Introduction to Vector Calculus

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October 15, 2008

5.4 Some Applications of Surface Integrals

In the development of the surface integral in Section 14.3, we parametrized the surface with a function $\vec{h}$ having as its domain a rectangle R. We then partitioned R into small rectangles $R_{i,j}$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and then used the $R_{i,j}$'s, via $\vec{h}$, to divide the surface S into nonoverlapping pieces $S_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, where $S_{i,j} = \vec{h}(R_{i,j})$. We then chose selection points $\vec{s}_{i,j}$ from $R_{i,j}$ and calculated

$$\underbrace{\sum\sum}_{\substack{\text{Sum of}\\ \text{a lot of}\\ \text{pieces of}\\ \text{the}\\ \text{surface}}} \underbrace{f(\vec{h}(\vec{s})_{i,j}))}_{\substack{f \text{ evaluated}\\ \text{at a point}\\ \text{in a piece of}\\ \text{the surface}}} \underbrace{[J(\vec{h}(\vec{s}_{i,j}))\text{Area}(R_{i,j})]}_{\substack{\text{Approximation of}\\ \text{the area of a small}\\ \text{piece of the surface}}} \text{ as our approximation for } \iint_S f \, dS.$$

While there is no rigor in the following, it is helpful in applications of surface integrals to associate $\iint_S f \; dS$ with the above mentioned sums; that is,

$$\underbrace{\sum\sum}_{\substack{\text{Sum of}\\ \text{many}\\ \text{pieces of}\\ \text{the}\\ \text{surface}}} \underbrace{f(\vec{h}(\vec{s}_{i,j}))}_{\substack{f \text{ evaluated}\\ \text{at a point}\\ \text{in a piece of}\\ \text{the surface}}} \underbrace{[J(\vec{h}(\vec{s}_{i,j}))\text{Area}(R_{i,j})]}_{\substack{\text{Approximation of}\\ \text{the area of a small}\\ \text{piece of the surface}}} \approx \underbrace{\iint_S}_{\substack{\text{Sum of}\\ \text{many}\\ \text{pieces of}\\ \text{the}\\ \text{surface}}} \quad \underbrace{f}_{\substack{f \text{ evaluated}\\ \text{at a point}\\ \text{in a piece of}\\ \text{the surface}}} \quad \underbrace{dS}_{\substack{\text{Approximation}\\ \text{of the area of}\\ \text{a small piece}\\ \text{of the surface}}}$$

A Rule of Thumb for Surface Integrals

Generally, surface integrals will be applied in the following situation:

(a) S is a simple surface and ρ is a real-valued function defined on S.

(b) Q is a physical quantity that can be approximated by:

(i) Breaking S into "very small" non overlapping pieces $S_{i,j}, i = 1, \ldots, n,\ j = 1,\ \ldots,\ m$.

(ii) Choosing an arbitrary "selection point" $\vec{s}_{i,j}$ from each $S_{i,j}$.

(iii) Calculating the sum $\sum\sum \rho(\vec{s}_{i,j})\ \text{Area}(S_{i,j})$ to approximate Q.

EXAMPLE 1: Suppose that S is a surface in $\mathbb{R}^3$ and $\rho(x, y, z)$ gives the mass density of S at (x, y, z). Assuming that ρ is continuous, if ΔS is a very small piece of S and $\vec{s}$ is a point of ΔS, then the product of $\rho(\vec{s})$ and $\text{Area}(\Delta S)$ will be an approximation of the mass of ΔS. Thus, using our rule of thumb above, $\iint_S \rho\ dS$ is the mass of S.

EXAMPLE 2: Let S be the surface in $\mathbb{R}^2$ bounded by the graphs of $y = x^2$, the x–axis, and the line $x = 1$. See Figure 1.

Let $\rho(x, y) = xy$ denote the mass density of S at (x, y). Then the mass of S is given by $\iint_S xy\ dS$. We can employ Theorem 1 of the previous section to calculate this integral without parametrizing S.

$$\begin{aligned}\iint_S xy\ dS &= \int_0^1 \int_0^{x^2} xy\ dy\, dx = \int_0^1 \frac{xy^2}{2}\Big|_{y=0}^{y=x^2}\ dx \\ &= \int_0^1 \frac{x^5}{2}\ dx = \frac{1}{12}. \qquad \blacksquare\end{aligned}$$

In Section 11.5 we computed the center of mass for a thin wire that was modeled by a fundamental curve. We can apply these same techniques to find the center of mass for a surface in $\mathbb{R}^3$. Let S be a surface parametrized by $\vec{r}$. Partition S by $\mathcal{P} = \{S_{i,j} \mid 1 \le i \le n, 1 \le j \le m\}$, and let $\mathcal{S} = \{\vec{s}_{1,1}, \ldots, \vec{s}_{n,m}\}$ be a selection from $\mathcal{P}$,

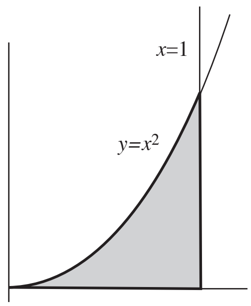

Figure 1. *The surface bounded by the graphs of $y = x^2$, the x–axis, and the line $x = 1$.*

where $\vec{s}_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})$. Assume that there is a density function $\rho(x, y, z)$ on S. If we let X be the number approximated by

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{m} x_{i,j}\rho(\vec{s}_{i,j})(\text{area of } S_{i,j})}{\text{total mass of } S},$$

then

$$X = \frac{\iint_S x\rho(\vec{r})\ dS}{\iint_S \rho(\vec{r})\ dS}.$$

Similarly, let

$$Y = \frac{\iint_S y\rho(\vec{r})\ dS}{\iint_S \rho(\vec{r})\ dS} \quad \text{and} \quad Z = \frac{\iint_S z\rho(\vec{r})\ dS}{\iint_S \rho(\vec{r})\ dS}.$$

The point (X, Y, Z) is called the *center of mass* of S. As in the case of curves, there is no reason that the center of mass need be on the surface.

