Stochastic solution of Cauchy problems

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Outline

Scaling limits and heat equation
Scaling limits and fractional diffusion
Fractional diffusion and iterated Brownian motions
Initial-Boundary value problems
“Probability is simply a branch of measure theory, with its own special emphasis and field of application ...”

J.L. Doob

Probability and transforms

If the random variable $Y$ has density $f(x)$ so that

$$P(a \leq Y \leq b) = \int_a^b f(x) dx$$

then $f(x)$ has Fourier transform

$$\hat{f}(k) = E(e^{-ikY}) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

$$= \int_{-\infty}^{\infty} (1 - ikx + \frac{1}{2!}(ikx)^2 + \cdots) f(x) dx$$

$$= 1 - ik\mu_1 - \frac{1}{2!}k^2\mu_2 + \cdots$$

where the $l$th moment is $\mu_l = \int_{-\infty}^{\infty} x^l f(x) dx$
Central limit theorem

If $\mu_1 = 0$ and $\mu_2 = 2$ then $\hat{f}(k) = 1 - k^2 + \cdots$

The IID sum $S(n) = Y_1 + \cdots + Y_n$ has FT $\hat{f}(k)^n$ and the
normalized sum $S(n)/\sqrt{n}$ has FT

$$\left(\hat{f}(k/\sqrt{n})\right)^n = (1 - (k/\sqrt{n})^2 + \cdots)^n$$

$$= \left(1 - \frac{k^2}{n} + \cdots\right)^n$$

$$\rightarrow e^{-k^2} \equiv \hat{g}(k) \text{ as } n \rightarrow \infty.$$ 

Inverting the Fourier transform reveals a Gaussian(Normal) density

$$g(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4}$$
Brownian motion

If $Y_n$ represents a particle jump at time $n$ then $S(n) = Y_1 + \cdots + Y_n$ is the location of the particle at time $n$. Expanding the time scale by a factor of $c > 0$ and taking limits as $c \to \infty$ shows that $c^{-1/2}S([ct]) \Rightarrow W(t)$ since

$$\hat{f}(c^{-1/2}k)[ct] = \left(1 - \frac{k^2}{c} + \cdots\right)[ct]$$

$$\to e^{-k^2t} \equiv \hat{p}(t, k) \text{ as } c \to \infty$$

for all $t > 0$. Inverting the FT shows that the density of the limiting Brownian motion process $W(t)$ is Gaussian (Normal)

$$p(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$
Brownian motion

Classical random walk

\[ S(t) = Y_1 + \cdots + Y_{[t]} \]

A particle takes a random jump \( Y_n \) at time \( t = n \). Particle location at time \( t \) is a simple random walk \( S(t) \) and scaling limit is a Brownian motion.

\[ c^{-1/2} S(ct) \Rightarrow W(t) \approx N(0, \sigma^2 t) \quad (c \to \infty) \]

- Contract spatial scale
- Expand time scale
- Normal limit density

Add an advective drift:

\[ L(t) = vt + W(t) \approx N(vt, \sigma^2 t) \]
Random walk simulation
Longer time scale
Scaling limit: Brownian motion

Random graph of fractal dimension 1.5 and no jumps.
Most likely shape of a Brownian path.

Microsoft stock-the last two years
Some history of Brownian motion (BM)

- Robert Brown (1827), a Botanist: was first to observe that pollen grains in water move continuously and very erratically.
- Louis Bachelier (1900): presented a stochastic analysis of the stock and option markets using BM
- Albert Einstein (1905): used BM to determine the law of the position of the particle...
- Norbert Wiener (1923): Mathematical foundations of BM
- Doob (1956): connections to analysis, heat equation
- Kolmogorov, Lévy, Khintchine, .....
If the Laplace transform of $f(t)$ is defined for $s > 0$ by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$$

then $\frac{df(t)}{dt}$ has Laplace transform $s\tilde{f}(s) - f(0)$.

