SOME EASY LOOKING PROBLEMS IN COMPLEX ANALYSIS AND APPROXIMATION THEORY

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Research Interests: Complex Variables, and Approximation Theory
What is Complex Analysis?
The Comlex Analysis is the study of functions of complex variable, that is, it is the study of functions $f : \mathbb{C} \rightarrow \mathbb{C}$.

What is Approximation Theory?
The study of Approximation Theory dates back to the famous result due to Weirstrass, called Weirstrass Approximation Theorem, although formally this term began in early 1960’s, when Oved Shisha started publishing the journal, the Journal of Approximation Theory.

THEOREM 0.1 (Weierstrass Approx. Theorem)
Let $f$ be a real function of a real variable defined on a compact set. Then $f$ can be approximated uniformly by a sequence of polynomials.

1. Polynomials
2. Inequalities for Polynomials, and Related Classes of Functions
3. Splines
4. Pade Approximations
5. Interpolation
6. Series Expansions
7. Special Functions
In view of growing dependence on computers, Approximation Theory has become a very important branch of Mathematics having applications in Computations, Numerical Analysis, Computer Aided Designs, Mathematical Physics, and many other areas.

Apart from its applications, Approximation Theory is a lively branch of Mathematical Analysis, because the topics of Approximation Theory arise frequently in such theoretical areas as Functional Analysis, Harmonic Analysis, and Function Theory.
By a **real polynomial** we will mean an expression of the form \( \sum_{\nu=0}^{n} c_{\nu} x^{\nu} \), \( c_{\nu} \) being real and \( x \) a real variable, and by a **complex polynomial** an expression of the form \( \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \), where \( a_{\nu} \) are complex and \( z \) a complex variable.

Polynomials have played a central role in Approximation Theory and Numerical Analysis for many years. To indicate why this is the case, we note that

1. The set of all polynomials of degree at most \( n \) is a finite dimensional linear space with a convenient basis.
2. Polynomials are smooth functions.
3. Polynomials are easy to store, manipulate, and evaluate on a computer.
4. The derivative and antiderivative of a polynomial are again polynomials whose coefficients can be found algebraically (even by a computer).
5. The number of zeros of a polynomial of degree \( n \) is precisely \( n \).
6. The sign structure and shape of a polynomial are intimately related to the sign structure of its set of coefficients.
7. Given any continuous function on an interval \([a, b]\), there exists a polynomial that is uniformly close to it.
8. Precise rates of convergence can be given for approximation of smooth functions by polynomials.
By a real polynomial we will mean an expression of the form \( \sum_{\nu=0}^{n} c_{\nu}x^{\nu} \), \( c_{\nu} \) being real and \( x \) a real variable, and by a complex polynomial an expression of the form \( \sum_{\nu=0}^{n} a_{\nu}z^{\nu} \), where \( a_{\nu} \) are complex and \( z \) a complex variable.

Although polynomials are among the nicest type of functions, but still there are many type of problems associated with polynomials and here we would be discussing mainly the following type of problems.

1. If a polynomial has zeros and if so where the zeros lie, that is, the problems concerning the **Location of the Zeros of a Polynomial**, **Eneström-Kakeya Theorem**, and Sandov’s conjecture.

2. How fast the maximum modulus of a polynomial can grow, that is, the problems concerning the **Growth of a Polynomial**.

3. Relationship between the growth of the derivative of a polynomial in terms of the growth of the polynomial, known as **Extremal Problems for Polynomials**.

4. We can consider all these problems for **larger classes of functions** that includes polynomials, like **Rational Functions**, **Splines**, and **Entire Functions of Exponential Type**.
1 Location of the Zeros of Polynomials

THEOREM 1.1 (Fundamental Theorem of Algebra) A polynomial \( p(z) = \sum_{v=0}^{n} a_v z^v \) of degree \( n \) has exactly \( n \) zeros. The zeros may however be coincident.

THEOREM 1.2 (Gauss, 1799) Let \( p(z) = z^n + A_1 z^{n-1} + \cdots + A_{n-1} z + A_n \) be a polynomial with real coefficients. Then \( p(z) \) has all its zeros in \( |z| \leq R \) where \( R = \max(1, S \sqrt{2}) \) where \( S \) is the sum of \( |A_k| \)'s.

