no zeros in \( |z| < K, K \geq 1 \), then on \( |z| = 1 \),

\[
|p'(z)| \leq \left( \frac{n}{1 + K^m} \right) \left( \|p\| - \min_{|z|=K} |p(z)| \right).
\]

(53)

The result is best possible and the equality holds for the polynomial \( p(z) = (z^m + K^m)^{n/m} \), \( n \) being a multiple of \( m \).

The above lemma is due to Govil [to appear].
Proof of the Theorem.

We first consider the case when \( p(z) \) is of degree \( n > 2 \). Note that for every \( \theta, 0 \leq \theta < 2\pi \)

\[
|p(Re^{i\theta}) - p(e^{i\theta})| = \left| \int_1^R p'(re^{i\theta})e^{i\theta} \, dr \right|
\]
\[
\leq \int_1^R \left| p'(re^{i\theta}) \right| \, dr.
\]  
(54)

Since \( p(z) \) is of degree \( n > 2 \), the polynomial \( p'(z) \) is of degree \( (n - 1) \geq 2 \), hence applying (40) to \( p'(z) \) in (43) we get

\[
|p(Re^{i\theta}) - p(e^{i\theta})| \leq \int_1^R \left( R^{n-1}||p'|| - (R^{n-1} - R^{n-3})|p'(0)| \right) \, dr
\]
\[
= \left( \frac{R^n - 1}{n} \right) ||p'||
\]
\[
- \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right) |p'(0)|.
\]  
(55)

Combining (44) with Lemma 12.2, we get for \( n > 2, R \geq 1 \) and \( 0 \leq \theta < 2\pi \),

\[
|p(Re^{i\theta}) - p(e^{i\theta})| \leq \left( \frac{R^n - 1}{1 + K^s} \right) (||p|| - m)
\]
\[
- \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right) |p'(0)|,
\]  
(56)

implying that for \( n > 2, R \geq 1 \) and \( 0 \leq \theta < 2\pi \),

\[
|p(Re^{i\theta})| \leq \left( \frac{R^n + K^s}{1 + K^s} \right) ||p|| - \left( \frac{R^n - 1}{1 + K^s} \right) m
\]
\[
- |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right).
\]  
(57)

which gives (38). The proof of (39) follows on the same lines as the proof of (38) except that instead of using (40), here we use (41).
7 $L^p$-Inequalities.

THEOREM 7.1 (Bernstein’s Inequality) If $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^\nu$ is a polynomial of degree $n$, then

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.$$  \hspace{1cm} (58)

The result is best possible and the equality holds for $p(z) = \lambda z^n$.

THEOREM 7.2 (Zygmund, Proc. L. M. S. 1932) If $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^\nu$ is a polynomial of degree $n$, then

$$\left(\frac{1}{2\pi} \int_{0}^{2\pi} |p'(e^{i\theta})|^\delta d\theta\right)^{1/\delta} \leq n \left(\frac{1}{2\pi} \int_{0}^{2\pi} |p(e^{i\theta})|^\delta d\theta\right)^{1/\delta}.$$  \hspace{1cm} (59)

The result is best possible with equality holding for $p(z) = \lambda z^n$.

THEOREM 7.3 (P. D. Lax, Bull. A. M. S. 1944) If $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^\nu$ is a polynomial of degree $n$, having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.$$  \hspace{1cm} (60)

Again the result is best possible with equality for $(z + 1)^n$.

THEOREM 7.4 (N. G. de Bruijn, 1947) If $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^\nu$ is a polynomial of degree $n$, having no zeros in $|z| < 1$, then for $\delta \geq 1$

$$\left(\frac{1}{2\pi} \int_{0}^{2\pi} |p'(e^{i\theta})|^\delta d\theta\right)^{1/\delta} \leq n(C_\delta)^{1/\delta} \left(\frac{1}{2\pi} \int_{0}^{2\pi} |p(e^{i\theta})|^\delta d\theta\right)^{1/\delta},$$  \hspace{1cm} (61)

where $C_\delta = 2^{-\delta} \sqrt{\pi} \Gamma\left(\frac{1}{2}\delta + 1\right) / \Gamma\left(\frac{1}{2}\delta + \frac{1}{2}\right)$. The result is best possible with equality holding for $p(z) = (\alpha + \beta z^n), |\alpha| = |\beta|$.

If we make $\delta \to \infty$ in (54), we get (53) and thus Theorem 7.4 is a generalization of Theorem 7.3.

Theorem 7.4 has been extended to the case $\delta \geq 0$ by Rahman and Schmeisser (Jour. Math. Anal. Appl, 1988).
THEOREM 8.1 (P. D. Lax, Bull. A. M. S. 1944.) If \( p(z) = \sum_{\nu=0}^{n} a_{\nu}z^{\nu} \) is a polynomial of degree \( n \), having no zeros in \(|z| < 1\), then
\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{62}
\]

The result is sharp with equality holding for \( p(z) = \lambda + \mu z^n, |\lambda| = |\mu| \).

THEOREM 8.2 (N. G. DeBruijn, 1947.) If \( p(z) \) is as in Theorem 12.1, then for \( \delta \geq 1 \),
\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |p'(e^{i\theta})|^\delta d\theta \right)^{\frac{1}{\delta}} \leq n(C_{\delta})^\frac{1}{2} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |p(e^{i\theta})|^\delta d\theta \right)^{\frac{1}{\delta}}. \tag{63}
\]
where \( C_{\delta} = \frac{2^{-\delta} \sqrt{\pi} \Gamma(\frac{1}{2}\delta + 1)}{\Gamma(\frac{1}{2}\delta + \frac{1}{2})} \). Here the equality holds again for the polynomials \( p(z) = (\lambda + \mu z^n), |\lambda| = |\mu| \).

Theorem 12.4 is a generalization of Theorem 10.1, because \( \lim_{\delta \to \infty} C_{\delta} = \frac{1}{2} \).

THEOREM 8.3 (M.A. Malik, Jour L.M.S., 1969.) If \( p(z) = \sum_{\nu=0}^{n} a_{\nu}z^{\nu} \) is a polynomial of degree \( n \), \( p(z) \neq 0 \) in \(|z| < K, K \geq 1 \), then
\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1+K} \max_{|z|=1} |p(z)|. \tag{64}
\]

Here the equality holds for \( p(z) = (z + K)^n \).

THEOREM 8.4 (N. K. Govil and Q. I. Rahman, Trans. A.M.S. 1969.) Under the hypothesis of Theorem 13.1, we have
\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |p'(e^{i\theta})|^\delta d\theta \right)^{\frac{1}{\delta}} \leq nE_{\delta} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |p(e^{i\theta})|^\delta d\theta \right)^{\frac{1}{\delta}}. \tag{65}
\]
where \( E_{\delta} = 2\pi / \int_{0}^{2\pi} |K + e^{i\alpha}|^\delta d\alpha \).

Since \( \lim_{\delta \to \infty} E_{\delta}^{\frac{1}{\delta}} = \frac{1}{1+K} \), Theorem 8.4 generalizes Theorem 8.3.
THEOREM 9.1 (M.A. Malik, Jour L.M.S., 1969) If \( p(z) = \sum_{\nu=0}^{n} a_{\nu}z^{\nu} \) is a polynomial of degree \( n \), \( p(z) \neq 0 \) in \( |z| < K, K \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1+K} \max_{|z|=1} |p(z)|.
\]

(66)

Here the equality holds for \( p(z) = (z + K)^{n} \).

THEOREM 9.2 (N. K. Govil and Q. I. Rahman, Trans. A.M.S. 1969) Under the hypothesis of Theorem 9.1, we have

\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |p'(e^{i\theta})|^\delta d\theta \right)^{\frac{1}{\delta}} \leq nE_{\delta} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |p(e^{i\theta})|^\delta d\theta \right)^{\frac{1}{\delta}},
\]

where \( E_{\delta} = 2\pi / \int_{0}^{2\pi} |K + e^{i\alpha}|^\delta d\alpha \).

When \( \delta \to \infty \), Theorem 9.2 reduces to Theorem 9.1.

THEOREM 9.3 (Robert Gardner and N.K. Govil, Jour. Math. Analy. & Appl., 1993) Let \( p(z) = a_{n}\Pi_{\nu=1}^{n} (z - z_{\nu}) \), \( a_{n} \neq 0 \) be a polynomial of degree \( n \). If \( |z_{\nu}| \geq K_{\nu} \geq 1, 1 \leq \nu \leq n \), then for \( \delta \geq 1 \),

\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |p'(e^{i\theta})|^\delta d\theta \right)^{\frac{1}{\delta}} \leq nF_{\delta} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |p(e^{i\theta})|^\delta d\theta \right)^{\frac{1}{\delta}},
\]

where \( F_{\delta} = \{2\pi/ \int_{0}^{2\pi} |t_{0} + e^{i\theta}|^\delta d\theta\} \) and \( t_{0} = \{1 + n/ \sum_{\nu=1}^{n} 1/K_{\nu-1}\} \).


Remarks:

1. If \( K_{\nu} = 1 \) for some \( \nu \) then Theorem 12.3 reduces to the theorem of de Bruijn.

2. If \( K_{\nu} \geq K \geq 1 \) for \( \nu = 1, 2, ..., n \) then \( t_{0} \geq K \) and so \( F_{\delta} \leq \{2\pi/ \int_{0}^{2\pi} |K + e^{i\theta}|^\delta d\theta\} = E_{\delta} \).

