Theorem 2. Let \( r \geq 2 \). If \( p_1, p_2, \ldots, p_r \) are the first \( r \) primes of the form \( p_i = 4k_i + 3 \), then the interval \( (p_r, \prod_{i=1}^{r} p_i) \) contains at least \( \lceil \log_2(4(r - 1)) \rceil \) primes congruent to 3 modulo 4.

Proof. For \( r = 2 \) or 3, the result can be checked by hand, so we assume that \( r \geq 4 \). Since \( p_4 = 19 > 4 \cdot 4 \), we have \( 4r < p_r \). If \( 1 \leq j \leq \lceil \log_2(4r) \rceil + 1 \), it follows that \( 2^{j+1} \leq p_r \). Hence

\[
\prod_{i=1}^{r} p_i - 2^{j+1} > 7p_r - 4p_r > p_r.
\]

If \( r \) is odd, set \( n_k = \prod_{i=1}^{r} p_i - 2^{k+1} \). Then \( n_k \equiv 3(4) \) (since \( 3^2 \equiv 1(4) \)). But no \( p_i \) divides \( n_k \) for \( 1 \leq i \leq r \), so the integer \( n_k \) has some prime factor \( q_k \equiv 3(4) \) with \( q_k < p_r \). If \( j \neq k \), say \( j > k \), the assumption that \( q_k \) also divides \( n_j \) leads to the same contradiction as earlier: since \( n_k - n_j = 2^{j+1} - 2^{k+1} = 2^{k+1}(2^{j-k} - 1) \), we have \( q_k | 2^{j-k} - 1 \) and hence \( q_k < p_r \). Thus, there are at least \( \lceil \log_2(4r) \rceil + 1 \) distinct primes of the form \( 4\ell + 3 \) in \( (p_r, \prod_{i=1}^{r} p_i) \). If \( r \) is even, the same argument applied to \( r - 1 \) shows that there are at least \( \lceil \log_2(4(r - 1)) \rceil + 1 \) distinct primes of the form \( 4\ell + 3 \) in \( (p_{r-1}, \prod_{i=1}^{r-1} p_i) \). Since the first of these is \( p_r \), there are at least \( \lceil \log_2(4(r - 1)) \rceil + 1 \) distinct primes of the form \( 4\ell + 3 \) in \( (p_r, \prod_{i=1}^{r-1} p_i) \), a subinterval of \( (p_r, \prod_{i=1}^{r} p_i) \).

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Smooth Interpolation, Hölder Continuity, and the Takagi–van der Waerden Function

Jack B. Brown and George Kozlowski

1. INTRODUCTION. Consider the classes of functions \( f : [0, 1] \to \mathbb{R} \) indicated in the following diagram (where \( 0 < \beta < \alpha < 1 \)):

\[
C^1 \subset Lip^1 \subset \bigcap_{0 < \gamma < 1} Lip^\gamma \subset Lip^\alpha \subset Lip^\beta \subset C,
\]

in which \( C \) denotes the class of continuous functions, \( C^1 \) the class of continuously differentiable functions, and

\[
Lip^\alpha = \{ f \in C : \exists K > 0 \exists |f(x) - f(y)| < K|x - y|^\alpha \text{ for } x, y \in [0, 1] \}
\]

the class of Lipschitz (or Hölder continuous) functions of order \( \alpha \). The symbol \( \lambda \) signifies Lebesgue measure. Marcinkiewicz [7] showed that there is a strong interpolation
link between the classes $C^1$ and $Lip^1$ when he proved: if $f$ belongs to $Lip^1$, then for every $\epsilon > 0$ there exists $g$ in $C^1$ such that $\lambda(\{x : f(x) \neq g(x)\}) < \epsilon$. Marcinkiewicz actually required only that $f$ be “pointwise” $Lip^1$ (i.e., that $f(x + t) = f(x) + O(t)$ at each $x$) and obtained similar results for higher order smoothness. Federer [5, p. 442] obtained the analogous $Lip^1$-$C^1$ result in higher dimensions, and Whitney [13] extended Federer’s results to higher order smoothness (see also [6, p. 227]). Whitney described an example of a function $f$ of one variable that was designed to show that one could not weaken the requirement of $f$ being in $Lip^1$ in the Marcinkiewicz theorem. He stated the following [13, p. 144]:

Let $\phi_0(t)$ be the distance from $t$ to the nearest integer. Using any sufficiently large integer $a$, set

$$
\phi_i(t) = 2^i \phi_0(\alpha^i t)/\alpha^i, \quad \phi(t) = \sum_{i=0}^{\infty} \phi_i(t).
$$

Then $\phi$ satisfies a Lipschitz condition of order $1 - \alpha$, for any $\alpha > 0$; but [the conclusion of Marcinkiewicz’s theorem] is not true for it.

