Introduction to Vector Calculus

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5.8 More on Triple Integrals

As we would expect, there is a change of variable theorem for solid integrals that is quite similar to the one for surface integrals.

Theorem 1 (Change of Variables) *Let P and Q be simple solids, and let $\vec{h}$ be a continuous function from P to Q such that:*

(*a*) *$\vec{h}$ is one-to-one on the interior of P;*

(*b*) *$J(\vec{h}(\vec{r}))$ is continuous; and*

(*c*) *$J(\vec{h}(\vec{r})) \neq 0$ on the interior of P.*

If f is a continuous function from Q into $\mathbb{R}$, then

$$\iiint_Q f\ dV = \iiint_P f(\vec{h}(\vec{r}))J(\vec{h}(\vec{r}))\ dV.$$

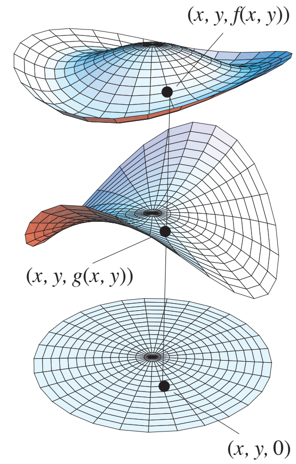

Figure 1.a *Two bounding functions over a surface in the xy–plane.*

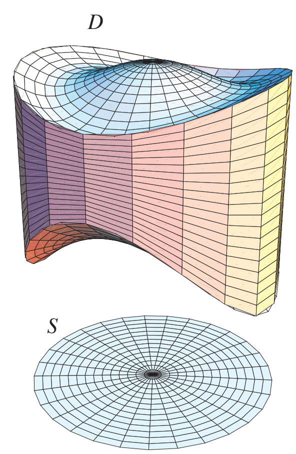

Figure 1.b *A solid bounded between two functions over a surface in the xy–plane.*

We can use the change of variables theorem to derive some formulas for integrating over certain types of solids without having to resort to parametrizations for the solid.

Theorem 2 *Suppose that S is a surface in xy–space and that f and g are defined on S such that $f(x,y) > g(x,y)$ for all (x,y) in S, and f and g have continuous first partial derivatives on S. The set of points (x,y,z) satisfying the conditions that (x,y) is in S and $g(x,y) \leq z \leq f(x,y)$ defines a solid D (see Figure 1). If ρ is a continuous function on D, then*

$$\iiint_D \rho\ dV = \iint_S \int_{g(x,y)}^{f(x,y)} \rho(x,y,z)\ dz\, dS.$$

Proof: Let $\vec{h}$ be a parametrization for S with domain the rectangle bounded by $a \leq u \leq b$ and $c \leq v \leq d$.

First, we prove the theorem for the special case that $f(x,y) = 1$ and $g(x,y) = 0$. Let V be the resulting solid. Let $\vec{h}(u,v) = (x(u,v), y(u,v))$ be a parametrization for S with domain the rectangle bounded by $a \leq u \leq b$ and $c \leq v \leq d$. Let B be the box bounded by the planes $u = a, u = b, v = c, v = d, w = 0$, and $w = 1$. Let $\vec{s}$ be the function from B into V defined by

$$\vec{s}(u,v,w) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ w \end{pmatrix}.$$

It is easily verified that $J(\vec{s}(u,v,w)) = J(\vec{h}(u,v))$. Thus

$$\begin{aligned}
\iiint_V \rho\, dV &= \int_c^d \int_a^b \int_0^1 \rho(u,v,w) J(\vec{s}(u,v,w))\, dw\, du\, dv \\
&= \int_c^d \int_a^b \int_0^1 \rho(u,v,w) J(\vec{h}(u,v))\, dw\, du\, dv \\
&= \iint_S \int_0^1 \rho(u,v,w)\, dw\, dS.
\end{aligned}$$