Thus Theorem 13.3 sharpens Theorem 13.2. If in (33), we make \( \delta \to \infty \), we get


\[
\max_{|z|=1} |p(z)| \leq \frac{n}{2} \left( 1 - \frac{1}{1 + \frac{2}{n} \sum_{\nu=1}^{n} 1/K_{\nu-1}} \right) \max_{|z|=1} |p(z)|.
\]

(69)

The result is best possible if \( K_{\nu} = K \geq 1 \) for \( 1 \leq \nu \leq n \) and the extremal polynomial is \( p(z) = (z + K)^{n} \).
10 Polar Derivatives for Polynomials.

THEOREM 10.1 (P. D. Lax, Bull. A. M. S. 1944.) If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \), having no zeros in \(|z| < 1\), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{70}
\]

Let \( \alpha \) be a complex number. We define \( D_{\alpha}\{p(z)\} \), the polar derivative of \( p(z) \), by

\[
D_{\alpha}\{p(z)\} = np(z) + (\alpha - z)p'(z). \tag{71}
\]

It is easy to see that \( D_{\alpha}\{p(z)\} \) is a polynomial of degree at most \((n - 1)\) and that \( D_{\alpha} p(z) \) generalizes the ordinary derivative in the sense that

\[
\lim_{\alpha \to \infty} \frac{D_{\alpha}\{p(z)\}}{\alpha} = p'(z)
\]

uniformly with respect to \( z \) for \(|z| \leq R, \ R > 0\).

The polynomial \( D_{\alpha}\{p(z)\} \) has been called by Laguerre the “émanant” of \( p(z) \); by Pólya and Szegö the “derivative of \( p(z) \) with respect to the point \( \alpha \)”, and by Marden simply “the polar derivative of \( p(z) \)”. It is obviously of interest to obtain estimates concerning growth of \( D_{\alpha}\{p(z)\} \) and one such estimate is due to Aziz, who extended the inequality due to Lax for \( D_{\alpha}\{p(z)\} \) by proving

THEOREM 10.2 (A. Aziz, Jour. Approx. Theory, 1988) If \( p_{n}(z) \) is a polynomial of degree \( n \) having no zeros in the disk \(|z| < 1\), then for every real or complex number \( \alpha \) with \(|\alpha| \geq 1\), we have

\[
\max_{|z|=1} |D_{\alpha}\{p(z)\}| \leq \frac{n}{2} (|\alpha| + 1) \max_{|z|=1} |p(z)|. \tag{72}
\]

The result is best possible and equality in (37) holds for \( p(z) = \lambda + \mu z^{n} \), where \(|\mu| = |\lambda| \) and \( \alpha \geq 1 \).
11

THEOREM 11.1 (P. D. Lax, Bull. A. M. S. 1944.) If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^\nu \) is a poly-

nomial of degree \( n \), having no zeros in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{73}
\]

THEOREM 11.2 (A. Aziz, Jour. Approx. Theory, 1988) If \( p_n(z) \) is a polynomial of degree \( n \) having no zeros in the disk \( |z| < 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \), we have

\[
\max_{|z|=1} |D_\alpha \{p(z)\}| \leq \frac{n}{2} (|\alpha| + 1) \max_{|z|=1} |p(z)|. \tag{74}
\]

The result is best possible and equality in (39) holds for \( p(z) = \lambda + \mu z^n \), where \( |\mu| = |\lambda| \) and \( \alpha \geq 1 \).

THEOREM 11.3 (N. G. DeBruijn, 1947.) If \( p(z) \) is a polynomial of degree \( n \), hav-
ing no zeros in \( |z| < 1 \), then for \( \delta \geq 1 \),

\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |p'(e^{i\theta})|^\delta d\theta \right)^\frac{1}{\delta} \leq n(C\delta)^\frac{1}{2} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |p(e^{i\theta})|^\delta d\theta \right)^\frac{1}{\delta}. \tag{75}
\]

where \( C\delta = \frac{2^{-\delta} \sqrt{\pi \Gamma(\frac{1}{2} \delta + 1)}}{\Gamma(\frac{1}{2} + \frac{1}{2})} \). Here the equality holds again for the polynomials \( p(z) = (\lambda + \mu z^n), |\lambda| = |\mu| \).

THEOREM 11.4 (Govil, Nyuydinkong and Tameru, Jour. Math. Anal. Appl., 2001.) If \( p(z) \) is a polynomial of degree \( n \), having no zeros in \( |z| < 1 \), then for \( \delta \geq 1 \) and for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \),

\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |D_\alpha \{p(e^{i\theta})\}|^\delta d\theta \right)^\frac{1}{\delta} \leq n(|\alpha| + 1)C\delta \left( \frac{1}{2\pi} \int_{0}^{2\pi} |D_\alpha \{p(e^{i\theta})\}|^\delta d\theta \right)^\frac{1}{\delta}. \tag{76}
\]

where \( C\delta = \frac{2^{-\delta} \sqrt{\pi \Gamma(\frac{1}{2} \delta + 1)}}{\Gamma(\frac{1}{2} + \frac{1}{2})} \). In the limiting case, when \( \delta \to \infty \), the above inequality is sharp and equality holds for the polynomials \( p(z) = (\lambda + \mu z^n), |\lambda| = |\mu| \).

THEOREM 11.5 (Govil, Nyuydinkong and Tameru, Jour. Math. Anal. Appl., 2001.) If \( p(z) \) is a polynomial of degree \( n \), having all its zeros in \( |z| \leq 1 \), then for \( \delta \geq 1 \) and for every real or complex number \( \alpha \) with \( |\alpha| \leq 1 \),

\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |D_\alpha \{p(e^{i\theta})\}|^\delta d\theta \right)^\frac{1}{\delta} \leq n(|\alpha| + 1)C\delta \left( \frac{1}{2\pi} \int_{0}^{2\pi} |D_\alpha \{p(e^{i\theta})\}|^\delta d\theta \right)^\frac{1}{\delta}. \tag{77}
\]

where \( C\delta = \frac{2^{-\delta} \sqrt{\pi \Gamma(\frac{1}{2} \delta + 1)}}{\Gamma(\frac{1}{2} + \frac{1}{2})} \). Here the equality holds again for the polynomials \( p(z) = (\lambda + \mu z^n), |\lambda| = |\mu| \).
THEOREM 12.1 (Govil, Nyuydinkong and Tameru, Jour. Math. Anal. Appl., 2001.) If $p(z)$ is a polynomial of degree $n$, having no zeros in $|z| < 1$, then for $\delta \geq 1$ and for every real or complex number $\alpha$ with $|\alpha| \geq 1$,

$$
\left( \frac{1}{2\pi} \int_0^{2\pi} |D_\alpha \{p(e^{i\theta})\}|^\delta d\theta \right)^{\frac{1}{\delta}} \leq n(|\alpha| + 1)C_\delta \left( \frac{1}{2\pi} \int_0^{2\pi} |D_\alpha \{p(e^{i\theta})\}|^\delta d\theta \right)^{\frac{1}{\delta}}.
$$

where $C_\delta = \frac{2^{-\delta} \sqrt{\pi \Gamma\left( \frac{1}{2}\delta + 1 \right)}}{\Gamma\left( \frac{1}{2}\delta + \frac{3}{2} \right)}$. In the limiting case, when $\delta \rightarrow \infty$, the above inequality is sharp and equality holds for the polynomials $p(z) = (\lambda + \mu z^n), |\lambda| = |\mu|$.

THEOREM 12.2 (Govil, Nyuydinkong and Tameru, Jour. Math. Anal. Appl., 2001.) If $p(z)$ is a polynomial of degree $n$, having all its zeros in $|z| \leq 1$, then for $\delta \geq 1$ and for every real or complex number $\alpha$ with $|\alpha| \leq 1$,

$$
\left( \frac{1}{2\pi} \int_0^{2\pi} |D_\alpha \{p(e^{i\theta})\}|^\delta d\theta \right)^{\frac{1}{\delta}} \leq n(|\alpha| + 1)C_\delta \left( \frac{1}{2\pi} \int_0^{2\pi} |D_\alpha \{p(e^{i\theta})\}|^\delta d\theta \right)^{\frac{1}{\delta}}.
$$

where $C_\delta = \frac{2^{-\delta} \sqrt{\pi \Gamma\left( \frac{1}{2}\delta + 1 \right)}}{\Gamma\left( \frac{1}{2}\delta + \frac{3}{2} \right)}$. Here the equality holds again for the polynomials $p(z) = (\lambda + \mu z^n), |\lambda| = |\mu|$.

If in Theorem 16.1, we make $\delta \rightarrow \infty$, we get the following theorem due to Aziz.

THEOREM 12.3 (A. Aziz, Jour. Approx. Theory, 1988) If $p_n(z)$ is a polynomial of degree $n$ having no zeros in the disk $|z| < 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq 1$, we have

$$
\max_{|z|=1} |D_\alpha \{p(z)\}| \leq \frac{n}{2}(|\alpha| + 1) \max_{|z|=1} |p(z)|.
$$

The result is best possible and equality in (80) holds for $p(z) = \lambda + \mu z^n$, where $|\mu| = |\lambda|$ and $\alpha \geq 1$.

Further, if we divide both sides of the inequality (43) in Theorem 16.1 by $|\alpha|$ and make $\alpha \rightarrow \infty$, we get the following theorem due to De Bruijn.