EXAMPLE 3: Let S be the upper hemisphere of the sphere of radius R with center at the origin. Let $\rho(x, y, z) = 1$ be the mass density function. Find the center of mass of S.

SOLUTION: We parametrize S by $\vec{r}(s, t) = (R\cos(s)\sin(t), R\sin(s)\sin(t), R\cos(t))$ for $0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}$. The Jacobian of $\vec{r}$ is $R^2\sin(t)$. Thus

$$X = \frac{\iint_S x\rho(\vec{r})\ dS}{\iint_S \rho(\vec{r})\ dS} = \frac{\int_0^{\pi/2}\int_0^{2\pi} R\cos(s)\sin(t)R^2\sin(t)\ ds\,dt}{\int_0^{\pi/2}\int_0^{2\pi} R^2\sin(t)\ ds\,dt} = \frac{0}{2\pi R^2};$$

$$Y = \frac{\iint_S y\rho(\vec{r})\ dS}{\iint_S \rho(\vec{r})\ dS} = \frac{\int_0^{\pi/2}\int_0^{2\pi} R\sin(s)\sin(t)R^2\sin(t)\ ds\,dt}{\int_0^{\pi/2}\int_0^{2\pi} R^2\sin(t)\ ds\,dt} = \frac{0}{2\pi R^2};$$

$$Z = \frac{\iint_S z\rho(\vec{r})\ dS}{\iint_S \rho(\vec{r})\ dS} = \frac{\int_0^{\pi/2}\int_0^{2\pi} R\cos(t)R^2\sin(t)\ ds\,dt}{\int_0^{\pi/2}\int_0^{2\pi} R^2\sin(t)\ ds\,dt} = \frac{\pi R^3}{2\pi R^2} = \frac{R}{2}.$$

$$(X, Y, Z) = \left(0, 0, \frac{R}{2}\right). \qquad \blacksquare$$

EXAMPLE 4: Suppose that S is a simple surface with a mass density function $\rho(\vec{r})$ and that S is rotating about the z–axis with an angular speed w rad/[unit of time]. If ΔS is a small piece of S

and $\vec{s} = (x, y, z)$ is a point of ΔS, then $w(x^2 + y^2)^{1/2}$ approximates the speed of ΔS and $\rho(x, y, z)$ area(ΔS) approximates the mass of ΔS. Thus

$$\frac{1}{2}\underbrace{\left[w\sqrt{x^2+y^2}\right]^2}_{\substack{\text{Approximation}\\ \text{of the speed}\\ \text{squared of a}\\ \text{piece of the}\\ \text{surface.}}}\underbrace{\rho(x,y,z)\ \text{area}(\Delta S)}_{\substack{\text{Approximation}\\ \text{of the mass}\\ \text{of a piece of}\\ \text{the surface.}}}$$

approximates the kinetic energy $\frac{mv^2}{2}$ of ΔS. If we break S up into small non overlapping pieces and add up the kinetic energy of each piece of S, then we have the total kinetic energy of the rotating mass. Therefore, by our rule of thumb,

> $\displaystyle\iint_S \frac{1}{2}[w^2(x^2+y^2)]\rho(x,y,z)\ dS$ is the kinetic energy of the surface rotating about the z–axis.

Of course we get similar formulas for the kinetic energy of a surface rotating about the x–axis or the y–axis:

> $\displaystyle\iint_S \frac{1}{2}[w^2(x^2+z^2)]\rho(x,y,z)\ dS$ is the kinetic energy of a surface rotating about the y–axis with angular speed w and:
>
> $\displaystyle\iint_S \frac{1}{2}[w^2(y^2+z^2)]\rho(x,y,z)\ dS$ is the kinetic energy of a surface rotating about the x–axis with angular speed w. ∎

Note that the formulas for the kinetic energy of a rotating surface parallel those for fundamental curves as given in Section 11.5. We can define the moments of inertia similarly.

Definition: Moment of Inertia

$I_x = \displaystyle\iint_S (y^2 + z^2)\rho(x, y, z)\ dS$ = moment of inertia about the x–axis.

$I_y = \displaystyle\iint_S (x^2 + z^2)\rho(x, y, z)\ dS$ = moment of inertia about the y–axis.

$I_z = \displaystyle\iint_S (x^2 + y^2)\rho(x, y, z)\ dS$ = moment of inertia about the z–axis.

The kinetic energy of a surface rotating about an axis with angular speed w is given by $\frac{1}{2}w^2 I_a$, where $a = x, y$, or z indicates the axis of rotation.