For the general case of the theorem, let g and f be arbitrary functions satisfying the conditions of the theorem, and let D be the resulting solid. Let $\vec{r}$ be the function defined by

$$\vec{r}(x,y,t) = \begin{pmatrix} x \\ y \\ g(x,y) + t[f(x,y) - g(x,y)] \end{pmatrix}.$$

It is a straightforward computation to show that $\vec{r}(x,y,t)$ takes the solid V (defined in the argument of the special case) onto the solid D, that $J(\vec{r}(x,y,t)) = [f(x,y) - g(x,y)]$, and that $\vec{r}$ satisfies the conditions of the Change of Variable Theorem. Therefore, we have from Theorem 1 that

$$\begin{aligned}
\iiint_D \rho\, dV &= \iiint_V \rho(\vec{r}(x,y,t)) J(\vec{r}(x,y,t))\, dV \\
&= \iint_S \int_0^1 \rho(x,y,g(x,y) + t[f(x,y) - g(x,y)]) \\
&\qquad [f(x,y) - g(x,y)] dt\, dS.
\end{aligned}$$

Letting $u(t) = g(x,y) + t[f(x,y) - g(x,y)]$, we see that $\frac{du}{dt} = f(x,y) - g(x,y)$, $u(0) = g(x,y)$, and $u(1) = f(x,y)$. Thus

$$\iiint_D \rho\, dV = \iint_S \int_{g(x,y)}^{f(x,y)} \rho(x,y,u)\, du\, dS. \qquad \blacksquare$$

EXAMPLE 1: Let D be the solid bounded between the graphs of $z = y,\ z = -y$, and $x^2 + y^2 = 1$ and such that $y \geq 0$. Calculate $\iiint_D z^2\, dV$.

SOLUTION: If we let S be the half disc parametrized by

$$\vec{r}(\rho,\theta) = (\rho\cos(\theta), \rho\sin(\theta))\,,\ 0 \leq \rho \leq 1,\ 0 \leq \theta \leq \pi,$$

then D is the set of points (x, y, z) bounded by the graphs of $z = y$ and $z = -y$ such that (x, y) is in S. See Figures 2a and 2b. Applying Theorem 1, we have

$$\begin{aligned}
\iiint_D z^2\, dV &= \iint_S \int_{-y}^{y} z^2 dz\, dS = \iint_S 2\frac{y^3}{3}\, dS \\
&= \frac{2}{3}\int_0^{\pi}\int_0^1 r^3 \sin^3(\theta) r\, dr\, d\theta \\
&= \frac{2}{15}\int_0^{\pi} \sin^3(\theta)\, d\theta \\
&= \frac{2}{15}\int_0^{\pi} (1-\cos^2(\theta)\sin(\theta)\, d\theta \qquad \text{let } u = \cos(\theta) \\
&= \frac{2}{15}\int_{u=1}^{u=-1} 1 - u^2 du = \frac{4}{45}.
\end{aligned}$$

■

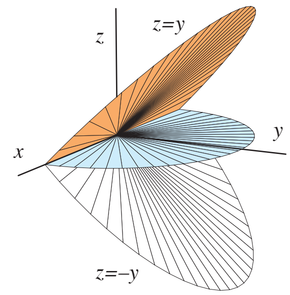

Figure 2.a *The bounding graphs for the solid in Example 1.*

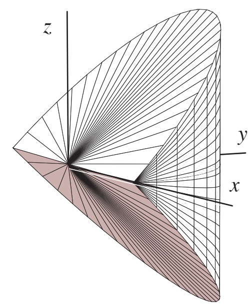

Figure 2.b *The solid in Example 1.*

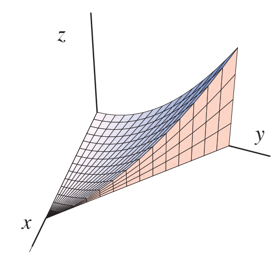

Figure 3. *The solid in Example 2.*

EXAMPLE 2: Let S be the triangle in xy–space with vertices $(0, 1), (1, 0)$, and $(0, 0)$. Let $f(x, y) = y^2$ and $g(x, y) = 0$. Let D be the solid consisting of the points (x, y, z) satisfying the conditions that (x, y) is in S and $0 \leq z \leq y^2$ (See Figure 3). Calculate $\iiint_D xyz\, dV$.