In 1816 he showed that \( R = \max_{1 \leq k \leq n} (n \sqrt{2} |A_k|)^{1/k} \) whereas in 1849, he gave a bound for polynomials with arbitrary real or complex \( A_k \)'s, and showed that \( R \) may be taken as the positive root of the equation \( z^n - 2^{1/2}(|A_1| z^{n-1} + \cdots + |A_n|) = 0 \). As a further indication of Gauss’ interest in the location of zeros of a polynomial, we have his letter to Schumacher in which he tells of having written enough on this topic to fill several volumes, but only results he published are those in his paper of 1850. Cauchy obtained more exact bounds for the zeros of a polynomial than those given by Gauss.

THEOREM 1.3 (Cauchy, 1829) All the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \) lie in

\[
|z| \leq 1 + A, \tag{1}
\]

where \( A = \max_{0 \leq j \leq n-1} |a_j| \).
THEOREM 1.4 (Cauchy, 1829) All the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \) lie in

\[ |z| \leq 1 + A, \]  

(2)

where \( A = \max_{0 \leq j \leq n-1} |a_j| \).

THEOREM 1.5 (Joyal, Labelle, and Rahman, Canadian Math. Bull. 1967) All the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \) lie in the disk

\[ |z| \leq \frac{1}{2} \left\{ 1 + |a_{n-1}| + \left[ (1 - |a_{n-1}|)^2 + 4\beta \right]^{1/2} \right\}, \]  

(3)

where \( \beta = \max_{0 \leq j \leq n-2} |a_j| \).

Since \( \beta \leq A \), and \( |a_{n-1}| \leq A \), in general the Theorem 1.5 sharpens Theorem 1.4.

However, if \( \beta = |a_{n-1}| \), then Theorem 1.5 fails to give an improvement over Theorem 1.4. The following theorem gives an improvement even in that situation.

THEOREM 1.6 (Datt and Govil, Jour. Approx. Theory 1978.) All the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \), where \( a_v \) may be complex, lie in

\[ |z| \leq 1 + \frac{1}{(1 + A)^n} A. \]  

(4)

Since, always we have \( (1 - \frac{1}{(1+A)^n}) < 1 \), Theorem 1.6 in every situation sharpens Theorem 1.4 due to Cauchy.
THEOREM 1.7 (Cauchy, 1829) All the zeros of the polynomial
\[ p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \] lie in the disk \( \{ z : |z| < 1 + A \} \), where \( A = \max_{0 \leq j \leq n-1} |a_j| \).

One has to assume that \( a_i \neq 0 \), for at least one \( i \in I = \{0, 1, 2, \cdots, n-1\} \), because if \( a_i = 0 \), for all \( i \in I = \{0, 1, 2, \cdots, n-1\} \), then the equation does not have any positive root, and so the above theorem is trivial, because in that case all the zeros lie at the origin.


THEOREM 1.8 (Sun and Hsieh, 1996) All the zeros of the polynomial
\[ p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \] lie in the disk \( \{ z : |z| < 1 + \delta_1 \} \subset \{ z : |z| < 1 + A \} \), where \( \delta_1 \) is the unique positive root of the equation
\[ z^3 + (2 - |a_{n-1}|)z^2 + (1 - |a_{n-1}| - |a_{n-2}|)z - A = 0. \]
Here \( A = \max_{0 \leq j \leq n-1} |a_j| \) is as above in Theorem 1.7.

THEOREM 1.9 (V.K. Jain, 2006) All the zeros of the polynomial \( p(z) = z^n + \sum_{v=0}^{n-1} a_v z^v \) lie in the disk
\( \{ z : |z| < 1 + \delta_0 \} \subset \{ z : |z| < 1 + \delta_1 \} \subset \{ z : |z| < 1 + A \} \),
where \( \delta_1 \), and \( A \) are as in the above Theorem, and \( \delta_0 \) is the unique positive root of the equation
\[ z^4 + (3 - |a_{n-1}|)z^3 + (3 - 2|a_{n-1}| - |a_{n-2}|)z^2 + (1 - |a_{n-1}| - |a_{n-2}| - |a_{n-3}|)z - A = 0. \]
2 Eneström-Kakeya Theorem

THEOREM 2.1 (Eneström-Kakeya) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying
\[
0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n,
\]
then \( p(z) \) has all its zeros in \( |z| \leq 1 \).