If the Fourier transform of $f(x)$ is defined for $k \in \mathbb{R}$ by

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-ikx} f(x) dx$$

then $\frac{df(x)}{dx}$ has Fourier transform $ik\hat{f}(k)$. 
The diffusion (heat) equation

Taking Fourier transforms in the classical diffusion equation

\[
\frac{\partial p(t, x)}{\partial t} = \frac{\partial^2 p(t, x)}{\partial x^2}
\]

yields

\[
\frac{\partial \hat{p}(t, k)}{\partial t} = (ik)^2 \hat{p}(t, k) = -k^2 \hat{p}(t, k)
\]

whose solution

\[
\hat{p}(t, k) = e^{-k^2 t}
\]

inverts to the same limit density for the Brownian motion $W(t)$. For a cloud of diffusing particles $p(t, x)$ is the particle density.
Let $W_t \in \mathbb{R}$ be Brownian motion started at $x$. Then the function (convolution of $f$ and $p(t,x)$)

$$u(t, x) = E_x[f(W(t))] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} f(y) dy$$

solves the heat equation

$$\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2 u(t, x)}{\partial x^2}, \quad t > 0, \quad x \in \mathbb{R}$$

$$u(0, x) = f(x), \quad x \in \mathbb{R}.$$ 

This is due to J.L. Doob (1956).

In this case we say, Brownian motion $W(t)$ is a stochastic solution of the heat equation.
Continuous time random walks

The CTRW is a random walk with jumps $X_n$ separated by random waiting times $J_n$. The random vectors $(X_n, J_n)$ are i.i.d.
Heavy tailed waiting times

Random wait $J_n$ between jumps, $n$th jump time given by a random walk

$$T(n) = J_1 + \cdots + J_n$$

Number of jumps by time $t$ is inverse $N(t) \geq n \iff T(n) \leq t$

For heavy tail waiting times

$$P(J_n > t) \approx Ct^{-\beta} \quad (0 < \beta < 1)$$

Inverse processes have inverse scaling

$$c^{-1/\beta} T(ct) \Rightarrow P(t) \\ c^{-\beta} N(ct) \Rightarrow Q(t)$$

$$P(ct) \approx c^{1/\beta} P(t) \\ Q(ct) \approx c^\beta Q(t)$$
Continuous time random walks (CTRW)

Particle jump random walk has scaling limit
\[ c^{-1/2} S([ct]) \xrightarrow{c} W(t). \]
Number of jumps has scaling limit \( c^{-\beta} N(ct) \xrightarrow{c} Q(t). \)
CTRW is a random walk subordinated to (a renewal process) \( N(t) \)
\[
S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}
\]
CTRW scaling limit is a subordinated process:
\[
c^{-\beta/2} S(N(ct)) = (c^\beta)^{-1/2} S(c^\beta \cdot c^{-\beta} N(ct)) \\
\approx (c^\beta)^{-1/2} S(c^\beta Q(t)) \xrightarrow{c} W(Q(t)).
\]
CTRW simulation with heavy tail waiting times
Longer time scale
Scaling limit: Subordinated motion

Limit retains long waiting times.
Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)
Fractional derivatives: An old idea gets new life

- Fractional derivatives $D^\beta f(x)$ for any $\beta > 0$ were invented by Leibniz (1695) soon after the more familiar integer derivatives.
- The Caputo fractional derivative of order $0 < \beta < 1$ defined by
  \[
  D_t^\beta g(t) = \frac{1}{\Gamma(1 - \beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t - s)^\beta}
  \]  
  was invented to properly handle initial values (Caputo 1967).
- Laplace transform of $D_t^\beta g(t)$ is $s^\beta \tilde{g}(s) - s^{\beta-1}g(0)$ incorporates the initial value in the same way as the first derivative.
examples

\[ D_t^\beta(t^p) = \frac{\Gamma(1 + p)}{\Gamma(p + 1 - \beta)} t^{p-\beta} \]

\[ D_t^\beta(e^{\lambda t}) = \lambda^\beta e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1 - \beta)} \]

\[ D_t^\beta(\sin t) = \sin(t + \frac{\pi \beta}{2}) \]
Nigmatullin (1986), Zaslavsky (1994) studied the Cauchy problem
\[ \frac{\partial^\beta}{\partial t^\beta} u(t, x) = \frac{\partial^2 u(t, x)}{\partial x^2}; \quad u(0, x) = f(x) \] (2)
that models particles that wait for a time \( J_n \) before the \( n \)th jump, where \( P(J_n > t) \approx t^{-\beta} \) for some \( 0 < \beta < 1 \).