SOLUTION: By Theorem 1,

$$\begin{aligned}
\iiint_D xyz\, dV &= \iint_S \int_0^{y^2} xyz\, dz\, dS \\
&= \iint_S xy\frac{z^2}{2}\Big|_{z=0}^{z=y^2} dS = \iint_S x\frac{y^5}{2}\, dS.
\end{aligned}$$

S is the surface in xy–space that is bounded by the lines $y = 1-x$, the x–axis and the y–axis. Thus

$$\begin{aligned}
&\iint_S x\frac{y^5}{2}\, dS = \int_0^1\int_0^{1-x} x\frac{y^5}{2}\, dy\, dx = \int_0^1 x\frac{y^6}{12}\Big|_{y=0}^{y=1-x} dx \\
&= \frac{1}{12}\int_0^1 x\left(1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6\right)\, dx \\
&= \frac{1}{672}.
\end{aligned}$$

■

EXAMPLE 3: Let S be the surface in xy–space bounded by the x–axis, the line $x = 1$, and the graph of $y = x^2$, and let V be the solid containing the points (x, y, z) that satisfy the conditions that (x, y) is in S and $0 \leq z \leq x + y$. Figures 4.a and 4.b display two views of S.

(a) Compute the volume of V; and

(b) Compute $\iiint_V (x + y + z)\, dV$.

SOLUTION:

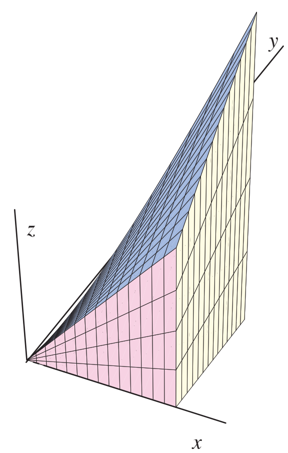

Figure 4.a *The solid in Example 3 viewed from the negative side of the xz–plane.*

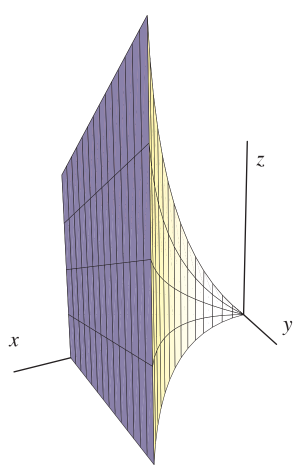

Figure 4.b *The solid in Example 3 viewed from the positive side of the xz–plane.*

(a) The volume of V is given by

$$\begin{aligned}\iiint_V dV &= \iint_S \int_0^{x+y} dz\, dS \\ &= \iint_S (x+y)\, dS = \int_0^1 \int_0^{x^2} (x+y)\, dy\, dx \\ &= \int_0^1 \left(x^3 + \frac{x^4}{2}\right) dx = \frac{1}{4} + \frac{1}{10} = \frac{7}{20}.\end{aligned}$$

(b)

$$\begin{aligned}\iiint_V (x+y+z)\, dV &= \iint_S \int_0^{x+y} (x+y+z)\, dz\, dS \\ &= \int_0^1 \int_0^{x^2} \int_0^{x+y} (x+y+z)\, dz\, dy\, dx \\ &= \int_0^1 \int_0^{x^2} \left(xz + yz + \frac{z^2}{2}\right)\Bigg|_{z=0}^{z=x+y} dy\, dx \\ &= \int_0^1 \int_0^{x^2} \left(2x^2 + 4xy + 2y^2\right) dy\, dx \\ &= \int_0^1 \left(2x^2 y + \frac{4xy^2}{2} + \frac{2y^3}{3}\right)\Bigg|_{y=0}^{y=x^2} dx \\ &= \int_0^1 \left(2x^4 + 2x^5 + \frac{2x^6}{3}\right) dx = \frac{29}{45}.\end{aligned}$$

■

Useful Observation

Examples 2 and 3 point toward a general situation where the problem of parametrizing certain types of solids can be avoided entirely. If

(a) a and b are numbers such that $a < b$.