THEOREM 12.4 (N. G. De Bruijn, 1947.) If $p(z)$ is a polynomial of degree $n$, having no zeros in $|z| < 1$, then for $\delta \geq 1$,

$$
\left( \frac{1}{2\pi} \int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta \right)^{\frac{1}{\delta}} \leq n(C_\delta)^{\frac{1}{\delta}} \left( \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta \right)^{\frac{1}{\delta}}.
$$

where $C_\delta = \frac{2^{-\delta} \sqrt{\pi \Gamma\left( \frac{1}{2}\delta + 1 \right)}}{\Gamma\left( \frac{1}{2}\delta + \frac{3}{2} \right)}$. Here the equality holds again for the polynomials $p(z) = (\lambda + \mu z^n), |\lambda| = |\mu|$.

There are some more consequences of these theorems which are given in the paper, which has just appeared.
THEOREM 13.1 (Bernstein Inequality) For any polynomial \( p(z) \) of degree \( n \), we have

\[
M(p, R) \leq R^n \| p \|.
\]  

(82)

THEOREM 13.2 (Frappier, Rahman and Ruscheweyh, Trans. A. M. S. 1985) If \( p(z) \) is a polynomial of degree at most \( n \), then

\[
\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - (R^n - R^{n-2})|p(0)| \text{ if } n \geq 2.
\]  

(83)

THEOREM 13.3 (N. K. Govil, Complex Variables, 2000) If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \), then for \( R \geq 1 \) we have

\[
M(p, R) \leq R^n \left\{ 1 - \frac{(\| p \| - |a_n|)^2 (R^2 - 1)}{|a_n| (\| p \| - |a_n|) + \| p \| |a_{n-1}| R + \| p \| (\| p \| - |a_n|) R^2} \right\} \| p \|. 
\]  

(84)

The above inequality is best possible and the equality holds for \( p(z) = \lambda z^n \), \( \lambda \) being a complex number.

THEOREM 13.4 (N. C. Ankeny, T. J. Rivlin, Pacific Jour. Math., 1955) If \( p(z) \) has no zeros in \( |z| < 1 \), then

\[
M(p, R) \leq \frac{(R^n + 1)}{2} \| p \|. 
\]  

(85)

THEOREM 13.5 (N. K. Govil, to appear in the Journal of Inequalities and Applications) If \( p(z) = a_0 + \sum_{\nu=m}^{n} a_{\nu} z^{\nu} \), where \( m \geq 1 \), is a polynomial of degree \( n \geq 2 \), having no zeros in \( |z| < K \), \( K \geq 1 \), then for \( R \geq 1 \),

\[
M(p, R) \leq \left( \frac{R^n + K^m}{1 + K^m} \right) \| p \| - \left( \frac{R^n - 1}{1 + K^m} \right) \min_{|z|=K} |p(z)| \\
- |a_1| \left( \frac{R^{n-1} - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right)
\]  

(86)

if \( n > 2 \), and

\[
M(p, R) \leq \left( \frac{R^n + K^m}{1 + K^m} \right) \| p \| - \left( \frac{R^n - 1}{1 + K^m} \right) \min_{|z|=K} |p(z)| - |a_1| \frac{(R-1)^n}{2},
\]  

(87)

if \( n = 2 \).
THEOREM 14.1 (N. K. Govil, Jour. Inequalities and Applications (To appear)) If \( p(z) = a_0 + \sum_{\nu=m} a_{\nu}z^{\nu} \), where \( m \geq 1 \), is a polynomial of degree \( n \geq 2 \), having no zeros in \( |z| < K, K \geq 1 \), then for \( R \geq 1 \),

\[
M(p, R) \leq \left( \frac{R^n + K^m}{1 + K^m} \right) \|p\| - \left( \frac{R^n - 1}{1 + K^m} \right) \min_{|z|=K} |p(z)|
- |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right)
\]

(88)

if \( n > 2 \), and

\[
M(p, R) \leq \left( \frac{R^n + K^m}{1 + K^m} \right) \|p\| - \left( \frac{R^n - 1}{1 + K^m} \right) \min_{|z|=1} |p(z)|
- |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right)
\]

(89)

if \( n = 2 \).

In particular on taking \( m = 1 \) and \( K = 1 \) in the above theorem, we get

COROLLARY 14.1 If \( p(z) = \sum_{\nu=0} a_{\nu}z^{\nu} \) is a polynomial of degree \( n \geq 2 \), having no zeros in \( |z| < 1 \), then for \( R \geq 1 \),

\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) \|p\| - \left( \frac{R^n - 1}{2} \right) \min_{|z|=1} |p(z)|
- |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right)
\]

(90)

if \( n > 2 \), and

\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) \|p\| - \left( \frac{R^n - 1}{2} \right) \min_{|z|=1} |p(z)|
- |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right)
\]

(91)

if \( n = 2 \). Both the above inequalities are best possible for the class of polynomials satisfying \( a_1 = 0 \) and \( \min_{|z|=1} |p(z)| = 0 \), and each becomes equality for the polynomial \( p(z) = \lambda + \mu z^n \), where \( \lambda \) and \( \mu \) are complex numbers with \( |\lambda| = |\mu| \).

The above corollary clearly sharpens the result of Ankeny and Rivlin, which states that if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then for \( R \geq 1 \)

\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) \|p\|.
\]

(92)

If in (54) and (55), we divide both the sides by \( R^n \) and make \( R \to \infty \), we get

COROLLARY 14.2 If \( p(z) = \sum_{\nu=0} a_{\nu}z^{\nu} \) is a polynomial of degree \( n \geq 2 \), having no zeros in \( |z| < K, K \geq 1 \), then

\[
|a_n| + \frac{|a_1|}{n} \leq \frac{1}{K + 1} \left( \|p\| - \min_{|z|=K} |p(z)| \right).
\]

(93)
For $K = 1$, it sharpens the well known result that if $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$ is a polynomial of degree $n \geq 2$, having no zeros in $|z| < 1$ then

$$|a_n| + \frac{|a_1|}{n} \leq \frac{1}{2} (\|p\|).$$

(94)
15  Polynomials Satisfying $p(z) \equiv z^n p\left(\frac{1}{z}\right)$.

In connection with Bernstein inequality

\[ \|p'\| \leq n\|p\|. \]  

(95)

Professor R. I. Rahman proposed to obtain an inequality analogous to the Bernstein inequality for the class $\Pi_n$, of polynomials satisfying $p(z) \equiv z^n p\left(\frac{1}{z}\right)$, and in this direction, the first result, perhaps is the following


If $p(z) \in \Pi_n$, has all its zeros either in the left half-plane or in the right half-plane, then

\[ \|p'\| \leq \frac{n}{\sqrt{2}}\|p\|. \]  

(96)

By considering $p(z) = z^n + 2iz^{n/2} + 1$, $n$ being even, they showed that if $p \in \Pi_n$ then the bound in (96) can not in general be smaller than $n/\sqrt{2}$.

By rather deep methods, Frappier, Rahman and Ruscheweyh, Trans. A. M. S. 1985, showed that by just assuming $p \in \Pi_n$, there is nearly no improvement for the derivative estimate in (95). In fact, they found a polynomial $p \in \Pi_n$ such that

\[ \|p'\| \geq (n - 1)\|p\|, \]

and at the same time they obtained, that if $p \in \Pi_n$, then

\[ \|p'\| \leq (n - \delta_n)\|p\|, \text{ where } \delta_n \to 2/5 \text{ as } n \to \infty. \]  

(97)


If $p(z) = \sum_{\nu=0}^{n} (\alpha_{\nu} + i\beta_{\nu})z^{\nu}$, $\alpha_{\nu} \geq 0, \beta_{\nu} \geq 0, 0 \leq \nu \leq n$, then

\[ \|p'\| \leq \frac{n}{\sqrt{2}}\|p\|. \]  

(98)

The equality in (98) holds for $p(z) = z^n + 2iz^{n/2} + 1$.

THEOREM 15.3 (N. K. Govil and David H. Vetterlein, Complex Variables, (1996))

If $p(z) \in \Pi_n$, and has all its coefficients lying in a sector of opening at most $\gamma$ where $0 \leq \gamma \leq 2\pi/3$, then

\[ \|p'\| \leq \left(\frac{n}{2 \cos \gamma/2}\right)\|p\|. \]  

(99)

The result is best possible for $0 \leq \gamma \leq \pi/2$, with equality holding for the polynomial $p(z) = z^n + 2e^{i\gamma}z^{n/2} + 1$, $n$ being even.


If $p(z)$ is a polynomial of degree $n$ satisfying $p(z) \equiv z^n p\left(\frac{1}{z}\right)$, then

\[ \max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \]  

(100)

The result is best possible and the equality holds for $p(z) = (z^n + 1)$. 
THEOREM 16.1 If $p(z)$ is a polynomial of degree $n$, then
\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)| \tag{101}
\]
\[
\max_{|z|=R \geq 1} |p(z)| \leq R^n \max_{|z|=1} |p(z)| \tag{102}
\]

Let $f(z)$ be an entire function, and let
\[
M(r) = \max_{|z|=r} |f(z)|
\]
\[
\rho = \text{order of } f = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}
\]
\[
\tau = \text{type of } f = \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho}
\]
\[
h_f(\theta) = \text{indicator function} = \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r}
\]

Definition. An entire function $f(z)$ is said to be of exponential type $\tau$ if it is either of order less than 1 or of type less than or equal to $\tau$ if of order equal to 1.

If $p(z)$ is a polynomial of degree $n$ then $f(z) = p(e^{iz})$ is an entire function of exponential type $n$. Thus the following theorem of S. Bernstein is a generalization of Bernstein’s inequalities for polynomials in Theorem 16.1.