The solution to time-fractional diffusion is given by
\[ u(t, x) = E_x(f(W(Q(t)))) = \int_0^\infty q(u, x) f_{Q(t)}(u) du \]
\( q(u, x) = E_x(f(W(u))) \) solution to the heat equation.
\( f_{Q(t)}(u) \) inverse stable density of \( u = Q(t) \).
In finance, \( Q(t) \) represents the number of trades by time \( t \).
Proof uses Fourier and Laplace transforms and inverting these transforms.
Taking Fourier-Laplace transform of the Equation (2) gives

\[ \bar{u}(s, k) = \frac{s^{\beta-1} \hat{f}(k)}{s^\beta + k^2} \]

\[ = s^{\beta-1} \int_0^\infty \exp(-[s^\beta + k^2]l) \hat{f}(k) dl \]  

The next step is to invert this Fourier-Laplace transform using the fact that \( Q(t) \) has density

\[ f_{Q(t)}(s) = \frac{t}{\beta} g_\beta \left( \frac{t}{s^{1/\beta}} \right) s^{-1/\beta-1}, \text{ and } \int_0^\infty e^{-su} g_\beta(u) = e^{-s^\beta}. \]

In the case \( \beta = 1/2 \),

\[ f_{Q(t)}(s) = \frac{2}{\sqrt{4\pi t}} e^{-s^2/4t} = f_{|W(t)|}(s) \]

This proof is due to Meerschaert, Benson, Scheffler and Baeumer (2002)
Equivalence to Higher order PDE’s

- For any \( m = 2, 3, 4, \ldots \) both the Cauchy problem

\[
\frac{\partial u(t, x)}{\partial t} = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} \Delta^j f(x) + \Delta^m u(t, x); \quad u(0, x) = f(x)
\]

and the fractional Cauchy problem:

\[
\frac{\partial^{1/m}}{\partial t^{1/m}} u(t, x) = \Delta u(t, x); \quad u(0, x) = f(x),
\]

have the same unique solution given by

\[
u(t, x) = \int_0^\infty p((t/s)^{1/m}, x) g_{1/m}(s) \, ds = E_x(f(W(Q(t))))
\]

- Due to Baeumer, Meerschaert, and Nane TAMS(2009).
Connections to iterated Brownian motions

Orsingher and Benghin (2004) and (2008) show that for \( \beta = 1/2^n \) the solution to

\[
\frac{\partial^{1/2n}}{\partial t^{1/2n}} u(t, x) = \Delta_x u(t, x); \quad u(0, x) = f(x),
\]  

is given by running

\[
I_{n+1}(t) = W_1(|W_2(|W_3(\cdots (W_{n+1}(t))\cdots |)|)|)
\]

Where \( W_j \)'s are independent Brownian motions, i.e.,

\[
u(t, x) = E_x(f(I_{n+1}(t)))\]
solves (6), and solves (4) for \( m = 2^n \).
Corollary

We obtain the equivalence of one dimensional distributions in the case $Q(t)$ is the inverse stable subordinator of index $\beta = 1/2^n$

$$I_{n+1}(t) = \mathcal{W}_1(\mathcal{W}_2(\mathcal{W}_3(\cdots (\mathcal{W}_{n+1}(t))\cdots )))^{(d)} = \mathcal{W}_1(Q(t))$$
Figure: Simulations of iterated Brownian motions
Heat equation in bounded domains

Heat equation in $D$ with Dirichlet boundary conditions:

$$\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x), \quad x \in D, \quad t > 0,$$
$$u(t, x) = 0, \quad x \in \partial D; \quad u(0, x) = f(x), \quad x \in D.$$ 

When $D = (0, M)$ can be solved by separation of variables: set $u(t, x) = \phi(x) T(t)$. Hence $\phi(x)$ satisfies

$$\frac{\partial^2 \phi(x)}{\partial x^2} = -\lambda \phi(x), \quad x \in (0, M), \quad \lambda > 0; \quad \phi(0) = 0, \quad \phi(M) = 0$$

and $T(t)$ satisfies $T'(t) = -\lambda T(t); \quad T(0) = 1$.

$$\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin\left(\frac{n\pi x}{M}\right), \quad \lambda_n = \frac{\pi^2 n^2}{M^2}, \quad T_n(t) = e^{-\lambda_n t}.$$ 

Same applies in any dimension $d \geq 1$. 