(b) f and g are functions defined on a subset of the real numbers such that

  (i) for each x between a and b, we have $g(x) < f(x)$; and

  (ii) the set of points in xy–space bounded by the graphs of f and g and the lines $x = a$ and $x = b$ is a simple surface S.

(c) F and G are real valued functions defined on the surface S such that:

  (i) for each (x, y) in the interior of S, we have that $G(x, y) < F(x, y)$; and

  (ii) the set of points in xyz–space satisfying $a \leq x \leq y, g(x) \leq y \leq f(x)$, and $G(x, y) \leq z \leq F(x, y)$ is a simple solid V,

then

$$\begin{aligned} \iiint_V \rho(x, y, z)\ dV &= \iint_S \int_{G(x,y)}^{F(x,y)} \rho(x, y, z)\ dz\, dS \\ &= \int_a^b \int_{g(x)}^{f(x)} \int_{G(x,y)}^{F(x,y)} \rho(x, y, z)\ dz\, dy\, dx. \end{aligned}$$

EXAMPLE 4: Describe the solid that is the domain of the integral

$$\int_0^1 \int_0^{x^2} \int_{\sin(xy)}^{1+x+y} f(x, y, z) dz\, dy\, dx.$$

Solution: The integral

$$\int_0^1 \int_0^{x^2} \int_{\sin(xy)}^{1+x+y} f(x,y,z)dz\,dy\,dx$$

is of the form

$$\iint_S \int_{\sin(xy)}^{1+x+y} f(x,y,z)dz\,dy\,dx,$$

where S is the region in the plane bounded by the x–axis and the graphs of $x=1$ and $y=x^2$. The solid is bounded above by the graph of $f(x,y)=1+x+y$ and below by $g(x,y)=\sin(xy)$. Thus the solid is the set of points (x,y,z) such that $\sin(xy) \leq z \leq (1+x+y)$, $0 \leq x \leq 1$ and $0 \leq y \leq x^2$. See Figure 5. ■

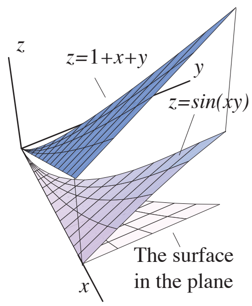

Figure 5. *The bounding surfaces for the solid in Example 4.*

EXAMPLE 5: The integral

$$\int_0^1 \int_{-z}^{z^2} \int_0^{y+z} f(x,y,z)dx\,dy\,dz$$

is of the form

$$\iint_S \int_0^{y+z} f(x,y,z)dx\,dS.$$

In this case, the surface S is the region in the yz–plane bounded between the graphs of $y=-z$, $y=z^2+1$ and the lines $z=0$ and $z=1$. The point (x,y,z) is in the domain of the integral provided $0 \leq z \leq 1$, $-z \leq y \leq 1+y^2$, and $0 \leq x \leq y+z$. The bounding surfaces and the solid are illustrated in Figure 6.