The function $\phi_0$ is the familiar “rooftop” function, with “roofs” having slopes $\pm 1$. For $n > 0$ the roofs of $\phi_n$ have slopes $\pm 2^n$. The graphs of $\phi_1$ and $\phi_2$ on $[0,1]$, with $a = 3$, are displayed in Figure 1. Whitney’s statement would seem to suggest that for sufficiently large $a$ in $\mathbb{N}$ the resulting function $\phi$ would belong to $\cap_{0 < \gamma < 1} Lip^\gamma$. Such, however, is not the case.

![Figure 1](image)

**Theorem 1.** Let $a$ in $\mathbb{N}$ be greater than 2, let $\phi$ be the function described by Whitney, and let $t_0 = 1 - \ln(2)/\ln(a)$. Then $\phi$ has the following properties:

1. $\phi$ is not a member of $Lip^\alpha$ for any $\alpha$ satisfying $t_0 < \alpha \leq 1$; in fact, for $\tau_N = 1/(2aN)$ the quotients

$$
|\phi(\tau_N) - \phi(0)|/|\tau_N|^\alpha
$$

are unbounded as $N \to \infty$.

2. $\phi$ belongs to $Lip^\beta$ if $0 < \beta \leq t_0$; in fact, with $L = 4(a - 1)/(a - 2)$, it is true that $|\phi(s) - \phi(t)| \leq L|s - t|^\beta$ for all such $\beta$ and for all $s$ and $t$ in $[0, 1]$.

**Proof.** Suppose that $t_0 < \alpha \leq 1$. If $i \leq N$, $\phi_i(\tau_N) = 2^i \phi_0(1/2a^{N-i})/\alpha^i = 2^i \tau_N$. If $i > N$, there are two possibilities: if $a$ is even, $\phi_i(\tau_N) = 0$; if $a$ is odd, $\phi_i(\tau_N) = 2^{i-1}/\alpha^i = (2/\alpha^{i-N})\tau_N$. For our purposes, it will be enough to know that $\phi_i(\tau_N) \geq 0$
for \( i > N \). Then

\[
\frac{|\phi(\tau_N) - \phi(0)|}{|\tau_N|^\alpha} = \sum_{i=0}^{\infty} \frac{\phi_i(\tau_N)}{\tau_N^\alpha} \geq 2^N \tau_N^{1-\alpha} \geq \frac{1}{2^{1-\alpha}} \left( \frac{2}{a^{1-\alpha}} \right)^N.
\]

Since \( 2a^{\alpha-1} > 1 \), these quotients tend to \( \infty \) as \( N \to \infty \).

Suppose next that \( 0 < \beta \leq t_0 \). For given \( s \) and \( t \) in \([0, 1]\), let \( N \) be the nonnegative integer such that

\[
\frac{1}{a^{N+1}} < |s - t| \leq \frac{1}{a^N}.
\]

Note that

\[
|\phi(s) - \phi(t)| \leq \sum_{i=0}^{N} |\phi_i(s) - \phi_i(t)| + \sum_{i=N+1}^{\infty} |\phi_i(s) - \phi_i(t)|
\]

and that each of the functions \( \phi_i \) is polygonal, with graph consisting of line segments of slope \( \pm 2^i \) and with values in the interval \([0, (2/a)^i]\). Majorizing terms of the first sum by \( 2^i |s - t| \) and terms of the second by \( (2/a)^i \) shows that

\[
|\phi(s) - \phi(t)| \leq \sum_{i=0}^{N} 2^i |s - t| + (2/a)^{N+1} \sum_{i=0}^{\infty} (2/a)^i
\]

\[
= (2^{N+1} - 1) |s - t| + (2/a)^{N+1} \frac{a}{a - 2}
\]

\[
\leq |s - t| 2^{N+1} \left( 1 + \frac{a}{a - 2} \right) = |s - t| 2^{N+2} a - 1
\]