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Stochastic solution of Cauchy problems
Denote the eigenvalues and the eigenfunctions of $\Delta_D$ by 
$\{\lambda_n, \phi_n\}_{n=1}^{\infty}$, where $\phi_n \in C^\infty(D)$. The corresponding heat kernel is given by

$$p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y).$$

The series converges absolutely and uniformly on $[t_0, \infty) \times D \times D$ for all $t_0 > 0$. In this case, the semigroup given by

$$T_D(t)f(x) = E_x[f(W(t))I(t < \tau_D(X))] = \int_D p_D(t, x, y)f(y)dy$$

$$= \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \bar{f}(n)$$

solves the Heat equation in $D$ with Dirichlet boundary conditions.
Fractional diffusion in bounded domains

\[
\frac{\partial^\beta}{\partial t^\beta} u(t, x) = \Delta u(t, x); \quad x \in D, \; t > 0
\]

\[
u(t, x) = 0, \; x \in \partial D, \; t > 0; \quad u(0, x) = f(x), \; x \in D.
\]

Separation of variables gives the unique (classical) solution as

\[
u(t, x) = \sum_{n=1}^{\infty} \bar{f}(n) \phi_n(x) M_\beta(-\lambda_n t^\beta)
\]

\[= E_x[f(W(Q(t)))I(\tau_D(W) > Q(t))]\]

\[= E_x[f(W(Q(t)))I(\tau_D(W(Q)) > t)]\]

\[= \frac{t}{\beta} \int_0^\infty T_D(l)f(x)g_\beta(tl^{-1/\beta})l^{-1/\beta-1} dl\]

Joint work with Meerschaert and Vellaisamy, AOP (2009).
Analytic solution in intervals $(0, M) \subset \mathbb{R}$ was obtained by Agrawal (2002).
In this case, eigenfunctions and eigenvalues are

$$
\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M), \quad \lambda_n = \frac{\pi^2 n^2}{M^2}
$$

The time fractional diffusion on $(0, M)$ has the solution

$$
u(t, x) = \sum_{n=1}^{\infty} \bar{f}(n) \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M) \mathcal{M}_\beta(-\lambda_n t^\beta)
$$

here

$$
\mathcal{M}_\beta(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(1 + \beta n)}
$$

For $\beta = 1$, $\mathcal{M}_1(-z) = e^{-z}$, and $\nu$ coincides with the solution of the heat equation on $(0, M)$. 
IBM in bounded domains

The (classical) solution of
\[
\frac{\partial}{\partial t}u(t, x) = \frac{\Delta f(x)}{\sqrt{\pi t}} + \Delta^2 u(t, x), \quad x \in D, \; t > 0; \tag{8}
\]
\[
\begin{align*}
\ u(t, x) &= \Delta u(t, x) = 0, \quad t \geq 0, \quad x \in \partial D; \\
\ u(0, x) &= f(x), \quad x \in D
\end{align*}
\]
is given by (running \( l_2(t) = W_1(|W_2(t)|) \), IBM)
\[
\begin{align*}
\ u(t, x) &= E_x[f(l_2(t))I(\tau_D(W_1) > |W_2(t)|)] \\
&= 2 \int_0^{\infty} T_D(l)f(x)h(t, l)dl, \tag{9}
\end{align*}
\]
where \( T_D(l) \) is the heat semigroup in \( D \), and \( h(t, l) \) is the transition density of one-dimensional Brownian motion \( \{W_2(t)\} \).
Proof: equivalence with fractional Cauchy problem for \( \beta = 1/2 \).
Extensions
Open problems

- Extension to Neumann boundary conditions...
- Extension to $\beta > 1$
- Extension to other operators: distributed order, Volterra-type integro-differential operators
- Work in progress for the Subordinated Brownian motions, e.g. symmetric stable process as the outer process....
- Applications-interdisciplinary research
Conclusion

- Math solves real world problems
- Probability is a useful area of Math in solving real world problems