EXAMPLE 5: Let S and ρ be as in Example 2. If S is rotating about the x–axis at 2π rad/sec, what is the total kinetic energy (where distance is measured in meters)?

SOLUTION:

$$\begin{aligned}
\text{Kinetic Energy} &= \frac{1}{2}w^2 I_x = \frac{1}{2}w^2 \iint_S (y^2 + z^2)\rho(x,y,z)\ dS \\
&= \frac{1}{2}w^2 \iint_S (y^2+z^2)xy\ dS \\
&= \frac{1}{2}(2\pi)^2 \int_0^1 \int_0^{x^2} y^2 xy\ dy\, dx \\
&= 2\pi^2 \int_0^1 \int_0^{x^2} y^2 xy\ dy\, dx \\
&= 2\pi^2 \int_0^1 \left(\frac{y^4}{4}\right) x \Big|_{y=0}^{y=x^2}\ dx \\
&= 2\pi^2 \int_0^1 \left(\frac{x^8}{4}\right) x\ dx = \frac{\pi^2}{20} \text{ Joules.}
\end{aligned}$$

■

Volumes of Solids Bounded Between Graphs of Functions

Suppose that S is a simple surface lying in xy–space, and that f and g are continuous functions from S into $\mathbb{R}$ such that $f(x,y) \geq g(x,y)$ for all (x,y) in S. Then the set of all points (x,y,z) such that (x,y) is in S and $f(x,y) \geq z \geq g(x,y)$ is a solid V as in Figures 2.a and 2.b.

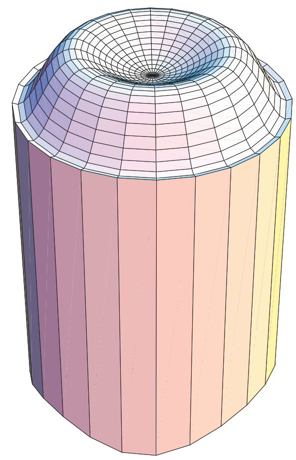

Figure 2.a *V is the set of all points (x,y,z) such that (x,y) is in S and $f(x,y) \geq z \geq g(x,y)$.*

Let ΔS be a small piece of S and let $\vec{s}$ be a point in ΔS. Then the part of V lying "above" ΔS can be approximated by $[f(\vec{s}) - g(\vec{s})]$ area(ΔS). By our rule of thumb, the total volume of V must be $\iint_S (f-g)\, dS$.

In order to calculate the volume of a solid using the above method, we must be able to identify the top and bottom boundaries of the solid. It is often useful to sketch the solid, but this is not aways easy, as illustrated by the next example.

EXAMPLE 6: Find the volume of the region inside the cylinder $x^2 + y^2 = 1$ and between $z = 1 - x$ and $z = -(x^2 + y^2)$.

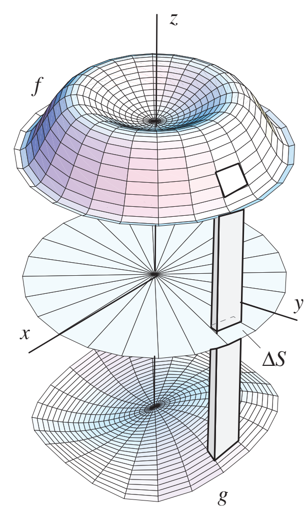

Figure 2.b *That part of V lying "above" ΔS.*

In Figure 3.a, we draw the top bounding function, and the bottom bounding graph is sketched in Figure 3.b. In Figures 4.a and 4.b, we give views of the solid from above and from below. It is the concave bottom bounding surface that makes the solid difficult to visualize.

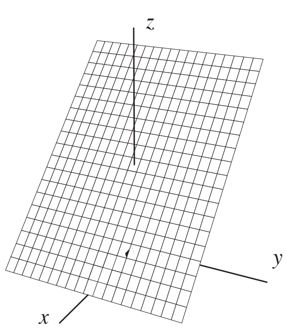

Figure 3.a *The top bounding function $z = 1 - x$.*

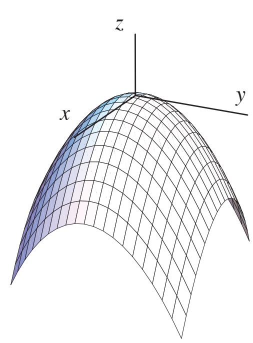

Figure 3.b *The bottom bounding function $z = -(x^2 + y^2)$.*

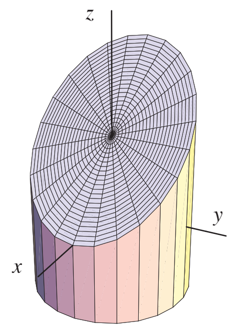

Figure 4.a *The solid viewed from above.*

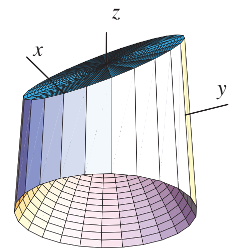

Figure 4.b *The solid viewed from below.*

Solution: Let S denote the disc $(x^2+y^2) \leq 1$. Note that $-(x^2+y^2) < 1-x$, for all x and y in S. Thus the volume is given by $\iint_S (f-g)\; ds$, where $f(x,y) = 1-x$, and $g(x,y) = -(x^2+y^2)$. The polar transformation gives a convenient parametrization for S:

$$\vec{\rho}(r,\theta) = (r\cos(\theta), r\sin(\theta)), 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi.$$