THEOREM 16.2 If $f(z)$ is an entire function of exponential type $\tau$, then
\[
\sup_{-\infty < x < \infty} |f'(x)| \leq \tau \sup_{-\infty < x < \infty} |f(x)|, \tag{103}
\]

and
\[
\sup_{-\infty < x < \infty} |f(x + iy)| \leq e^{\tau|y|} \sup_{-\infty < x < \infty} |f(x)|, \tag{104}
\]

for $-\infty < y \leq 0$.

Both the above inequalities are best possible and the equality holds for the function $f(z) = e^{iz}$, which is of exponential type $\tau$. 
THEOREM 17.1 If \( p(z) \) is a polynomial of degree \( n \), \( p(z) \neq 0 \) in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)| \tag{105}
\]

\[
\max_{|z|=R \geq 1} |p(z)| \leq \frac{(R^n + 1)}{2} \max_{|z|=1} |p(z)|. \tag{106}
\]

Definition. An entire function \( f(z) \) of exponential type which is bounded on the real axis, does not vanish in \( \text{Im } z > 0 \) and for which \( h_f(\pi/2) = 0 \) is called asymmetric.

If \( p(z) \) is a polynomial of degree \( n \) then \( f(z) = p(e^{iz}) \) is an entire function of exponential type \( n \). Further if \( p(z) \neq 0 \) in \( |z| < 1 \), then \( f(z) = p(e^{iz}) \neq 0 \) in \( \text{Im } z > 0 \). Also it can be easily verified that \( h_f(\pi/2) = 0 \), that is, \( f(z) \) is asymmetric. Thus the following result is a generalization of Theorem 17.1.

THEOREM 17.2 (R. P. Boas, Illinois J. Math. 1957). If \( f(z) \) is an asymmetric entire function of type \( \tau \), then

\[
\sup_{-\infty < x < \infty} |f'(x)| \leq \frac{\tau}{2} \sup_{-\infty < x < \infty} |f(x)|, \tag{107}
\]

\[
\sup_{-\infty < x < \infty} |f(x + iy)| \leq \left( \frac{e^{|y|} + 1}{2} \right) \sup_{-\infty < x < \infty} |f(x)|. \tag{108}
\]

for \( -\infty < y \leq 0 \).

In 1959, Professor R. P. Boas proposed the problem of finding inequalities analogous to (66) and (67) for entire functions \( f \) of exponential type which are bounded on the real axis, do not vanish in \( \text{Im } z > k \), \( k \) being real and for which \( h_f(\pi/2) = 0 \).

A partial answer to the above problem was given by N. K. Govil and Q. I. Rahman.

THEOREM 17.3 (N. K. Govil and Q. I. Rahman, Trans. Amer. Math. Soc. 1969.) Let \( f(z) \) be an entire function of exponential type \( \tau \) having all its zeros in \( \text{Im } z \leq k \leq 0 \). If \( h_f(\pi/2) = 0 \), \( h_f(\pi/2) \leq c < 0 \), and also \( h_g(\pi/2) \leq -c < 0 \) where \( g(z) = e^{iz} \{ f(\bar{z}) \} \), then \( |f(x)| \leq 1 \) for real \( x \), implies

\[
|f'(x)| \leq \tau/(1 + e^{c|k|}) , \quad -\infty < x < \infty.
\]

The result is best possible and the equality holds for \( f(z) = \left( \frac{e^{ic_1z} - e^{-c_1k}}{1 + e^{-c_1k}} \right)^{\frac{\pi}{c_1}} \). For \( k = 0 \) it reduces to Boas inequality (66).
THEOREM 18.1 (R. P. Boas, Illinois J. Math. 1957) \textit{If } f(z) \textit{ is an asymmetric entire function of type } \tau, \|f\| = \sup_{-\infty < x < \infty} |f(x)|, \textit{ then }

\[ \|f\| \leq \left( \frac{\tau}{2} \right) \|f\| \] \quad (109)

\[ \sup_{-\infty < x < \infty} |f(x + iy)| \leq \left( \frac{e^{\tau|y|} + 1}{2} \right) \|f\|, -\infty < y \leq 0. \] \quad (110)

Here we present another generalization of the above inequalities.

Definition. \textit{Let } f \textit{ be an entire function of exponential type } \tau. \textit{ The polar derivative of } f \textit{ with respect to a complex number } \zeta \textit{ is defined to be }

\[ D_{\zeta}[f] = \tau f(z) + i(1 - \zeta)f'(z) \]

This definition is due to Rahman and Schmeisser (Jour.Math.Anal. & Appl.,1987).

Since \( \lim_{\zeta \to \infty} \frac{|D_{\zeta}[f(z)]|}{|\zeta|} = |f'(z)| \), the polar derivative defined above is a generalization of the ordinary derivative.

THEOREM 18.2 (Robert Gardner and N. K. Govil, Proc. Amer. Math. Soc. 1995) \textit{Let } f \textit{ be an entire function of exponential type } \tau, h_f(\pi/2) = 0, \|f\| = \sup_{-\infty < x < \infty} |f(x)| = 1 \textit{ and } f(z) = f(x + iy) \neq 0 \textit{ for } Im z > 0. \textit{ Then }

\[ |D_{\zeta}[f(z)]| \leq \frac{\tau}{2}(|\zeta|e^{\tau|y|} + 1) \] \quad (111)

\textit{for } -\infty < y = Im z \leq 0, \textit{ and } |\zeta| \geq 1. \textit{ The result is best possible and the equality holds } f(z) = \left( \frac{e^{i\tau z} + 1}{2} \right). \]

Remarks:

1. For \( \zeta = 1 \), the above theorem clearly reduces to the inequality (69) of Boas

2. If we divide both the sides of (70) by \( |\zeta| \) and make \( |\zeta| \to \infty \), we get

COROLLARY 18.1 \textit{Under the hypotheses of Theorem 22.2, we have }

\[ |f'(x + iy)| \leq \frac{\tau}{2} e^{\tau|y|}, -\infty < y \leq 0, -\infty < x < \infty \] \quad (112)

If we take \( y = 0 \) in Theorem 18.2, we get

COROLLARY 18.2 \textit{Let } f \textit{ be as in Theorem 18.2, then for } |\zeta| \geq 1, \textit{ we have }

\[ \|D_{\zeta}[f]\| \leq \frac{\tau}{2}(|\zeta| + 1). \] \quad (113)

Both the above corollaries generalize inequality (68) of Boas. To obtain (68) from Corollary 1, take \( y = 0 \) and to obtain it from Corollary 2, simply divide both the sides of (72) by \( |\zeta| \) and make \( |\zeta| \to \infty \).
Let $P_n$ denote the set of all complex algebraic polynomials $p(z) = \sum_{\nu=0}^{n} b_{\nu} z^\nu$ of degree at most $n$ and let $p'(z)$ be the derivative of $p(z)$.

Let $T := \{z : |z| = 1\}$, the unit circle in the complex plane $\mathbb{C}$.

For $f$ defined on $T$, we set $\|f\| = \sup_{z \in T} |f(z)|$, the Chebyshev norm of $f$ on $T$.

Let $D_\pm = \{z : |z| < 1\}$, the region inside $T$, and let $D_+ = \{z : |z| > 1\}$, the region outside $T$. For $a_\nu \in \mathbb{C}, \nu = 1, 2, \ldots, n$, let

$$w(z) = \prod_{\nu=1}^{n} (z - a_\nu)$$

and

$$B(z) = \prod_{\nu=1}^{n} \left(\frac{1 - \bar{a}_\nu z}{z - a_\nu}\right),$$

the Blaschke Product.

Let $R_n = R_n(a_1, a_2, \ldots, a_n) := \{\frac{p(z)}{w(z)} : p \in P_n\}$. Then $R_n$ is the set of rational functions with possible poles at $a_1, a_2, \ldots, a_n$ and having a finite limit at $\infty$. Also note that $B(z) \in R_n$.

Definition. (i) For $p(z) = \sum_{\nu=0}^{n} b_{\nu} z^\nu$, the conjugate transpose (reciprocal) $p^*$ of $p$ is defined by

$$p^*(z) = z^n \{\overline{p(\frac{1}{z})}\} = \bar{b}_0 z^n + \bar{b}_1 z^{n-1} + \cdots + \bar{b}_n.$$

(ii) For $r(z) = \frac{p(z)}{w(z)} \in R_n$, the conjugate transpose, $r^*$, of $r$ is defined by

$$r^*(z) = B(z)\{\overline{r(\frac{1}{z})}\}.$$

(iii) The polynomial $p \in P_n$ is called self-inversive, if $p^*(z) = \lambda p(z)$ for some $\lambda \in T$.

(iv) The rational function $r \in R_n$ is called self-inversive if $r^*(z) = \lambda r(z)$ for some $\lambda \in T$.

Note that if $r \in R_n$ and $r = \frac{p}{w}$, then $r^* = \frac{p^*}{w}$ and hence $r^* \in R_n$. So $r = \frac{p}{w}$ is self-inversive if and only if $p$ is self-inversive. Bernstein type inequalities for rational functions have appeared in the study of rational approximation problems by Petrushev and Popov [Encyclopedia of Mathematics & its Applications 28 (1987)]. These inequalities contain some constants which are not optimal. Recently Borwein, Erdélyi and Zhang [J. London Math. Soc. (2) 51 (1994)] have obtained Bernstein-Markov type inequalities for real rational functions for both algebraic and trigonometric polynomials on a finite interval.