Thus,

\[
|\phi(s) - \phi(t)| \leq |s - t|^{1-\beta} 2^N L \leq \left( \frac{2}{a^{1-\beta}} \right)^N L \leq L,
\]

because the hypothesis on \( \beta \) implies that \( 2a^{\beta-1} \leq 1 \).

\[\square\]

2. THE TAKAGI–VAN DER WAERDEN FUNCTION. Since

\[
\lim_{a \to \infty} 1 - \frac{\ln(2)}{\ln(a)} = 1
\]

and since it can be shown (the proof is similar to the proof of Theorem 2) that \( \lambda(\{ x : \phi(x) = g(x) \}) = 0 \) for each \( g \) in \( C^1 \), it would follow that one could not replace the hypothesis that \( f \) belongs to \( Lip^{\alpha} \) in Marcinkiewicz’s theorem with the weaker requirement that \( f \) be in \( Lip^{\alpha} \) for any specific choice of \( \alpha \) in \((0, 1)\). However, it does not show that one could not replace that requirement with the assumption that \( f \) belongs to \( \cap_{0 < \gamma < 1} Lip^{\gamma} \). It is conceivable that the leading \( 2^i \) in Whitney’s description of the function \( \phi_i \) is actually a typographical error and that Whitney was really referring to the so-called van der Waerden function

\[
v(t) = \sum_{i=0}^{\infty} \frac{\phi_i(a^i t)}{a^i}
\]
with parameter $a$ in $\mathbb{N}$, $a > 1$. B. L. van der Waerden [12] described this function (with $a = 10$) in 1930, providing a simple example of a continuous, nowhere differentiable function. The function $v$ was described independently by other authors, including T. Takagi [11] in 1903 (as was pointed out in [11]). We refer to it as the Takagi–van der Waerden function. It has been shown [2], [4] that $v$ has more pathological nondifferentiability properties than were pointed out in [12]. It turns out that $v$ does indeed satisfy the conditions described by Whitney.

**Theorem 2.** The Takagi–van der Waerden function $v$ belongs to $\cap_{0 < \gamma < 1} \text{Lip}^\gamma$, but $v$ agrees with no function $g$ from $C^1$ on any set of positive measure. In fact, if $M$ is a subset of $[0,1]$ with $\lambda(M) > 0$, then the set

$$D(v, M) = \left\{ \frac{v(y) - v(x)}{y - x} : x \in M, \ y \in M, \ y \neq x \right\}$$

is unbounded.

**Proof.** The fact that $v$ is a member of $\cap_{0 < \gamma < 1} \text{Lip}^\gamma$ was established by Shidfar and Sabetfakhri for $a = 2$ in [9], and the same fact for all integers $a$ greater than 2 follows from the theorem in [10].

For each $i$ in $\mathbb{N}$, let $f_i(t) = \phi_0(a^i t)/a^i$, and let

$$v_n(t) = \sum_{i=0}^{n} f_i(t)$$

for $0 \leq t \leq 1$. Then $f_i$ is a polygonal function with each segment in its graph having slope $\pm 1$, and $v_n$ is likewise a polygonal function, each segment in its graph having slope some integer in the interval $[-n,n]$.

In order to prove the second claim in the theorem, let $M$ be a subset of $[0,1]$ with $\lambda(M) > 0$ and suppose that $b > 0$ (assume without loss of generality that $b$ is an integer). Let $z$ in $M$ be a Lebesgue density point of $M$ (i.e., for every $\epsilon > 0$ there exists $\delta > 0$ such that for every subinterval $[c, d]$ of $[0,1]$ that contains $z$ and has length less than $\delta$, it is true that $\lambda([c, d] \setminus M) < \epsilon \cdot (d - c)$ [8, pp. 12–13], [3, pp. 315–316]). Choose $\delta > 0$ corresponding to $\epsilon = a^{-b}/8$ in this definition, and consider an arbitrary subinterval $[c, d]$ of $[0,1]$ that contains $z$ and has $|d - c| < \delta$, so that $\lambda([c, d] \setminus M) < \epsilon \cdot (d - c)$. Note that this ensures that if a subinterval $(u, v)$ of $[c, d]$ has length $v - u \geq \epsilon \cdot (d - c)$, then $(u, v)$ must contain a point $x$ of $M$. Fix $n$ in $\mathbb{N}$ such that $1/a^n < \delta$. We now specify $[c, d]$ to be the base (of length $1/a^n$) of one of the “roofs” of $f_n$ that contains $z$. Write $m = n + b$ and $e = c + a^{-n-1}$, so that $[c, e]$ is the base of one of the “roofs” (illustrated in Figure 2) of $f_{n+1}$.