Joyal, Labelle and Rahman dropped the hypothesis that coefficients be all positive and proved

THEOREM 2.2 (Joyal, Labelle and Rahman, Canad. Math. Bull. 1967) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying
\[
a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n,
\]
then \( p(z) \) has all its zeros in
\[
|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.
\]  

THEOREM 2.3 (Dewan and Govil, Jour. Approx. Theory, 1983) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients such that
\[
a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq a_n
\]
then \( p(z) \) has all its zeros in the annulus
\[
R_1 \leq |z| \leq R_2
\]
where \( R_1 \) and \( R_2 \) are constants depending on the coefficients \( a_0, a_1, a_{n-1} \) and \( a_n \). Moreover
\[
0 \leq R_1 \leq 1 \leq R_2 \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.
\]  

Obviously, we would like to generalize the above results when the coefficients \( a_v \) of the polynomial are complex.
**THEOREM 2.4** Let \( p(z) = \sum_{v=0}^{n} a_v z^v \), \( a_n \neq 0 \), \( Re \ a_j = \alpha_j, Im \ a_j = \beta_j \) for \( j = 0, 1, 2, \ldots, n \). If 
\[
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \text{ and } \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.
\]

In particular, if all the coefficients are real, it gives an improvement of the result of Joyal, Labelle and Rahman. Further if the coefficients are as well positive it gives the Eneström-Kakeya Theorem.

**THEOREM 2.5** If 
\[
\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \text{ and } \beta_0 \geq \beta_1 \geq \cdots \geq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)}{|a_n|}.
\]

**THEOREM 2.6** If 
\[
\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \text{ and } \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| + \alpha_0 - \beta_0 - \alpha_n + \beta_n} \leq |z| \leq \frac{|a_0| + \alpha_0 - \beta_0 - \alpha_n + \beta_n}{|a_n|}.
\]

**THEOREM 2.7** If 
\[
\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \text{ and } \beta_0 \geq \beta_1 \geq \cdots \geq \beta_n,
\]
then \( p(z) \) has all its zeros in
\[
\frac{|a_0|}{|a_n| - \alpha_0 + \beta_0 + \alpha_n - \beta_n} \leq |z| \leq \frac{|a_0| - \alpha_0 + \beta_0 + \alpha_n - \beta_n}{|a_n|}.
\]
All the above four results are in fact corollaries of a more general theorem due to Gardner & Govil, Journal of Approximation Theory 78 (1994), 286-292.
Let \( p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) be a polynomial of degree \( n \), and let \( \Re a_{\nu} = \alpha_{\nu} \), and \( \Im a_{\nu} = \beta_{\nu} \). If we have the information only about the real or imaginary parts of the coefficients, we can use the following results.

**THEOREM 2.8** If \( \alpha_{0} \leq \alpha_{1} \leq \cdots \leq \alpha_{n} \) then all the zeros of \( p(z) \) lie in \( R_{1} \leq |z| \leq R_{2} \)

where \( R_{1} = |a_{0}|/\left(\alpha_{n} - \alpha_{0} + |a_{n}| + |\beta_{0}| + |\beta_{n}| + 2 \sum_{j=1}^{n-1} |\beta_{j}|\right) \)

and \( R_{2} = \left(|a_{0}| - \alpha_{0} + \alpha_{n} + |\beta_{0}| + |\beta_{n}| + 2 \sum_{j=1}^{n-1} |\beta_{j}|\right)/|a_{n}|. \)

**THEOREM 2.9** If \( \alpha_{0} \geq \alpha_{1} \geq \cdots \geq \alpha_{n} \) then all the zeros of \( p(z) \) lie in \( R_{1} \leq |z| \leq R_{2} \)

where \( R_{1} = |a_{0}|/\left(\alpha_{0} - \alpha_{n} + |a_{n}| + |\beta_{0}| + |\beta_{n}| + 2 \sum_{j=1}^{n-1} |\beta_{j}|\right) \)

and \( R_{2} = \left(|a_{0}| + \alpha_{0} - \alpha_{n} + |\beta_{0}| + |\beta_{n}| + 2 \sum_{j=1}^{n-1} |\beta_{j}|\right)/|a_{n}|. \)

**THEOREM 2.10** If \( \beta_{0} \leq \beta_{1} \leq \cdots \leq \beta_{n} \) then all the zeros of \( p(z) \) lie in \( R_{1} \leq |z| \leq R_{2} \)

where \( R_{1} = |a_{0}|/\left(\beta_{n} - \beta_{0} + |a_{n}| + |\alpha_{0}| + |\alpha_{n}| + 2 \sum_{j=1}^{n-1} |\alpha_{j}|\right) \)

and \( R_{2} = \left(\beta_{n} - \beta_{0} + |a_{0}| + |\alpha_{0}| + |\alpha_{n}| + 2 \sum_{j=1}^{n-1} |\alpha_{j}|\right)/|a_{n}|. \)