Their main result is

THEOREM 19.1 If $z \in T$, and $a_j \in \mathbb{C} \setminus T, \ j = 1, 2, \ldots, n$, then for $r \in R_n$

$$|r'(z)| \leq \max \left\{\sum_{|a_j| > 1} \frac{|a_j|^2 - 1}{|a_j - z|^2}, \sum_{|a_j| < 1} \frac{1 - |a_j|^2}{|a_j - z|^2}\right\} \|r\|.$$

The above inequality is sharp.
THEOREM 20.1 If \( p \in \mathcal{P}_n \), then
\[ \| p' \| \leq n \| p \|, \]  \hspace{1cm} (114)
\[ \max_{|z|=R \geq 1} |p(z)| \leq R^n \| p \|. \]  \hspace{1cm} (115)

THEOREM 20.2 If \( p \in \mathcal{P}_n \) and has all its zeros in \( \mathbb{T} \cup \mathbb{D}_+ \), then
\[ \| p' \| \leq \frac{n}{2} \| p \|, \]  \hspace{1cm} (116)
\[ \max_{|z|=R \geq 1} |p(z)| \leq \left( \frac{R^n + 1}{2} \right) \| p \|. \]  \hspace{1cm} (117)

Li, Mohapatra and Rodriguez [J. London Math. Soc. 5 (1995)] proved inequalities analogous to inequalities (19.1) and (19.3). Their results are

THEOREM 20.3 (Li, Mohapatra & Rodriguez, 1995) For a rational function \( r \in \mathbb{R} \) with \( |a_\nu| > 1, 1 \leq \nu \leq n \), we have
\[ |r'(z)| \leq \| r \| |B'(z)|, \quad |z| \geq 1. \]  \hspace{1cm} (118)
The above result is best possible and the equality holds for \( r(z) = \lambda B(z) \), where \( B(z) \) is the Blaschke product and \( \lambda \in \mathbb{C} \).

THEOREM 20.4 (Li, Mohapatra & Rodriguez, 1995) Let \( r \in \mathbb{R} \) be a rational function with \( |a_\nu| > 1, 1 \leq \nu \leq n \), and having all its zeros in \( \mathbb{T} \cup \mathbb{D}_+ \). Then for \( |z| \geq 1 \), we have
\[ |r'(z)| \leq \| r \| \left( \frac{|B'(z)|}{2} \right), \quad |z| \geq 1. \]  \hspace{1cm} (119)
The above result is best possible and the equality holds for \( r(z) = \lambda B(z) + \mu \), \( |\lambda| = |\mu| \).

N. K. Govil and R. N. Mohapatra proved the analogous of the inequalities (19.2) and (19.4) for rational functions, which as well generalize (19.2) and (19.4). Their results are

THEOREM 20.5 (N. K. Govil & R. N. Mohapatra, 1998) For a rational function \( r(z) \in \mathbb{R}_n \), with \( |a_\nu| > 1, 1 \leq \nu \leq n \), we have
\[ |r(z)| \leq \| r \| |B(z)|, \quad |z| \geq 1. \]  \hspace{1cm} (120)
The above result is best possible and the equality holds for \( r(z) = \lambda B(z) \), where \( \lambda \in \mathbb{C} \).

THEOREM 20.6 (N. K. Govil & R. N. Mohapatra, 1998) Let \( r(z) \in \mathbb{R}_n \) with \( |a_\nu| > 1, 1 \leq \nu \leq n \). If all the zeros of \( r(z) \) lie in \( \mathbb{T} \cup \mathbb{D}_+ \), then for \( |z| \geq 1 \),
\[ |r(z)| \leq \| r \| \left( \frac{|B(z)| + 1}{2} \right). \]  \hspace{1cm} (121)
The result is best possible and the equality holds for the rational function \( r(z) = \alpha B(z) + \beta \), \( |\alpha| = |\beta| \).
Borwein and Erdélyi [J. London Math. Soc. 51 (1994)] have obtained Bernstein inequalities for rational spaces of complex algebraic polynomials. Their main result is

**Theorem C** If \( z \in \mathbb{T} \), and \( a_j \in \mathbb{C} \setminus \mathbb{T}, j = 1, 2, \ldots, n \), then for \( r \in \mathcal{R}_n \)

\[
|r'(z)| \leq \max \left\{ \sum_{|a_j| > 1} \frac{|a_j|^2 - 1}{|a_j - z|^2}, \sum_{|a_j| < 1} \frac{1 - |a_j|^2}{|a_j - z|^2} \right\} \|r\|.
\]

The above inequality is sharp.

The main point of the above theorem is that the poles are not necessarily restricted to be inside \( \mathbb{T} \) or outside \( \mathbb{T} \).

Also Li, Mohapatra and Rodriguez [J. London Math. Soc. 5 (1995)] have obtained Bernstein type inequalities for rational functions \( r \in \mathcal{R}_n \) with all the poles \( a_1, \ldots, a_n \) in \( \mathbb{D}_+ \). In particular for this class of rational functions they obtain inequalities analogous to inequalities (1) and (3). Their results depend upon the following identity for rational functions.

**Theorem D** Suppose that \( \lambda \in \mathbb{T} \). Then the following hold: The equation \( B(z) = \lambda \) has exactly \( n \) simple roots, say \( t_1, t_2, \ldots, t_n \), which lie on the unit circle \( \mathbb{T} \); and if \( r \in \mathcal{R}_n \) and \( z \in \mathbb{T} \), then

\[
B'(z)r(z) - r'(z)[B(z) - \lambda] = \frac{B(z)}{z} \sum_{k=1}^{n} c_k r(t_k) \left| \frac{B(z) - \lambda}{z - t_k} \right|^2,
\]

where \( c_k = c_k(\lambda) \) is defined by

\[
c_k^{-1} = \sum_{j=1}^{n} \frac{|a_j|^2 - 1}{|t_k - a_j|^2} \text{ for } k = 1, 2, \ldots, n.
\]

Moreover, for \( z \in \mathbb{T} \), we have

\[
z \frac{B'(z)}{B(z)} = \sum_{k=1}^{n} c_k \left| \frac{B(z) - \lambda}{z - t_k} \right|^2.
\]

In this talk we will present inequalities analogous to inequalities (2) and (4) for the class of rational functions considered by Li, Mohapatra and Rodriguez, that is, for rational functions \( r \in \mathcal{R}_n \) with all the poles \( a_1, a_2, \ldots, a_n \) in \( \mathbb{D}_+ \).
As stated earlier, Li, Mohapatra and Rodriguez [J. London Math. Soc. 5 (1995)] proved inequalities analogous to inequalities (1) and (3). Their results are

**Theorem E (Li, Mohapatra & Rodriguez)**  
*For a rational function* $r \in \mathcal{R}$ *with* $|a_\nu| > 1$, $1 \leq \nu \leq n$, *we have*

$$|r'(z)| \leq \|r\| |B'(z)|, \quad |z| \geq 1. \quad (122)$$

*The above result is best possible and the equality holds for* $r(z) = \lambda B(z)$, *where* $B(z)$ *is the Blashke product and* $\lambda \in \mathbb{C}$.

**Theorem F (Li, Mohapatra & Rodriguez)**  
*Let* $r \in \mathcal{R}$ *be a rational function with* $|a_\nu| > 1$, $1 \leq \nu \leq n$, *and having all its zeros in* $T \cup \mathbb{D}_+$. *Then for* $|z| \geq 1$, *we have*

$$|r'(z)| \leq \|r\| \frac{|B'(z)|}{2}, \quad |z| \geq 1. \quad (123)$$

*The above result is best possible and the equality holds for* $r(z) = \lambda B(z) + \mu$, $|\lambda| = |\mu|$.

Our first result that is presented below provides an inequality analogous to (2) for rational functions.

**THEOREM 20.7**  
*For a rational function* $r(z) \in \mathcal{R}_n$, *with* $|a_\nu| > 1$, $1 \leq \nu \leq n$, *we have*

$$|r(z)| \leq \|r\| |B(z)|, \quad |z| \geq 1. \quad (124)$$

*The above result is best possible and the equality holds for* $r(z) = \lambda B(z)$, *where* $\lambda \in \mathbb{C}$.

Our next result, provides an inequality analogous to (4) for rational functions, and is given by

**THEOREM 20.8**  
*Let* $r(z) \in \mathcal{R}_n$ *with* $|a_\nu| > 1$, $1 \leq \nu \leq n$. *If all the zeros of* $r(z)$ *lie in* $T \cup \mathbb{D}_+$, *then for* $|z| \geq 1$,

$$|r(z)| \leq \|r\| \left(\frac{|B(z)| + 1}{2}\right). \quad (125)$$

*The result is best possible and the equality holds for the rational function* $r(z) = \alpha B(z) + \beta$, $|\alpha| = |\beta|$.
21

It may be noted that inequalities (19.2) and (19.4) can be deduced from inequalities (19.7) and (19.8) respectively and before we go to the proofs of these theorems, we will show how inequality (19.4) can be deduced from (19.8). The deduction of the inequality (19.2) from (19.7) follows on the same lines.