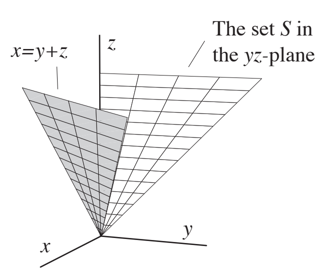

Figure 6.a *The bounding surfaces for the domain of the integral in Example 5.*

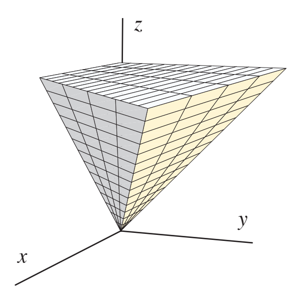

Figure 6.b *The domain of the integral in Example 5.*

In general:

> An expression of the form $\int_S \int_\alpha^\beta f(x,y,z)dz\,dS$ only makes sense if S is a surface in the xy–coordinate plane. α and β must be functions of x and y.

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EXAMPLE 6: The expression $\int_0^1 \int_{-x}^1 \int_0^z dz\,dy\,dx$ does not make sense. This would be expected to be an integral of the form $\int_S \int_\alpha^\beta f(x, y,z) dz\,dS$. However, β is not a function of x and y. ■

EXERCISES 5.8

In Exercises 1–4, evaluate the triple integral.

1. $\displaystyle\int_0^1 \int_{-z}^0 \int_{-y}^z dx\,dy\,dz.$

2. $\displaystyle\int_{-1}^1 \int_0^z \int_{x+z}^{z^2} x\ dy\,dx\,dz.$

3. $\displaystyle\int_0^1 \int_{-x}^{x^2} \int_0^{y+x} x+y\ dz\,dy\,dx.$

4. $\displaystyle\int_0^1 \int_{y^3}^{y^2} \int_{-x}^y xy\ dz\,dx\,dy.$

In Exercises 5–9 the integral is of the form $\iint_S \int_{F(y,z)}^{G(y,z)} f(x,y,z)\,dx\,dS$, $\iint_S \int_{F(x,z)}^{G(x,z)} f(x,y,z)\,dy\,dS$, *or* $\int_S \int_{F(x,y)}^{G(x,y)} f(x,y,z)\,dz\,dS$. *Describe the set S and determine which of the coordinate planes it is in.*

5. $\displaystyle\int_0^1 \int_0^2 \int_{-1}^2 f(x,y,z)\ dx\,dy\,dz.$

6. $\displaystyle\int_0^1 \int_{-z}^0 \int_{-y}^z f(x,y,z)\ dx\,dy\,dz.$

7. $\displaystyle\int_0^2 \int_{-x}^{2+x^2} \int_{-y}^{x+y} f(x,y,z)\ dz\,dy\,dx.$

8. $\displaystyle\int_0^2 \int_{-x}^{2+x^2} \int_{-x}^{x+z} f(x,y,z)\ dy\,dz\,dx.$

9. $\displaystyle\int_0^2 \int_{-z}^{2+z^2} \int_{-z}^{y+z} f(x,y,z)\ dx\,dy\,dz.$

In Exercises 10–15, $f(x,y,z)$ is a continuous function. Determine whether the expression makes sense. If the expression is integrable, describe the domain of the integral.

10. $\displaystyle\int_0^1 \int_{-z}^{0} \int_{-y}^{z} f(x,y,z)\,dx\,dy\,dz.$

11. $\displaystyle\int_0^1 \int_{-z}^{0} \int_{-y}^{z} f(x,y,z)\,dz\,dx\,dy.$

12. $\displaystyle\int_0^1 \int_{-z}^{0} \int_{-y}^{z} f(x,y,z)\,dy\,dx\,dz.$

13. $\displaystyle\int_0^1 \int_{-z}^{0} \int_{0}^{x+y+z} f(x,y,z)\,dx\,dy\,dz.$

14. $\displaystyle\int_0^1 \int_{-z}^{0} \int_{-y^2}^{y+z} f(x,y,z)\,dz\,dy\,dx.$

15. $\displaystyle\int_0^1 \int_{-z}^{0} \int_{-y^2}^{y+x} f(x,y,z)\,dx\,dy\,dz.$

In Exercises 16–22, find the volume of the solid V consisting of all points (x,y,z) such that (x,y) is in the surface S in xy–space and $G(x,y) \le z \le F(x,y)$.