Since $J(\vec{\rho}(r,\theta)) = r$, we have

$$\begin{aligned}
\iint_S (f-g)\, dS &= \iint_S (1-x+x^2+y^2)\; dS \\
&= \int_0^{2\pi}\int_0^1 (1-r\cos(\theta)+r^2)r\; dr\, d\theta \\
&= \int_0^{2\pi} \left[\frac{r^2}{2} - \left(\frac{r^3}{3}\right)\cos(\theta) + \frac{r^4}{4}\right]\Bigg|_{r=0}^{r=1} d\theta \\
&= \int_0^{2\pi} \left[\frac{3}{4} - \left(\frac{1}{3}\right)\cos(\theta)\right]\; d\theta \\
&= \frac{3\pi}{2}.
\end{aligned}$$

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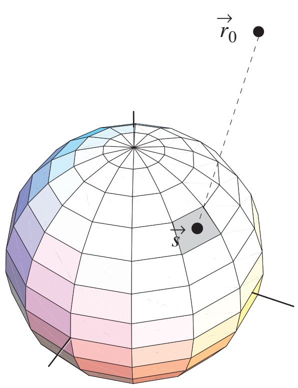


Figure 5. *The electric potential at $\vec{r}_0$ due to the charged piece of surface ΔS is approximated by*

$$\frac{\kappa\rho Area(\Delta S)}{\|\vec{s}-\vec{r}_0\|}.$$

EXAMPLE 7: Recall that if q is a point charge at $\vec{r}_0$, then the electric potential at a position $\vec{r}$ is $V(\vec{r}) = \frac{k}{\|\vec{r}-\vec{r}_0\|}$. Suppose that the unit sphere carries a charge that is uniformly distributed on the sphere. The charge density is a constant ρ measured in Coulombs/m^2. Let $\vec{r}_0$ be a fixed point in space and let $\vec{s}$ be a point in a small piece of the sphere ΔS, as illustrated in Figure 5. Then the electric potential at $\vec{r}_0$ due to the charged piece of surface ΔS is approximated by

$$\frac{\kappa\rho\text{Area}(\Delta S)}{\|\vec{s}-\vec{r}_0\|}.$$

Therefore, if we divide the sphere into non overlapping pieces $S_{i,j}$ and pick selection points $\vec{s}_{i,j}$, then the electric field at $\vec{r}_0$ due to the charged sphere will be approximated by the sum

$$\sum\sum\left(\frac{\kappa\rho}{\|\vec{s}_{i,j}-\vec{r}_0\|}\right)\text{Area}(S_{i,j}).$$

We conclude that the potential at $\vec{r}_0$ due to the charges sphere is

$$\iint_S \frac{\rho}{\|\vec{r}_0-\vec{r}\|}\,dS = \kappa\iint_S \frac{\rho}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}\,dS$$

$$\int_0^{\pi}\int_0^{2\pi} \frac{1}{\sqrt{(x_0-\cos(\theta)\sin(\phi))^2+(y_0-\sin(\theta)\sin(\phi))^2+(z_0-\cos(\phi))^2}}\sin(\phi)\,d\theta\,d\phi.\blacksquare$$

EXERCISES 5.4

1. The mass of a surface is 10 kg. The mass is uniformly distributed over the surface. The surface area is 5 m^2. What is the mass density of the surface?

2. The mass of a surface is M. The mass is uniformly distributed over the surface. The surface area is A. What is the mass density of the surface?

In Exercises 3–11, S is the region in the xy–plane bounded by the graph of $y = x$, the x–axis, and the line $x = 2$. The mass density of S is given by $\rho(x, y) = x + y$.

3. Find the mass of S.

4. Find the center of mass of S.

5. Find the kinetic energy of S if it is rotating about the x–axis with an angular speed of π rad/sec.

6. Find the kinetic energy of S if it is rotating about the y–axis with an angular speed of $\frac{\pi}{2}$ rad/sec.

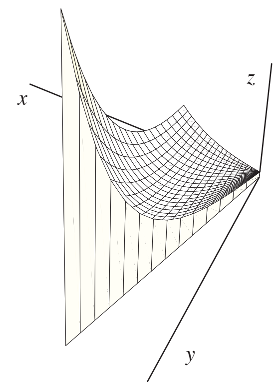
7. Find the kinetic energy of S if it is rotating about the origin in the xy–plane with an angular speed of 2π rad/sec.

8. Find the kinetic energy of S if it is rotating about the line $x = 2$ with an angular speed of 2π rad/sec.

9. Find the kinetic energy of S if it is rotating about the point $(-1, -1)$ in the xy–plane with an angular speed of 2π rad/sec.

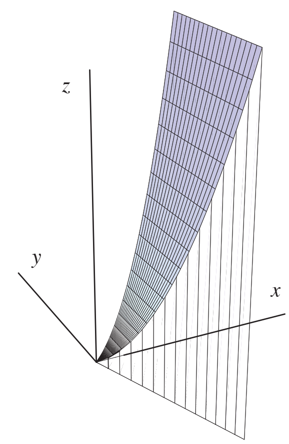
10. Find the volume of the solid consisting of all points (x, y, z) where (x, y) is in S and $0 \le z \le e^{x+y}$.