By inequality (19.8), we have for $|z| \geq 1$

$$|r(z)| = \left| \frac{p(z)}{\prod_{\nu=1}^{n} (z - a_{\nu})} \right| \leq \|r\| \left( \frac{\prod_{\nu=1}^{n} \left| \frac{1 - \bar{a}_{\nu} z}{z - a_{\nu}} \right| + 1}{2} \right). \quad (126)$$

Let $z^{*}$ be a point on $|z| = 1$, such that $|r(z^{*})| = \|r\|$. Then (19.8) is equivalent to that for $|z| \geq 1$,

$$\left| \frac{p(z)}{\prod_{\nu=1}^{n} (z - a_{\nu})} \right| \leq \left| \frac{p(z^{*})}{\prod_{\nu=1}^{n} (z^{*} - a_{\nu})} \right| \left( \frac{\prod_{\nu=1}^{n} \left| \frac{1 - \bar{a}_{\nu} z}{z - a_{\nu}} \right| + 1}{2} \right). \quad (127)$$

If we multiply both the sides of (20.2) by $a_{1}a_{2} \cdots a_{n}$, we get

$$\left| \frac{a_{1}a_{2} \cdots a_{n} p(z)}{\prod_{\nu=1}^{n} (z - a_{\nu})} \right| \leq \left| \frac{a_{1}a_{2} \cdots a_{n} p(z^{*})}{\prod_{\nu=1}^{n} (z^{*} - a_{\nu})} \right| \left( \frac{\prod_{\nu=1}^{n} \left| \frac{1 - \bar{a}_{\nu} z}{z - a_{\nu}} \right| + 1}{2} \right), \quad (128)$$

which is equivalent to that for $|z| \geq 1$,

$$\left| \frac{p(z)}{\prod_{\nu=1}^{n} \left( \frac{z}{a_{\nu}} - 1 \right)} \right| \leq \left| \frac{p(z^{*})}{\prod_{\nu=1}^{n} \left( \frac{z^{*}}{a_{\nu}} - 1 \right)} \right| \left( \frac{\prod_{\nu=1}^{n} \left| \frac{1 - \bar{a}_{\nu} z}{z - a_{\nu}} \right| + 1}{2} \right).$$

On making each $a_{\nu} \to \infty$, we get that for $|z| \geq 1$,

$$|p(z)| \leq |p(z^{*})| \left( \frac{|z|^{n} + 1}{2} \right),$$

which clearly gives that for $|z| = R \geq 1$,

$$|p(z)| \leq \|p\| \left( \frac{R^{n} + 1}{2} \right),$$

and, which is the inequality (19.4).
22 Proof of Theorem 19.5.

Theorem 19.5 (Govil & Mohapatra, 1998)) For a rational function \( r \in \mathcal{R} \) with \( |a_\nu| > 1, 1 \leq \nu \leq n \), we have

\[
|r(z)| \leq \|r\| |B(z)|, \quad |z| \geq 1.
\]

The above result is best possible and the equality holds for \( r(z) = \lambda B(z) \), where \( B(z) \) is the Blaschke product and \( \lambda \in \mathbb{C} \).

The proof of Theorem 19.5 is fairly easy and follows from the following result which is an immediate consequence of the maximum modulus principle for unbounded domains.

Lemma 1 If \( f(z) \) is analytic in \( \{z \in \mathbb{C} : |z| \geq 1\} \) and \( f(z) \) tends to a finite limit as \( z \) tends to infinity, then \( |f(z)| \leq \|f\| \) for \( |z| \geq 1 \).

Proof of Theorem 19.5 Since \( r \in \mathcal{R}_n \), hence

\[
r(z) = \frac{p(z)}{w(z)} = \frac{\prod_{\nu=1}^{m} (z - z_\nu)}{\prod_{\nu=1}^{n} (z - a_\nu)}, \quad m \leq n
\]

and therefore

\[
\frac{r(z)}{B(z)} = \left\{ \frac{\prod_{\nu=1}^{m} (z - z_\nu)}{\prod_{\nu=1}^{n} (z - a_\nu)} \right\} / \left\{ \frac{\prod_{\nu=1}^{n} (1 - a_\nu z)}{\prod_{\nu=1}^{n} (z - a_\nu)} \right\}
\]

\[
= \frac{\prod_{\nu=1}^{m} (z - z_\nu)}{\prod_{\nu=1}^{n} (1 - \bar{a}_\nu z)}, \quad m \leq n.
\]

Since \( r(z) \) has all its poles in \( \mathbb{D}_+ \), we have \( |a_\nu| > 1 \) for \( 1 \leq \nu \leq n \) and hence the function \( r(z)/B(z) \) is analytic for \( |z| \geq 1 \). Also \( |B(z)| = 1 \) on \( |z| = 1 \), and therefore on \( |z| = 1 \),

\[
\left| \frac{r(z)}{B(z)} \right| = \frac{|r(z)|}{|B(z)|} \leq \|r\|.
\]

Further, since \( m \leq n \), we get, by Lemma 1

\[
\left| \frac{r(z)}{B(z)} \right| \leq \|r\| \text{ for } |z| \geq 1,
\]

from which Theorem 19.5 follows.
23 Proof of Theorem 19.6

Theorem 19.6 (N. K. Govil & R. N. Mohapatra, 1998) Let \( r(z) \in \mathcal{R}_n \) with \( |a_\nu| > 1, 1 \leq \nu \leq n \). If all the zeros of \( r(z) \) lie in \( \mathbb{T} \cup \mathbb{D}_+ \), then for \( |z| \geq 1 \),

\[
|r(z)| \leq \|r\| \left( \frac{|B(z)| + 1}{2} \right).
\]

The result is best possible and the equality holds for the rational function \( r(z) = \alpha B(z) + \beta \), \( |\alpha| = |\beta| \).

For the proof of Theorem 19.6, we will need

Lemma 1 If \( f(z) \) is analytic in \( \{ z \in \mathbb{C} : |z| \geq 1 \} \) and \( f(z) \) tends to a finite limit as \( z \) tends to infinity, then \( |f(z)| \leq \|f\| \) for \( |z| \geq 1 \).

Lemma 2 Let \( r \in \mathcal{R}_n \) with all its poles in \( \mathbb{D}_+ \). Then for \( |z| \geq 1 \),

\[
|r^*(z)| \leq \|r\| |B(z)|.
\]

Lemma 3 Let \( r \in \mathcal{R}_n \) with all its poles in \( \mathbb{D}_+ \). If \( r(z) \) has all its zeros in \( \mathbb{T} \cup \mathbb{D}_+ \), then for \( |z| \geq 1 \),

\[
|r(z)| \leq |r^*(z)|.
\]

Lemma 4 Let \( r \in \mathcal{R}_n \) with all its poles in \( \mathbb{D}_+ \). Then for \( |z| \geq 1 \),

\[
|r(z)| + |r^*(z)| \leq \|r\| (|B(z)| + 1).
\]

The proof of the Theorem 19.6 now follows immediately on combining Lemmas 3 and 4.

From Lemma 4, one can immediately obtain

Corollary Let \( r(z) = \frac{p(z)}{w(z)} \in \mathcal{R}_n \) with all its poles in \( \mathbb{D}_+ \). If \( r(z) \) is self-inversive, that is, \( r(z) \equiv \lambda r^*(z) \), for some \( \lambda \in \mathbb{T} \), then for \( |z| \geq 1 \),

\[
|r(z)| \leq \|r\| \left( \frac{|B(z)| + 1}{2} \right). \quad (129)
\]

The result is best possible with equality holding for the rational function \( r(z) = \alpha B(z) + \beta \), where \( \alpha \) and \( \beta \) are complex numbers with \( |\alpha| = |\beta| \).
24 Some New Results.

If $p(z)$ is a polynomial of degree $n$, then

$$M(p, R) \leq R^n \|p\|. \quad (130)$$

Further, if $p(z)$ has no zeros in $|z| < 1$, then

$$M(p, R) \leq \frac{(R^n + 1)}{2} \|p\|. \quad (131)$$

THEOREM 24.1 (N.K. Govil and R.N. Mohapatra, 1998) For a rational function $r(z) \in \mathcal{R}_n$, with $|a_\nu| > 1$, $1 \leq \nu \leq n$, we have

$$|r(z)| \leq \|r\| |B(z)|, \quad |z| \geq 1. \quad (132)$$

The above result is best possible and the equality holds for $r(z) = \lambda B(z)$, where $\lambda \in \mathbb{C}$.

THEOREM 24.2 (N.K. Govil and R.N. Mohapatra, 1998) Let $r(z) \in \mathcal{R}_n$ with $|a_\nu| > 1, 1 \leq \nu \leq n$. If all the zeros of $r(z)$ lie in $\mathbb{T} \cup \mathbb{D}_+$, then for $|z| \geq 1$,

$$|r(z)| \leq \|r\| \left( \frac{|B(z)| + 1}{2} \right). \quad (133)$$

The result is best possible and the equality holds for the rational function $r(z) = \alpha B(z) + \beta$, $|\alpha| = |\beta|$.

The following theorem provides a refinement of Theorem 23.1.

THEOREM 24.3 (N.K. Govil, 1999) For a rational function $r(z) \in \mathcal{R}_n$, with $|a_\nu| > 1, 1 \leq \nu \leq n$, we have

$$|r(z)| \leq \|r\| |B(z)| \left\{ 1 - \frac{(\|r\| - |r^*(0)|)(|z| - 1)}{(|r^*(0)| + |z| \|r\|)} \right\}, \quad |z| \geq 1. \quad (134)$$

The above result is best possible and the equality holds for $r(z) = \lambda B(z)$, where $\lambda \in \mathbb{C}$.