The polygonal function $v_n$ has constant slope $M_n$ over the interval $[c, e]$. There will be two cases to consider. Assume first that $M_n \geq 0$, and consider $h = c + (1/2a^m)$. Note that $h$ is the abscissa of the top of a “roof” of $f_m$. The graph of $v_m$ has constant slope $b + M_n$ on $[c, h]$. Set

$$\Delta = \frac{1}{4}(h - c) = \frac{1}{4} \frac{1}{2 \cdot a^m} = \frac{1}{8} \frac{1}{a^{n+b}} = \frac{1}{8} \frac{1}{a^b} \cdot \frac{1}{a^n} = \epsilon \cdot (d - c).$$

The interval $(u, v)$ with left endpoint $c$ and length $\Delta = \epsilon \cdot (d - c)$ must contain a point $x$ of $M$. There must also exist an element $y$ of $M$ in the open interval of length $\Delta$ with right endpoint $h$ (see Figure 2). We then have
so we can estimate

$$|v(y) - v(x)| \geq (b + M_n) \cdot (y - x) - \sum_{i=m+1}^{\infty} |f_i(y) - f_i(x)|$$

$$\geq (b + M_n) \frac{h - c}{2} - \sum_{i=m+1}^{\infty} \frac{1}{2 \cdot a^i}$$

$$= (b + M_n) \frac{h - c}{2} - \frac{1}{2 \cdot a^m + 1} \cdot \frac{a}{a - 1}$$

$$= \left( b + M_n \right) \frac{2}{2} - \frac{1}{a - 1} \cdot (h - c).$$

Accordingly, for each integer $b > 0$ there exist points $x$ and $y$ of $M$ for which

$$\left| \frac{v(y) - v(x)}{y - x} \right| \geq \left| \frac{v(y) - v(x)}{h - c} \right| \geq \frac{b + M_n}{2} - \frac{1}{a - 1} \geq \frac{b}{2} - \frac{1}{a - 1}.$$
\[ \frac{|v(y) - v(x)|}{y - x} \geq \frac{|v(y) - v(x)|}{d - h} \geq \frac{b + |M_n|}{2} - \frac{1}{a - 1} \geq \frac{b}{2} - \frac{1}{a - 1} \]

in this case as well. Again, the unboundedness of \( D(v, M) \) follows.

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When Does the Position Vector of a Space Curve Always Lie in Its Rectifying Plane?

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1. INTRODUCTION. Let \( \mathbb{E}^3 \) denote Euclidean three-space, with its inner product \( \langle \ , \ \rangle \). Consider a unit-speed space curve \( \mathbf{x} : I \rightarrow \mathbb{E}^3 \), where \( I = (\alpha, \beta) \) is a real interval, that has at least four continuous derivatives. Let \( \mathbf{t} \) denote \( \mathbf{x}' \). It is possible, in general, that \( \mathbf{t}'(s) = 0 \) for some \( s \); however, we assume that this never happens. Then we can introduce a unique vector field \( \mathbf{n} \) and positive function \( \kappa \) so that \( \mathbf{t}' = \kappa \mathbf{n} \). We call \( \mathbf{t}' \) the curvature vector field, \( \mathbf{n} \) the principal normal vector field, and \( \kappa \) the curvature of the given curve. Since \( \mathbf{t} \) is a constant length vector field, \( \mathbf{n} \) is orthogonal to \( \mathbf{t} \). The binormal vector field is defined by \( \mathbf{b} = \mathbf{t} \times \mathbf{n} \). It is a unit vector field orthogonal to both \( \mathbf{t} \) and \( \mathbf{n} \). One defines the torsion \( \tau \) by the equation \( \mathbf{b}' = -\tau \mathbf{n} \). The famous Frenet-Serret equations are given by (see, for instance, [4] or [6]):