Figure 6
Solid in Exercise 10.



11. Find the volume of the solid consisting of all points (x, y, z), where (x, y) is in S and $-x \le z \le x^2$.

Figure 7
Solid in Exercise 11.



In Exercises 12–19, S is the region in the xy–plane bounded by the graphs of $y = x$ and $y = x^2$. The mass density of S is given by $\rho(x, y) = x^2 y$.

12. Find the mass of S.

13. Find the center of mass of S.

14. Find the kinetic energy of S if it is rotating about the x–axis with an angular speed of 2π rad/sec.

15. Find the kinetic energy of S if it is rotating about the y–axis with an angular speed of 2π rad/sec.

16. Find the kinetic energy of S if it is rotating about the z–axis with an angular speed of 2π rad/sec.

17. Find the kinetic energy of S if it is rotating about the line $x = 1$ in the xy–plane with an angular speed of 2π rad/sec.

18. Find the volume of the solid consisting of all points (x, y, z), where (x, y) is in S and $0 \le z \le x + y$.

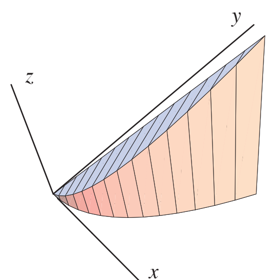


Figure 8
The solid in Exercise 18.

19. Find the volume of the solid consisting of all points (x, y, z), where (x, y) is in S and $x^2 \le z \le x$.

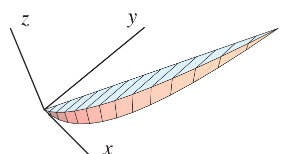


Figure 9
The solid in Exercise 19.

20. Let S be the region in the xy–plane bounded by $xy = 4$ and $x + y = 5$. Find the volume of the solid containing (x, y, z) if (x, y) is in S and $0 \le z \le x + y$.

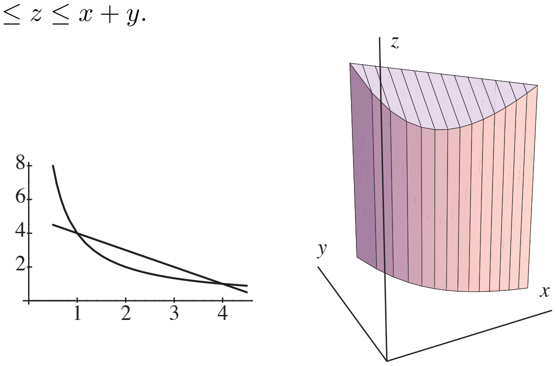


Figure 10. *The region S and the solid for Exercise 20.*

In Exercises 21–25, S is the parallelogram with two adjacent edges the line segments $[\vec{A}, \vec{B}]$ and $[\vec{A}, \vec{C}]$, where $\vec{A} = (1, 2, 1)$, $\vec{B} = (0, 1, 0)$, and $\vec{C} = (1, 1, 1)$.

21. Find the mass of S if the mass density of S is given by $\rho(x, y, z) = x + z$.

22. Find the center of mass of S if the mass density of S is given by $\rho(x, y, z) = x + z$.

23. Find the kinetic energy of S if it is rotating about the z–axis with angular speed of 2π rad/sec and if it has a constant mass density k.

24. Find the kinetic energy of S if it is rotating about the x–axis with an angular speed of 2π rad/sec and if it has a constant mass density k.

25. Find the kinetic energy of S if it is rotating about the z–axis with an angular speed of 2π rad/sec and if its mass density is as in Exercise 21.

26. The surface of a sphere of radius 1 m with a constant mass density ρ is rotating about a line passing through the center of the sphere at a rate of f rotations/sec. Find its kinetic energy.

In Exercises 27 and 28, a disc of radius R with constant mass density ρ is rotating about a line passing through its center at a rate of f rotations/sec.

27. Find the disc's kinetic energy if the axis of rotation lies in the plane containing the disc.

28. Find the disc's kinetic energy if the axis of rotation is perpendicular to the disc.

29. Let P be the plane passing through the origin that is normal to the vector $(1, 0, 1)$. Let V be the solid containing the set of points (x, y, z) that are below P, above the xy–coordinate plane, and lying inside the cylinder $x^2 + y^2 \le 1$. Find the volume of V.

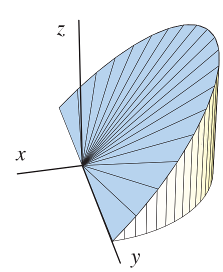
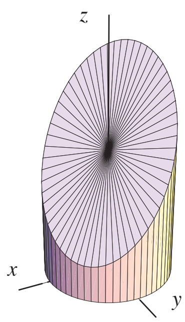


Figure 11
The solid V for Exercise 29.

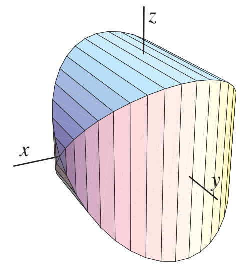
30. Let P be the plane passing through the point $(0, 0, 2)$ that is normal to the vector $(1, 1, 1)$. Let V be the solid containing the set of points (x, y, z) that are below P, above the xy–coordinate plane, and lying inside the cylinder $x^2 + y^2 \le 1$. Find the volume of V.