As refinement of Theorem 23.2 we can prove

THEOREM 24.4 (N.K. Govil, 1999) Let $r(z) \in \mathcal{R}_n$ with $|a_\nu| > 1, 1 \leq \nu \leq n$. If all the zeros of $r(z)$ lie in $\mathbb{T} \cup \mathbb{D}_+$, then for $|z| \geq 1$,

$$|r(z)| \leq \|r\| \left( \frac{|B(z)| + 1}{2} \right) - \min_{|z|=1} \left( \frac{|B(z)| - 1}{2} \right). \quad (135)$$

The result is best possible and the equality holds for the rational function $r(z) = \alpha B(z) + \beta$, $|\alpha| = |\beta|$.
For any polynomial $p(z)$ of degree $n$, we have

$$M(p, R) \leq R^n \|p\|. \quad (136)$$

Further, if $p(z)$ has no zeros in $|z| < 1$, then

$$M(p, R) \leq \frac{(R^n + 1)}{2} \|p\|. \quad (137)$$

**THEOREM 25.1** If $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$ is a polynomial of degree $n$, then for $R \geq 1$ we have

$$M(p, R) \leq R^n \{ 1 - \frac{(|p| - |a_n|)^2 (R^a - 1)}{|a_n|(|p| - |a_n|) + |p| |a_{n-1}| R + |p| (|p| - |a_n|) R^2} \} \|p\|. \quad (138)$$

The above inequality is best possible and the equality holds for $p(z) = \lambda z^n$, $\lambda$ being a complex number.

**THEOREM 25.2** If $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$ is a polynomial of degree $n \geq 2$, having no zeros in $|z| < K$, $K \geq 1$, then for $R \geq 1$,

$$M(p, R) \leq \left( \frac{R^n + K}{1 + K} \right) \|p\| - \left( \frac{R^n - 1}{1 + K} \right) \min_{|z|=K} |p(z)| - |a_1| \left( \frac{R^n - 1}{n - R^{n-2} - 1} \right) \quad (139)$$

if $n > 2$, and

$$M(p, R) \leq \left( \frac{R^n + K}{1 + K} \right) \|p\| - \left( \frac{R^n - 1}{1 + K} \right) \min_{|z|=K} |p(z)| - |a_1| \frac{(R-1)^n}{2} \quad (140)$$

if $n = 2$.

In particular on taking $K = 1$ in the above theorem, we get

**COROLLARY 25.1** If $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$ is a polynomial of degree $n \geq 2$, having no zeros in $|z| < 1$, then for $R \geq 1$,

$$M(p, R) \leq \left( \frac{R^n + 1}{2} \right) \|p\| - \left( \frac{R^n - 1}{2} \right) \min_{|z|=1} |p(z)| - |a_1| \frac{(R-1)^n}{2} \quad (141)$$

if $n > 2$, and

$$M(p, R) \leq \left( \frac{R^n + 1}{2} \right) \|p\| - \left( \frac{R^n - 1}{2} \right) \min_{|z|=1} |p(z)| - |a_1| \frac{(R-1)^n}{2} \quad (142)$$

if $n = 2$. Both the above inequalities are best possible for the class of polynomials satisfying $a_1 = 0$ and $\min_{|z|=1} |p(z)| = 0$, and each becomes equality for the polynomial $p(z) = \lambda + \mu z^n$, where $\lambda$ and $\mu$ are complex numbers with $|\lambda| = |\mu|$.
For polynomials of degree greater than 1, the above corollary clearly provides a refinement of the inequality (24.2). Also it is an improvement over the known result due to Aziz and Dawood [J. Approx. Theory, 1988] that if \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then

\[
M(p, R) \leq \left( \frac{R^n + 1}{2} \right) \|p\| - \left( \frac{R^n - 1}{2} \right) \min_{|z|=1} |p(z)|. \tag{143}
\]

If in (24.4) and (24.5) we divide both the sides by \( R^n \) and make \( R \to \infty \), we get

**COROLLARY 25.2** If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \geq 2 \), having no zeros in \( |z| < K, \ K \geq 1 \), then

\[
|a_n| + \frac{|a_1|}{n} \leq \frac{1}{1 + K} (\|p\| - \min_{|z|=K} |p(z)|). \tag{144}
\]

If we take \( K = 1 \) in the above corollary, it reduces to

**COROLLARY 25.3** If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \geq 2 \) having no zeros in \( |z| < 1 \), then

\[
|a_n| + \frac{|a_1|}{n} \leq \frac{1}{2} (\|p\| - \min_{|z|=1} |p(z)|). \tag{145}
\]

For the class of polynomials satisfying \( a_1 = 0 \) and \( \min_{|z|=1} |p(z)| = 0 \), the above inequality (24.10) is best possible and becomes equality for the polynomial \( p(z) = \lambda + \mu z^n \), where \( \lambda \) and \( \mu \) are complex numbers with \( |\lambda| = |\mu| \).
THEOREM 26.1 If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \), then for \( R \geq 1 \) we have

\[
M(p, R) \leq R^{n} \{1 - \frac{(\|p\| - |a_n|)^2 (R^2 - 1)}{|a_n| (\|p\| - |a_n|) + \|p\| |a_n| R + \|p\| (\|p\| - |a_n|) R^2} \|p\|. \tag{146}
\]

The above inequality is best possible and the equality holds for \( p(z) = \lambda z^{n} \), \( \lambda \) being a complex number.

For the proof of the above Theorem, we shall need the following lemmas.

LEMMA 26.1 (Govil, Rahman and Schmeisser, 1979) If \( f(z) \) is analytic and \( |f(z)| \leq 1 \) in \( |z| \leq 1 \), then for \( |z| < 1 \)

\[
|f(z)| \leq \frac{(1 - |a|)|z|^2 + |b| |z| + |a|(1 - |a|)}{|a|(1 - |a|)|z|^2 + |b| |z| + (1 - |a|)}, \tag{147}
\]

where \( a = f(0) \), \( b = f'(0) \). The example

\[
f(z) = \frac{a + \frac{b}{1 + a} z - z^2}{1 - \frac{b}{1 + a} z - a z^2}
\]

shows that the estimate is best possible.

If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \), then applying Lemma 25.1 to the function \( f(z) = p(z)/\|p\| \), one immediately gets

LEMMA 26.2 Let \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) be a polynomial of degree \( n \). Then for \( |z| \leq 1 \),

\[
|p(z)| \leq \left\{ \frac{\|p\||\|p\| - |a_0||z|^2 + \|p\||a_1||z| + |a_0||\|p\| - |a_0||}{|a_0||\|p\| - |a_0||z|^2 + \|p\||a_1||z| + \|p\||\|p\| - |a_0||} \right\} \|p\|. \tag{148}
\]

Proof of Theorem 25.1. Since \( R \geq 1 \) we have \( \frac{1}{R} \leq 1 \), and therefore on applying Lemma 25.2 to the polynomial \( q(z) = z^{n} p(\frac{1}{z}) \) on \( |z| = \frac{1}{R} \), we get

\[
M(q, \frac{1}{R}) \leq \left\{ \frac{\|q\||(q) - |a_n|\frac{1}{R^2} + \|q\||a_{n-1}|\frac{1}{R} + |a_n||(q) - |a_n||}{|a_n||(q) - |a_n||\frac{1}{R^2} + \|q\||a_{n-1}|\frac{1}{R} + \|q\||(q) - |a_n||} \right\} \|q\|

= \left\{ \frac{\|q||q|| - |a_n|| + \|q||a_{n-1}| R + |a_n||(q) - |a_n]|R^2}{|a_n||(q) - |a_n|| + \|q||a_{n-1}| R + \|q||(q) - |a_n||} \right\} \|q\|. \tag{149}
\]
Now noting that $M(q, \frac{1}{R}) = \frac{1}{R} M(p, R)$ and $\|q\| = \|p\|$, we get from (3.1) that for $R \geq 1$,

$$M(p, R) \leq R^n \left\{ \frac{\|p\| (\|p\| - |a_n|) + \|p\| |a_{n-1}| R + |a_n| (\|p\| - |a_n|) R^2}{|a_n| (\|p\| - |a_n|) + \|p\| |a_{n-1}| R + \|p\| (\|p\| - |a_n|) R^2} \right\} \|p\|. \tag{150}$$

Since, as is easy to verify that

$$1 - \frac{\|p\| (\|p\| - |a_n|) + \|p\| |a_{n-1}| R + |a_n| (\|p\| - |a_n|) R^2}{|a_n| (\|p\| - |a_n|) + \|p\| |a_{n-1}| R + \|p\| (\|p\| - |a_n|) R^2} = 1 - \left( \frac{|a_n|^2 (R^2 - 1)}{\|p\| (\|p\| - |a_n|) + \|p\| |a_{n-1}| R + \|p\| (\|p\| - |a_n|) R^2} \right)^2, \tag{151}$$

we now get Theorem 25.1, on combining (25.5) and (25.6).
THEOREM 27.1 If \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) is a polynomial of degree \( n \geq 2 \), having no zeros in \( |z| < K \), \( K \geq 1 \), then for \( R \geq 1 \),

\[
M(p, R) \leq \left( \frac{R^{n} + K}{1 + K} \right) ||p|| - \left( \frac{R^{n} - 1}{1 + K} \right) \min_{|z|=K} |p(z)| - |a_1| \left( \frac{R^{n} - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right)
\]

if \( n > 2 \), and

\[
M(p, R) \leq \left( \frac{R^{n} + K}{1 + K} \right) ||p|| - \left( \frac{R^{n} - 1}{1 + K} \right) \min_{|z|=K} |p(z)| - |a_1| \frac{(R - 1)^{n}}{2},
\]

if \( n = 2 \).