**THEOREM 2.11** If \( \beta_{0} \geq \beta_{1} \geq \cdots \geq \beta_{n} \) then all the zeros of \( p(z) \) lie in \( R_{1} \leq |z| \leq R_{2} \)

where \( R_{1} = |a_{0}|/\left(\beta_{0} - \beta_{n} + |a_{0}| + |\alpha_{0}| + |\alpha_{n}| + 2 \sum_{j=1}^{n-1} |\alpha_{j}|\right) \) and

\[ R_{2} = \left(\beta_{0} - \beta_{n} + |a_{0}| + |\alpha_{0}| + |\alpha_{n}| + 2 \sum_{j=1}^{n-1} |\alpha_{j}|\right)/|a_{n}|. \)
All the above results are the corollaries of a more general result due to Gardner and Govil [Acta Mathematica Hungarica, 1996].
3 Some Recent Extensions of Eneström Kakeya Theorem.

THEOREM 3.1 (Eneström-Kakeya) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying

\[
0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n,
\]

then \( p(z) \) has all its zeros in \( |z| \leq 1 \).

Joyal, Labelle and Rahman dropped the hypothesis that coefficients be all positive and proved

THEOREM 3.2 (Joyal, Labelle and Rahman, Canad. Math. Bull. 1967) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) with real coefficients satisfying

\[
a_0 \leq a_n \leq a_2 \leq \cdots \leq a_n,
\]

then \( p(z) \) has all its zeros in \( |z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|} \).

THEOREM 3.3 (Aziz and Zarga, 1996) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) such that for some \( K \geq 1 \),

\[
K a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0,
\]

then all zeros of \( p(z) \) lie in

\[
|z + (K - 1)| \leq \frac{K a_n + |a_0| - a_0}{|a_n|}.
\]

Now what about when the coefficients of the polynomial \( p(z) = \sum_{v=0}^{n} a_v z^v \) satisfy the condition \( K_1 a_n \geq K_2 a_{n-1} \geq a_{n-2} \geq a_{n-3} \geq \cdots \geq a_1 \geq a_0 \), where \( K_1, K_2 \geq 1 \) are constants satisfying \( K_1 \geq 1, K_2 \), and surprisingly the answer to this problem is not easy to obtain, and the problem is open.
Theorem 4.3. Let \( p(z) \) be a polynomial of degree \( n \) with \( \text{Re}(a_v) = \alpha_v \) and \( \text{Im}(a_v) = \beta_v \). If for some \( K \geq 1 \)

\[
K\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,
\]

then \( p(z) \) has all its zeros in

\[
|z + (K - 1)| \leq \frac{K\alpha_n - \alpha_0 + |\alpha_0| + 2 \sum_{v=0}^{n} |\beta_v|}{|\alpha_n|}.
\]

Theorem 4.4. Let \( p(z) \) be a polynomial of degree \( n \) with \( \text{Re}(a_v) = \alpha_v \) and \( \text{Im}(a_v) = \beta_v \). If for some \( K \geq 1 \)

\[
K\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0,
\]

then \( p(z) \) has all its zeros in

\[
|z + (K - 1)| \leq \frac{K\beta_n - \beta_0 + |\beta_0| + 2 \sum_{v=0}^{n} |\alpha_v|}{|\beta_n|}.
\]
4  The Conjecture of Sendov.

Blagovest Sendov, a Bulgarian mathematician has enriched Approximation Theory by numerous important results, he is best known for his work on Hausdorff metric, positive and monotonic operators, splines, segment analysis, Whitney constants and several other topics. Although amongst his more than 150 publications one can hardly find any result on zeros and critical points of polynomials, his name has become famous in the Analytic Theory of Polynomials since it has been Sendov to whom this discipline owes its most challenging conjecture, which has resisted a proof for almost fifty years.

**CONJECTURE 4.1 (Sendov)** Let \( p(z) = \prod_{v=0}^{n} (z - z_v) \) be a polynomial of degree \( n \geq 2 \) with all its zeros in the closed unit disk. Then each of the disks \( \{ z : |z - z_v| \leq 1 \} \) contains at least one critical point, that is at least one zero of the derivative \( p'(z) \).

For a long time, the origins of Sendov’s conjecture remained in obscurity. Neither a date for the statement, nor a motivation and not even the true author were known.

According to the several testimonies, it now appears that Sendov made his conjecture already in 1959 to Obreschkoff (who ignored in his book), in 1962 at the International Congress of mathematicians in Stockholm to Marden, and independently to other mathematicians including Illief. In 1965, at the conference on Complex Analysis and Applications in Yerevan (Armenia), Illief spoke about this conjecture at an informal meeting between participants. This way Walter K. Hayman learnt about the problem, but he misunderstood its origin. In 1967, when Hayman published his book *Open problems in Complex Analysis*, he included Sendov’s problem as a conjecture of Illief.