Figure 12
The solid V for Exercise 30.



31. Express the volume of the intersection of the cylinders $x^2 + y^2 \le 1$ and $x^2 + z^2 \le 1$ as a double integral.

Figure 13
The solid V for Exercise 31.



32. An electric charge of 5 Coulombs is uniformly distributed on a surface with area 3 m^2. What is the charge density?

33. An electric charge is uniformly distributed on a sphere as in Example 7. Show that the electric potential is zero at the center of the sphere.

34. The vectors $(1, 2, 3)$ and $(1, 0, 0))$ are drawn emanating from the the vector $(1, 1, 1)$ to form adjacent edges of a parallelogram P. An electric charge is uniformly distributed on P. If $\vec{r}$ is **not** on the parallelogram, $V(\vec{r})$ denotes the electric potential at $\vec{r}$ due to the charge on P. If ρ denotes the charge density on P, express $V(\vec{r})$ as a double integral.

5.5 Change of Variables

In Section 6.2 we developed the chain rule in reverse for integration over a single variable. We now extend this result for integrals over a surface. The derivative of a parametrization for a fundamental curve, which is used in the chain rule in reverse, generalizes to the Jacobian of a parametrization for a simple surface. Thus it should not be too surprising to see the Jacobian in the following Change of

Variables Theorem.

Theorem 1 (Change of Variables) *Let P and Q be surfaces, and let $\vec{h}$ be a differentiable function from P to Q such that*

(a) *$\vec{h}$ is one-to-one on the interior of P;*

(b) *$J(\vec{h}(\vec{r}))$ is continuous; and*

(c) *$J(\vec{h}(\vec{r})) \neq 0$ on the interior of P.*

If f is a continuous function from Q into $\mathbb{R}$, then

$$\iint_Q f \, dS = \iint_P f(\vec{h}(\vec{r}))J(\vec{h}(\vec{r})) \, dS.$$

Notice that this theorem allows you to evaluate the surface integral by relating the surface to some other surface over which you can integrate. This is more flexible than using the definition of surface integral, which required that the surface in question be parametrized over a rectangle.

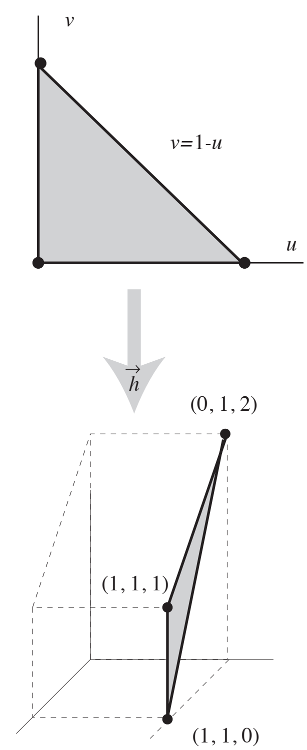


Figure 1. *The triangle in uv–space is taken onto the triangle in xy–space.*

EXAMPLE 1: Let Q be the triangle with vertices $\vec{A} = (1,1,1)$, $\vec{B} = (0,1,2)$, and $\vec{C} = (1,1,0)$. Calculate $\iint_Q x + y + z \, dS$.

Solution: We let P be the triangle in uv–space with vertices $(0,0), (1,0)$, and $(0,1)$, and let $\vec{T}$ be the linear transformation that takes $(1,0)$ onto $\vec{B}-\vec{A} = (-1,0,1)$ and $(0,1)$ onto $\vec{C}-\vec{A} = (0,0,-1)$. If $\vec{h}$ is defined by $\vec{h}(\vec{r}) = \vec{A} + \vec{T}(\vec{r})$, then $\vec{h}$ takes the unit square onto the parallelogram with adjacent edges the line segments $[\vec{A},\vec{B}]$ and $[\vec{A},\vec{C}]$, and $\vec{h}$ takes the triangle P in uv–space with vertices $(0,0), (1,0)$, and $(0,1)$ onto Q. See Figure 1.

$$\vec{h}(u,v) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1-u \\ 1 \\ 1+u-v \end{pmatrix}$$

and

$$J(\vec{h}(u,v)) = 1.$$

Thus

$$\begin{aligned} \iint_Q f \, dS &= \iint_P f(\vec{h}(\vec{r}))J(\vec{h}(\vec{r})) \, dS \\ &= \int_0^1 \int_0^{1-u} (1-u) + (1) + (1+u-v) \, dv \, du \end{aligned}$$

$$= \int_0^1 \int_0^{1-u} 3 - v \; dv\, du = \int_0^1 3(1-u) - \frac{(1-u)^2}{2} \; du$$

$$= \int_0^1 \frac{5}{2} - 2u - \frac{u^2}{2} \; du = \frac{5}{2} - 1 - \frac{1}{6} = \frac{4}{3}. \qquad \blacksquare$$

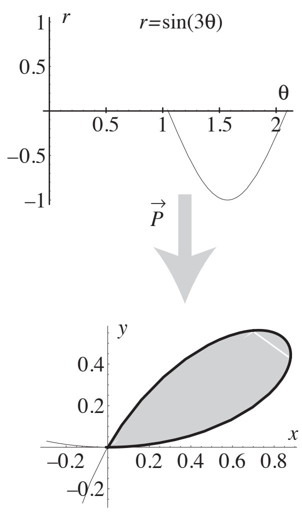


Figure 2. *The polar function $\vec{P}$ takes the region in $r\theta$–space onto the region in xy–space.*

EXAMPLE 2: Find the area of the region in xy–space that is bounded by the set with polar equation $r = \sin(3\theta), 0 \le \theta \le \frac{\pi}{3}$.