Proof of Theorem 26.1. We first consider the case when the polynomial \( p(z) \) is of degree \( n > 2 \). Since the polynomial \( p(z) \) has no zeros in \( |z| < K, K \geq 1 \), for every \( \alpha \) with \( |\alpha| < 1 \) the polynomial \( p(z) - \alpha \min_{|z|=K} |p(z)| \) has no zeros in \( |z| < K, K \geq 1 \) and therefore applying inequality [N. K. Govil, 1991]

\[
M(p, R) \leq \left( \frac{R^{n} + K}{1 + K} \right) ||p|| - \left( \frac{R^{n} - 1}{1 + K} \right) \min_{|z|=K} |p(z)| - |a_1| \left( \frac{R^{n} - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right)
\]

to the polynomial \( p(z) - \alpha \min_{|z|=K} |p(z)| \), we get

\[
\max_{|z|=R} |p(z) - \alpha \min_{|z|=K} |p(z)|| \leq \left( \frac{R^{n} + K}{1 + K} \right) ||p - \alpha \min_{|z|=K} |p(z)|| - |a_1| \left( \frac{R^{n} - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right).
\]

Let \( z_0 \) on \( |z| = 1 \), be such that

\[
||p - \alpha \min_{|z|=K} |p(z)|| = |p(z_0) - \alpha \min_{|z|=K} |p(z)||,
\]

and let \( z_1 \) on \( |z| = R, (R \geq 1) \) be such that \( |p(z_1)| = \max_{|z|=R} |p(z)| \). Then the inequalities (26.4) and (26.5) together imply

\[
\max_{|z|=R} |p(z) - \alpha \min_{|z|=K} |p(z)|| \leq \left( \frac{R^{n} + K}{1 + K} \right) |p(z_0) - \alpha \min_{|z|=K} |p(z)|| - |a_1| \left( \frac{R^{n} - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right),
\]

for every \( \alpha \) with \( |\alpha| < 1 \).
which in particular gives that

\[ |p(z_1) - \alpha \min_{|z|=K} |p(z)|| \leq \left( \frac{R^n + K}{1 + K} \right) |p(z_0) - \alpha \min_{|z|=K} |p(z)|| \]

\[ - |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right). \]  

(158)

If we choose \( \arg \alpha \), so that

\[ |p(z_0) - \alpha \min_{|z|=K} |p(z)|| = |p(z_0)| - |\alpha| \min_{|z|=K} |p(z)|, \]

we get from (26.7) that

\[ |p(z_1)| - |\alpha| \min_{|z|=K} |p(z)| \leq \left( \frac{R^n + K}{1 + K} \right) (|p(z_0)| - |\alpha| \min_{|z|=K} |p(z)|) \]

\[ - |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right). \]  

(159)

The fact that the quantity \(|p(z_0)| - |\alpha| \min_{|z|=K} |p(z)||\) in the right hand side of (26.8) is positive, is obvious. The inequality (26.8) clearly gives

\[ |p(z_1)| - |\alpha| \min_{|z|=K} |p(z)| \leq \left( \frac{R^n + K}{1 + K} \right) |p(z_0)| - \left( \frac{R^n + K}{1 + K} \right) |\alpha| \min_{|z|=K} |p(z)| \]

\[ - |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right), \]  

(160)

which implies

\[ |p(z_1)| - |\alpha| \min_{|z|=K} |p(z)| \leq \left( \frac{R^n + K}{1 + K} \right) \|p\| - \left( \frac{R^n + K}{1 + K} \right) |\alpha| \min_{|z|=K} |p(z)| \]

\[ - |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right), \]  

(161)

and which is clearly equivalent to

\[ |p(z_1)| \leq \left( \frac{R^n + K}{1 + K} \right) \|p\| - \left( \frac{R^n - 1}{1 + K} \right) |\alpha| \min_{|z|=K} |p(z)| \]

\[ - |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right). \]  

(162)

Since \( z_1 \) was chosen so that \( |p(z_1)| = \max_{|z|=R} |p(z)| \), the inequality (26.11) gives

\[ \max_{|z|=R \geq 1} |p(z)| \leq \left( \frac{R^n + K}{1 + K} \right) \|p\| - \left( \frac{R^n - 1}{1 + K} \right) |\alpha| \min_{|z|=K} |p(z)| \]

\[ - |a_1| \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right). \]  

(163)

If in the above inequality we now make \( |\alpha| \to 1 \), we get (26.1). The proof of the inequality (26.2) is similar. The proof of Theorem 2 is thus complete.
28 Location of the Zeros of Polynomials

Let \( p(z) = \sum_{v=0}^{n} a_v z^v \) be a polynomial of degree \( n \). By Fundamental Theorem of Algebra, \( p(z) \) has exactly \( n \) zeros in the complex plane. But this theorem does not say anything regarding the location of the zeros of the polynomial, i.e., the region which contains some or all of the zeros of the polynomial. The problems of this kind were first studied by Gauss and Cauchy.

**THEOREM 28.1 (Gauss, 1799.)** Let \( p(z) = z^n + A z^{n-1} + \cdots + A_{n-1} z + A_n \) be a polynomial with real coefficients. Then \( p(z) \) has all its zeros in \( |z| \leq R \) where \( R = \max(1, S \sqrt{2}) \) where \( S \) is the sum of positive \( A_k \)'s.

In 1816 he showed that \( R = \max_{1 \leq k \leq n}(n \sqrt{2} |A_k|)^{\frac{1}{n}} \) while in 1849, he gave a bound in case of arbitrary real or complex \( A_k \)'s.

Cauchy obtained more exact bounds for the moduli of the zeros of a polynomial than those given by Gauss.

**THEOREM 28.2 (Cauchy, 1829)** All the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \) lie in

\[
|z| \leq 1 + A, \tag{164}
\]

where \( A = \max_{0 \leq j \leq n-1} |a_j| \).

**THEOREM 28.3 (B. Datt and N. K. Govil, Jour. Approx. Theory 1988.)** All the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \), where \( a_v \) may be complex, lie in

\[
|z| \leq 1 + (1 - \frac{1}{(1 + A)^n}) A. \tag{165}
\]

Here \( A = \max_{0 \leq j \leq n-1} |a_j| \) is as in Theorem 17.2.

Theorem 28.3 clearly sharpens Theorem 28.2 due to Cauchy, because

\[
(1 - \frac{1}{(1 + A)^n}) < 1.
\]

**THEOREM 28.4 (Eneström-Kakeya)** If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying

\[
0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n,
\]

then \( p(z) \) has all its zeros in \( |z| \leq 1 \).
THEOREM 29.1 (Eneström-Kakeya) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying
\[
0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n,
\]
then \( p(z) \) has all its zeros in \( |z| \leq 1 \).

Joyal, Labelle and Rahman dropped the hypothesis that coefficients be all positive and proved

THEOREM 29.2 (Joyal, Labelle and Rahman, Canad. Math. Bull. 1967) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying
\[
a_0 \leq a_n \leq a_2 \leq \cdots \leq a_n,
\]
then \( p(z) \) has all its zeros in
\[
|z| \leq \frac{a_n - a_0 + a_0}{a_n} \tag{166}
\]

THEOREM 29.3 (K. K. Dewan and N. K. Govil, Jour. Approx. Theory, 1984) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients such that
\[
a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq a_n
\]
then \( p(z) \) has all its zeros in the annulus
\[
R_1 \leq |z| \leq R_2
\]
where \( R_1 \) and \( R_2 \) are constants depending on the coefficients \( a_0, a_1, a_{n-1} \) and \( a_n \). Moreover
\[
0 \leq R_1 \leq 1 \leq R_2 \leq \frac{a_n - a_0 + |a_0|}{|a_n|}. \tag{167}
\]

THEOREM 29.4 (N. K. Govil and V. K. Jain, Jour. Approx. Theory, 1978) Let \( p(z) = \sum_{v=0}^{n} a_v z^v (\neq 0) \) be a polynomial with complex coefficients such that
\[
|\text{arg} a_v - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad v = 0, 1, 2, \ldots, n.
\]
for some real \( \beta \), and
\[
|a_0| \leq |a_1| \leq |a_2| \leq \cdots \leq |a_n|
\]
then \( p(z) \) has all its zeros in \( R_1 \leq |z| \leq R_2 \), where \( R_1 \) and \( R_2 \) are constants depending on \( a_0, a_{n-1}, a_n \) and \( \alpha \).
If instead of having information about the moduli of the coefficients we have the information about their real and imaginary parts, the following results could be of interest.

**THEOREM 30.1** Let \( p(z) = \sum_{v=0}^{n} a_v z^v, a_n \neq 0, \text{Re} \ a_j = \alpha_j, \text{Im} \ a_j = \beta_j \) for \( j = 0, 1, 2, \ldots, n. \) If
\[
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \text{ and } \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.
\]

In particular if all the coefficients are real, it gives an improvement of the result of Joyal, Labelle and Rahman. Further if the coefficients are as well positive it gives the Eneström-Kakeya Theorem.

**THEOREM 30.2** If
\[
\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \text{ and } \beta_0 \geq \beta_1 \geq \cdots \geq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)}{|a_n|}.
\]

**THEOREM 30.3** If
\[
\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \text{ and } \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| + \alpha_0 - \beta_0 - \alpha_n + \beta_n} \leq |z| \leq \frac{|a_0| + \alpha_0 - \beta_0 - \alpha_n + \beta_n}{|a_n|}.
\]

**THEOREM 30.4** If
\[
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \text{ and } \beta_0 \geq \beta_1 \geq \cdots \geq \beta_n,
\]
then \( p(z) \) has all itz zeros in
\[
\frac{|a_0|}{|a_n| - \alpha_0 + \beta_0 + \alpha_n - \beta_n} \leq |z| \leq \frac{|a_0| - \alpha_0 + \beta_0 + \alpha_n - \beta_n}{|a_n|}.
\]

All the above four results are in fact corollaries of a more general theorem due to Gardner & Govil (the paper to appear in the *Journal of Approximation Theory*).