**Since then this conjecture has been known to the public, but for than 10 years it was erroneously attributed to Illief.**
Often a conjecture arises from a missing step in an approach towards a desired result, and so when some mathematicians met Sendov and tried to learn about the possible backgrounds of Sendov’s conjecture, it turned out that there was nothing that was a motivation, and it seems that Sendov just wanted to puzzle the mathematical community with something that might look quite easy at a first glance.

In fact, except for the trivial case when $z_\nu = 0$, the disks $\{ z : |z - z_\nu| \leq 1 \}$ contain lot of points that cannot be critical points of the polynomial $p$. As a consequence of Gauss-Gauss-Lucas Theorem, Sendov’s conjecture would imply that already each of the lens-shaped domains

$$\{ z : |z - z_\nu| \leq 1 \} \cap \{ z : |z| \leq 1 \} \quad (\nu = 1, 2, \ldots, n)$$

contains a critical point. Several non-trivial refinements of Sendov’s conjecture have been proposed, and some disproved with the help of computational methods. The most promising refinement is due to Phelps and Rodriguez.

**CONJECTURE 4.2 (Sendov’s conjecture, Stengthened Form)**

Let $p(z) = \prod_{\nu=0}^{n} (z - z_\nu)$ be a polynomial of degree $n \geq 2$ with all its zeros in the closed unit disk. Then each of the open disks $\{ z : |z - z_\nu| < 1 \}$, for $1 \leq \nu \leq n$ contains at least one critical point of the polynomial $p(z)$, unless $p(z) = z^n - c$, $|c| = 1$. 
The Classes for Which the Conjectures Were Proved

6.1 Polynomials of Small Degree: A natural way of approaching a conjecture on polynomials is to try a verification for small degrees. While Sendov’s conjecture is trivial for polynomials of degree $n = 2$, it is not so obvious for $n = 3$.

In 1968, Branan presented a proof for $n = 3$, and in the same year Rubinstein verified the conjecture for $3 \leq n \leq 4$.

In 1969, Joyal, and Schmeisser obtained the stronger form of the conjecture for $3 \leq n \leq 4$, and in the same year Meir and Sharma verified Sendov’s conjecture for $n = 5$.

In 1971, Gacs extended the stronger conclusion by Schmeisser to $n = 5$.

The case $n = 6$ had to wait for more than twenty years till Brown (P.A.M.S. 1991) made some progress. In the meantime, the rumors had been spread, saying that in 1986, the conjecture was proved for all degrees $n$. This had blocked further research for some years.

In 1992, Katsoprinakis (Bull. London Math. Soc. 1992) published a proof for $n = 6$, but he used a lemma that was incorrectly stated in a book, and so his proof contained a gap. In 1996, Borcea, and Katsoprinakis who filled the gap in his former proof, both gave correct proofs for $n = 6$.

In the same year, Borcea (Analysis, 1996) obtained a proof for $n = 7$, and in 1999, Brown and Xiang settled the case for $n = 8$. 
This seems to be all that one knows, and in all the cited cases, it turns out that Sendov’s conjecture holds in the strengthened form of Phelps-Rodriguez. We may summarize these results in the following theorem.

THEOREM 5.1 Let \( p(z) = \prod_{v=0}^{n} (z - z_v) \) be a polynomial of degree \( 2 \leq n \leq 8 \) with all its zeros in the closed unit disk. Then each of the open disks \( \{ z : |z - z_v| < 1 \} \) contains at least one critical point, that is at least one zero of the derivative \( p'(z) \), unless \( p(z) = z^n - c, \ |c| = 1 \).

In fact, for \( 2 \leq n \leq 5 \), each of the disks

\[ \{ z : |z - z_v| < 1 \} \]

can be replaced by the largest open disk contained in the lens-shaped domain. Hopes that this statement may extend to to \( n > 5 \) were destroyed by Miller (Trans. A.M.S. 1990). Employing computational methods, he constructed polynomials of degrees 6, 8, 10, and 12, for which the above result does not hold. Kumar and Shenoy (1991) added counterexamples for the degrees 7, 9, and 11.
6.2 Polynomials with Real Zeros: For polynomials with real zeros, the location of the critical points can be well described with the help of Rolle’s theorem. It is not difficult to verify the Sendov’s conjecture for such polynomials, and Phelps and Rodriguez did it. In fact, one can easily obtain the following refined statement.