SOLUTION: We start by sketching the region in $r\theta$–space and in xy–space as shown in Figure 2. Let Q be the region in xy–space, and let P be the region in $r\theta$–space bounded by the graph of $r = \sin(3\theta), 0 \le \theta \le \frac{\pi}{3}$, and the r–axis. Then the polar transformation $\vec{P}$ takes P onto Q.

$$\text{Area} = \iint_Q dS = \iint_P J(\vec{P}(\vec{r}))\; dS = \int_0^{\pi/3} \int_0^{\sin(3\theta)} r\; dr\, d\theta$$

$$= \int_0^{\pi/3} \frac{\sin^2 3\theta}{2} \; d\theta = \frac{\pi}{12}. \qquad \blacksquare$$

EXAMPLE 3: A region S lying in the xy–plane is bounded by the set with polar equation $r = 1 - \cos\theta$. Its mass density is the constant ρ, and it is rotating about the z–axis with an angular frequency of f rotations/sec. Find its kinetic energy.

SOLUTION: Let P be the region in $r\theta$–space bounded by the r–axis and the graph of $r = 1 - \cos\theta, 0 \le \theta \le 2\pi$, and let S be the associated region in xy–space as in Figure 3.

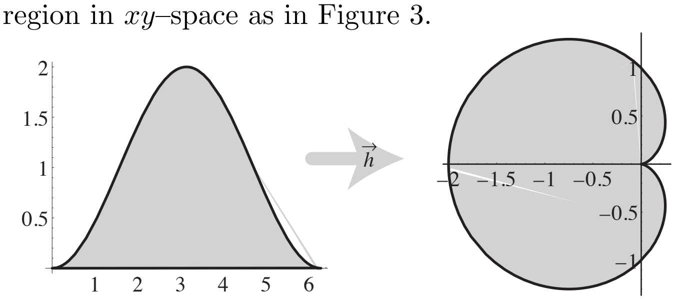


Figure 3. *The polar function takes the region P in $r\theta$–space onto the surface S in xy–space.*

The angular speed of S is $2\pi f$. Therefore, its kinetic energy is given by

$$\text{K. E.} = \frac{1}{2}(2\pi f)^2 \iint_S \left(x^2 + y^2\right) \rho \; dS$$

$$
\begin{aligned}
&= \frac{1}{2}(2\pi f)^2\rho \iint_P \left(r^2\cos^2\theta + r^2\sin^2\theta\right) J(\vec{P}(\vec{r}))\ dS \\
&= \frac{1}{2}(2\pi f)^2\rho \int_0^{2\pi}\int_0^{1-\cos\theta} r^3\ dr\, d\theta \\
&= \frac{1}{2}(2\pi f)^2\frac{1}{4}\rho \int_0^{2\pi} (1-\cos\theta)^4\ d\theta \\
&= \frac{1}{2}(2\pi f)^2\frac{15\pi}{16}\rho = \frac{15\pi^3 f^2\rho}{8}. \qquad \blacksquare
\end{aligned}
$$

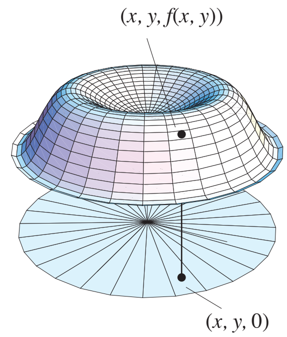


Figure 4. *S is the graph of f and P is the domain of f.*

EXAMPLE 4: Let P be a surface in xy–space, and let f be a function from P into $\mathbb{R}$ that is continuous and that has continuous partial derivatives. Let S be the set of all points of the form (x, y, z), where (x, y) is in P and $z = f(x, y)$. Let $\vec{h}$ be the function from P into S defined by $\vec{h}(x, y) = (x, y, f(x, y))$. (See Figure 4.)

It is easily seen that

$$J(\vec{h}(x,y)) = \sqrt{\left(\frac{\partial f(x,y)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y)}{\partial y}\right)^2 + 1}.$$

Thus the area of such a surface S is given by

$$
\begin{aligned}
\text{Area} &= \iint_S dS = \iint_P J(\vec{h}(\vec{r}))\ dS \\
&= \iint_P \sqrt{\left(\frac{\partial f(x,y)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y)}{\partial y}\right)^2 + 1}\ dS. \qquad \blacksquare
\end{aligned}
$$

EXAMPLE 5: Let T be the triangle in the plane with vertices $(0, 0), (1, 0)$, and $(1, 1)$. Find the surface area of the set of points (x, y, z) satisfying $z = y + x^2$ for (x, y) in T.