16. S is the region bounded by the x–axis, the graph of $y = x^2$, and the line $x = 2$. $G(x,y) = -x$ and $F(x,y) = x + y$.

17. S is the region bounded by the graphs of $y = x^2$ and $y = -x^2$, and the line $x = 2$. $G(x,y) = -x$ and $F(x,y) = x + y$.

18. S is the region bounded by the graphs of $y = x$ and $y = x^2$. $G(x,y) = -xy$ and $F(x,y) = x^2$.

19. S is the region bounded by the circle $x^2+y^2 = 1$. $G(x,y) = 0$ and $F(x,y) = 2 - x$.

20. S is the parallelogram with two adjacent edges the line segments $[\vec{A},\vec{B}]$ and $[\vec{A},\vec{C}]$, where $\vec{A} = (0,0), \vec{B} = (1,1)$, and $\vec{C} = (1,2)$. $G(x,y) = -2$ and $F(x,y) = x + y$.

21. S is the parallelogram with two adjacent edges the line segments $[\vec{A},\vec{B}]$ and $[\vec{A},\vec{C}]$, where $\vec{A} = (0,0), \vec{B} = (1,1)$, and $\vec{C} = (1,3)$. $G(x,y) = x - y$ and $F(x,y) = x^2 + y + 1$.

22. S is the triangle with vertices $\vec{A} = (0,0), \vec{B} = (1,1)$, and $\vec{C} = (1,2)$. $G(x,y) = -2$ and $F(x,y) = x + y$.

Find the volume of the solid V in Exercises 23–28.

23. S is the surface in xz–space bounded by the x–axis, the graph of $z = x^2$, and the line $x = 2$. V is the set of points (x,y,z) such that (x,z) is in S and $-x \le y \le x + z$.

24. S is the surface in yz–space bounded by the y–axis, the graph of $z = y^2$, and the line $y = 2$. V is the set of points of (x,y,z) such that (y,z) is in S and $-y \le x \le y + z$.

25. S is the region in xz–space bounded by the graphs of $x = z^2$, $x = -z^2$, and the line $z = 2$. V is the set of points (x,y,z) such that (x,z) is in S and $-z \le y \le z$.

26. S is the region in yz–space bounded by the graphs of $y = z^2$ and $y = 8 - z^2$. V is the set of points (x,y,z) such that (y,z) is in S and $-y \le x \le y$.

27. S is the triangle in yz–space with vertices $(1,0), (2,2)$, and $(3,1)$. V is the set of points (x,y,z) such that (y,z) is in S and $-y \le x \le z$.

28. S is the set of points in xz–space satisfying $\frac{x^2}{4} + z^2 \le 1$, and V is the set of points (x,y,z) such that (x,z) is in S and $0 \le y \le x + 5$.

In Exercises 29–33, find the volume of the solid bounded by the graphs (in xyz–space) of the given equations.

29. $y = x^2, y + z = 1$, and $z = 0$.

30. $x + y + z = 1$ and the three coordinate planes.

31. $z = x^2, z = 1 - x^2, y = 0$, and $y = z$.

32. $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.

33. $x^2 + y^2 - 1 = z$ and $1 - x^2 - y^2 = z$.

In Exercises 34–40, calculate $\iiint_V f(x, y, z)\, dV$.

34. V is defined in Exercise 16, $f(x, y, z) = x$.

35. V is defined in Exercise 16, $f(x, y, z) = xy$.

36. V is defined in Exercise 17, $f(x, y, z) = x+y+z$.

37. V is defined in Exercise 20, $f(x, y, z) = x^2 + y^2$.

38. V is defined in Exercise 22, $f(x, y, z) = xy + z$.

39. V is defined in Exercise 23, $f(x, y, z) = x+y+z$.

40. V is defined in Exercise 25, $f(x, y, z) = x+y+z$.