**THEOREM 5.2** Let \( p(z) = \prod_{v=0}^{n} (z - z_v) \) be a polynomial of degree \( n \geq 2 \) with real zeros ordered as \(-1 \leq x_1 \leq \ldots \leq x_n \leq 1\). Then each of the intervals

\[
\left[ x_1, x_1 + \frac{2}{n} \right], \left[ x_n - \frac{2}{n}, x_n \right], \text{ and } \left[ -1 + x_v + |x_v|, 1 + x_v - |x_v| \right],
\]

for \( 2 \leq \nu \leq (n - 1) \), contains a critical point of \( p(z) \).

6.3 Polynomials with Real Coefficients: The class of polynomials with real coefficients covers the monic polynomials with only real zeros. So far, no one has succeeded in verifying Sendov’s conjecture for polynomials with real coefficients. However for the subclass of the so-called Cauchy polynomials, the conjecture was proved in the strengthened form of Phelps and Rodriguez.

**THEOREM 5.3** (Analytic Theory of Polynomials, p.408) Let \( p(z) = \prod_{v=0}^{n} (z - z_v) = z^n - \sum_{\nu=0}^{n-1} a_{\nu} z^\nu \quad (n \geq 2) \) have all its zeros in the closed unit disk, and suppose that \( a_0, \ldots, a_{n-1} \) are all nonnegative. Then each of the open disks \( \{ z : |z - z_\nu| < 1 \} \), for \( 1 \leq \nu \leq n \) contains a critical point of \( p \), unless \( p(z) = z^n - 1 \).
6.4 Polynomials Having Zeros on a Circle: If in the situation of Sendov’s conjecture, the polynomial $p$ has a zeros on the unit circle then the statement on the location of a critical point relative to this zeros not only holds but can be refined, and this was done by Rubinstein.

**THEOREM 5.4** (Rubinstein, Pacific J. Math. 1968) Let 
\[ p(z) = \prod_{v=0}^{n} (z - z_v) \]
be a polynomial of degree $n \geq 2$ having all its zeros in the closed unit disk. Suppose that $|z_1| = 1$. Then $p$ has a critical point in the open disk \( \{z : |z - z_1| < 1\} \), unless \( p(z) = z^n - z_1^n \).

As an immediate consequence of the above theorem, one gets

**THEOREM 5.5** Let 
\[ p(z) = \prod_{v=0}^{n} (z - z_v) \]
be a polynomial of degree $n \geq 2$ having all its zeros on the unit circle. Then for each $\nu \in \{1, \ldots, n\}$, $p$ has a critical point in the open disk \( \{z : |z - z_\nu| < 1\} \), unless all the critical points lie on the unit circle. In fact one can prove that each disk \( \{z : |z - z_\nu/2| < 1/2\} \) contains a critical point.

6.5 Polynomials Having a Zero at the Origin: It is curious phenomenon that Sendov’s conjecture becomes an easy problem as soon as the polynomial has a zero at the origin, while its other zeros may be arbitrarily located in the unit disk. This was first observed by Schmeisser [Math. Z. 1969]. An alternative proof of this was given by Gacs [JMAA, 1971].

**THEOREM 5.6** Let 
\[ p(z) = \prod_{v=0}^{n} (z - z_v) \]
be a polynomial of degree $n \geq 2$ having all its zeros in the closed unit disk, and suppose that \( f(0) = 0 \). Then each of the open disks \( \{z : |z - z_\nu| < 1\} \), for \( \nu = 1, \ldots, n \) contains at least one critical point of the polynomial $p(z)$. 
6 Extremal Problems for Polynomials

THEOREM 6.1 (D. I. Mendeleev) If \( p(x) = \sum_{v=0}^{2} c_v x^v \) is a real polynomial of degree 2, then

\[
\max_{-1 \leq x \leq 1} |p'(x)| \leq 4 \max_{-1 \leq x \leq 1} |p(x)|
\]

THEOREM 6.2 (A. A. Markov (1889)) If \( p(x) = \sum_{v=0}^{n} c_v x^v \) is a real polynomial of degree \( n \), then

\[
\max_{-1 \leq x \leq 1} |p'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |p(x)|.
\]

The result is best possible and the equality holds for Tchebychev’s polynomial of first kind \( T_n(x) = \cos(n \cos^{-1} x) \).

The above result was generalized by A. A. Markov’s brother, W. W. Markov, who proved

THEOREM 6.3 (W. W. Markov, Math. Annalen 77 (1916)) If \( p(x) \) is a real polynomial of degree \( n \), then

\[
\max_{-1 \leq x \leq 1} |p^{(s)}(x)| \leq \frac{n^2(n^2 - 1^2)(n^2 - 2^2) \ldots (n^2 - s - 1^2)}{1 \cdot 3 \cdot 5 \ldots (2s - 1)} \max_{-1 \leq x \leq 1} |p(x)|.
\]

Again the equality holds for the polynomial \( T_n(x) = \cos(n \cos^{-1} x) \).

THEOREM 6.4 (S. Bernstein, 1912) If \( p(x) \) is a real trigonometric polynomial of degree \( n \), then

\[
|p'(x)| \leq \frac{n}{\sqrt{1 - x^2}} \max_{-1 \leq x \leq 1} |p(x)|.
\]
**Remark:** In the neighborhood of origin, Theorem 6.4 gives a better bound than Theorem 6.2 while in the neighborhood of \( x = \pm 1 \), the bound obtained by Theorem 6.2 is better than that from Theorem 6.4.
THEOREM 7.1 (S. Bernstein, Memoire de l’Académie Royale) If \( p(x) \) is a real polynomial of degree at most \( n \), then
\[
|p'(x)| \leq \frac{n}{\sqrt{1 - x^2}} \max_{-1 \leq x \leq 1} |p(x)|.
\]

Since \( p(\cos \theta) = \sum_{v=0}^{n} c_v \cos^v \theta \) can be written as a trigonometric polynomial of degree \( n \), the following result is a generalization of Theorem 7.1.

THEOREM 7.2 (S. Bernstein) If \( t(\theta) = \sum_{v=0}^{n} (a_v \cos v\theta + b_v \sin v\theta) \) is a trigonometric polynomial of degree \( n \), then for real \( \theta \),
\[
|t'(\theta)| \leq n \max_{-\pi \leq \theta \leq \pi} |t(\theta)|.
\]

The following inequality which follows readily from Theorem 7.2 is also known as Bernstein’s inequality.

THEOREM 7.3 (Bernstein’s Inequality) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \), then
\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.
\]
The result is best possible and the equality holds for \( p(z) = \lambda z^n \), \( \lambda \) being a complex number.

Bernstein in fact proved Theorem 7.2 with \( 2n \) in place of \( n \). Theorem 7.2 in the present form appeared in print for the first time in a paper of Feketé who attributes the proof to Fejer. Alternate proofs of this theorem were given by Rogosinski, de la Vallee Poisson and others.

To obtain Theorem 7.3 from Theorem 7.2, simply apply Theorem 7.2 to \( t(\theta) = p(e^{i\theta}) = \sum_{v=0}^{n} a_v e^{iv\theta} \), which is a trigonometric polynomial of degree \( n \).
THEOREM 8.1 (S. Bernstein, 1926) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree at most \( n \), then
\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|
\]
(13)

THEOREM 8.2 (S. Bernstein) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree at most \( n \), then
\[
\max_{|z|=R \geq 1} |p(z)| \leq R^n \max_{|z|=1} |p(z)|.
\]
(14)

THEOREM 8.3 (R. S. Varga, 1957) If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \), then
\[
\max_{|z|=r \leq 1} |p(z)| \geq r^n \max_{|z|=1} |p(z)|
\]
(15)

Theorem 8.2 is a simple deduction from the Maximum Modulus Principle, while Theorem 8.3 can be easily derived from Theorem 8.2. Theorem 8.1 is better known as S. Bernstein’s Inequality, although, it appeared in print for the first time in a paper of Fejter who attributes the proof to Fejer. Alternative proofs of this inequality were given by M. Riesz, Rogosinski, de la Vallee Poussin, and others. Bernstein had proved Theorem 8.1 with \( 2n \) in place of \( n \). Bernstein in 1930 observed that Theorem 8.1 can be obtained from Theorem 8.2 as well by making use of Gauss-Lucas Theorem.

In fact, it can be shown that all the above three inequalities are equivalent, in the sense that anyone can be derived from the other.
THEOREM 9.1 If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \), then

\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{16}
\]

\[
\max_{|z|=R \geq 1} |p(z)| \leq R^n \max_{|z|=1} |p(z)|, \tag{17}
\]

\[
\max_{|z|=r \leq 1} |p(z)| \geq r^n \max_{|z|=1} |p(z)|. \tag{18}
\]

In all above inequalities, the equality holds for \( p(z) = \lambda z^n \).

THEOREM 9.2 (Frappier, Rahman and Ruscheweyh, Trans. A. M. S. 1985)

If \( p(z) \) is a polynomial of degree at most \( n \), then

\[
\max_{|z|=1} |p'(z)| \leq \begin{cases} 
 n \max_{|z|=1} |p(z)| - \frac{2n}{n+2}|p(0)| & \text{if } n \geq 2 \\
 \max_{|z|=1} |p(z)| - |p(0)| & \text{if } n = 1 
\end{cases} \tag{19}
\]

\[
\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - (R^n - R^{n-2})|p(0)| \text{ if } n \geq 2. \tag{20}
\]

THEOREM 9.3 If \( p(z) \) is a polynomial of degree \( n \), \( p(z) \neq 0 \) in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|, \tag{21}
\]

\[
\max_{|z|=R \geq 1} |p(z)| \leq \frac{(R^n + 1)}{2} \max_{|z|=1} |p(z)|, \tag{22}
\]

\[
\max_{|z|=r \leq 1} |p(z)| \geq \left( \frac{r + 1}{2} \right)^n \max_{|z|=1} |p(z)|. \tag{23}
\]
THEOREM 10.1 (P. D. Lax, Bull. AMS. 1944) If \( p(z) \) is a polynomial of degree \( n \), \( p(z) \neq 0 \) in \( |z| < 1 \), then
\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{24}
\]

R. P. Boas raised the following question that “How large can the bound in (24) be if \( p(z) \) has \( k \) zeros on or outside the unit circle?”

One would expect the answer to be \((n - \frac{k}{2})\).

THEOREM 10.2 (Giroux and Rahman, Trans. AMS, 1974) For every positive integer \( n \), there exists a polynomial \( p(z) \) of degree \( n \) having a zero on \(|z| = 1\), such that
\[
\max_{|z|=1} |p'(z)| \geq (n - c/n) \max_{|z|=1} |p(z)|.
\]

On the other hand for an arbitrary polynomial \( p(z) \) of degree \( n \) having a zero on \(|z| = 1\), he proved
\[
\max_{|z|=1} |p'(z)| \leq (n - \frac{1 - \sin 1}{4\pi n}) \max_{|z|=1} |p(z)|.
\]

THEOREM 10.3 (Ruscheweyh, Complex Variables, 1986) There exist polynomials \( p(z) \) of degree \( n \) having all but one zero on \(|z| = 1\), such that
\[
\max_{|z|=1} |p'(z)| = [An + o(n)] \max_{|z|=1} |p(z)|,
\]
where \( A \approx 0.884 \).

The result of Ruscheweyh thus shows that even if we assume that all but one zeros lie on \(|z| = 1\), the bound in the Bernstein’s inequality cannot really be very significantly improved.
**THEOREM 11.1** (P. D. Lax, Bull. AMS. 1944) If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then
\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.
\] (25)

Here is another question raised by R. P. Boas.

"How large can the bound in (25) be if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, \, K > 0 \)?"

**THEOREM 11.2** (Malik, Jour. Lond. Math. Soc. 1969) If \( p(z) \) is a polynomial of degree \( n \) having no zero in \( |z| < K, \, K \geq 1 \), then
\[
\max_{|z|=1} |p'(z)| \leq \left( \frac{n}{1+K} \right) \max_{|z|=1} |p(z)|.
\] (26)

Equality holds for \( p(z) = (z + K)^n \).

**THEOREM 11.3** (Govil and Rahman, Trans. AMS 1969) If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < K, \, K \geq 1 \), then
\[
\max_{|z|=1} |p^{(s)}(z)| \leq \frac{n(n-1) \ldots (n-s+1)}{1+K^s} \max_{|z|=1} |p(z)|.
\] (27)

The solution to this problem when the polynomial does not have zeros in \( |z| < K \), where \( 0 < K < 1 \) is still unknown, and is thus an open problem.
Remark 1. The bound in Theorem 12.2 depends on the moduli of the zeros of smallest modulus. For example, for both polynomials $(z + K)^n$ and $(z + K)(z + K + 1)^{n-1}$, the inequality (22) gives the same bound, $\frac{n}{1 + K}$.


Let $p(z) = a_n \prod_{v=1}^{n} (z - z_v), a_n \neq 0$. If $|z_v| \geq K_v \geq 1, 1 \leq v \leq n$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left\{ 1 - \frac{1}{1 + \frac{2}{n} \sum_{v=1}^{n} \frac{1}{K_v - 1}} \right\} \max_{|z|=1} |p(z)| \quad (28)$$

If $K_v = 1$ for some $v$, it reduces to Lax’s result. If $K_v \geq K \geq 1$, it reduces to Malik’s result, Theorem 12.2.

Remark 2. One need not know the location of all the zeros of the polynomial in order to apply this theorem. No doubt the usefulness of the theorem will be heightened if we have information about all the zeros.