SOLUTION: We employ the formula developed in Example 4. Let $f(x, y) = y + x^2$. Then $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 1$, and

$$\sqrt{\left(\frac{\partial f(x,y)}{\partial x}\right)^2 + \left(\frac{\partial f(x,y)}{\partial y}\right)^2 + 1} = \sqrt{2+4x^2}.$$

Thus the area is

$$
\begin{aligned}
\iint_S dS &= \iint_T \sqrt{2+4x^2}\ dS = \int_0^1\int_0^x \sqrt{2+4x^2}\ dy\, dx \\
&= \int_0^1 \left(y\sqrt{2+4x^2}\right)\Big|_{y=0}^{y=x} dx = \frac{1}{12}\left(6^{3/2} - 2^{3/2}\right).
\end{aligned}
$$

EXERCISES 5.5

In Exercises 1–4, T is the triangle in $\mathbb{R}^n$ with vertices $\vec{A}, \vec{B}$, and $\vec{C}$. Compute $\iint_T f\ dS$.

1. $\vec{A} = (0,0), \vec{B} = (1,1), \vec{C} = (2,0)$, and $f(x,y) = xy^2$.
2. $\vec{A} = (-1,0), \vec{B} = (1,1), \vec{C} = (2,-1)$, and $f(x,y) = e^{(x+y)}$.
3. $\vec{A} = (1,0,0), \vec{B} = (0,1,0), \vec{C} = (0,0,1)$, and $f(x,y,z) = e^{(x+y+z)}$.
4. $\vec{A} = (1,0,0), \vec{B} = (0,1,0), \vec{C} = (0,0,1)$, and $f(x,y,z)$ is the square of the distance from (x,y,z) to the origin.

In Exercises 5–7, T is the triangle in $\mathbb{R}^n$ with vertices $\vec{A} = (0,1,0)$, $\vec{B} = (1,1,0)$, and $\vec{C} = (1,2,1)$. Assume that the mass density of T is a constant $\rho\,\mathrm{kg/m}^2$.

5. Find the kinetic energy if T is rotating about the x–axis at a rate of 2 rotations/sec.
6. Find the kinetic energy if T is rotating about the y–axis at a rate of 2 rotations/sec.
7. Find the kinetic energy if T is rotating about the z–axis at a rate of 2 rotations/sec.

In Exercises 8–12, sketch the region in xy–space bounded by the curve with the given polar equation, and find its area.

8. $r = 3\sin\theta$.
9. $r = \sin(2\theta)$.
10. $r = 3 - \sin\theta$.
11. $r = 1 + \cos(2\theta)$.
12. $r = 4 + \sin\theta$.

In Exercises 13–17, R is the region in $r\theta$–space bounded by the given curves. Sketch R (as it appears in $r\theta$–space) and $\vec{P}(R)$, the image of R in xy–space. Find the area of the region in xy–space.

13. $r = \theta,\ \theta = \pi,$ and $r = 0$.
14. $r = e^\theta,\ \theta = 2\pi,$ and $r = 0$.
15. $r = \sin\theta,\ 0 \le \theta \le \pi,$ and $r = 0$.
16. $r = 1 + \sin\theta,\ 0 \le \theta \le 2\pi,$ and $r = 0$.
17. $r = \sin(3\theta),\ 0 \le \theta \le \frac{\pi}{3},$ and $r = 0$.

In Exerecises 18–22, S is the region in xy–space bounded by the curve with the given polar equation. Find the volume of the solid that contains all points (x,y,z) that satisfy the conditions that (x,y) is in S and $g(x,y) \le z \le f(x,y)$.

18. $r = 1 + \sin\theta, g(x,y) = 0,\ f(x,y) = \sqrt{x^2+y^2}$.

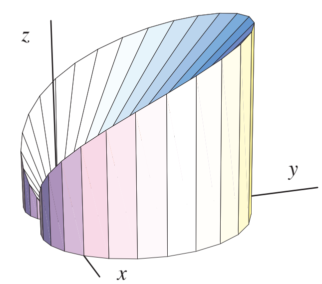


Figure 5
The solid in Exercise 18.

19. $r = \sin\theta, g(x,y) = 0, f(x,y) = y$.

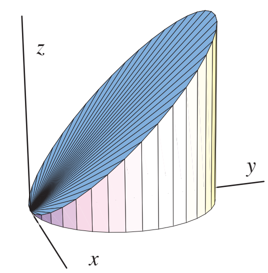


Figure 6
The solid in Exercise 19.

20. $r = \sin\theta, g(x,y) = -y, f(x,y) = x^2 + y^2$.

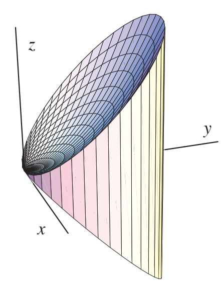
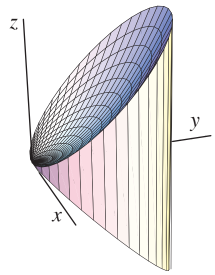


Figure 7
The solid in Exercise 20.

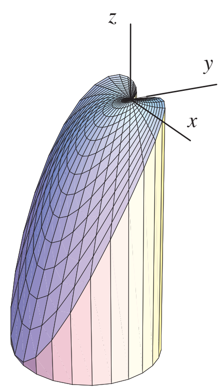
21. $r = \cos\theta, g(x,y) = 1-(x^2+y^2), f(x,y) = 1+x^2+y^2$.

Figure 8
The solid in Exercise 21.



22. $r = 1-\sin\theta, g(x,y) = -4, f(x,y) = y\sqrt{x^2+y^2}$.

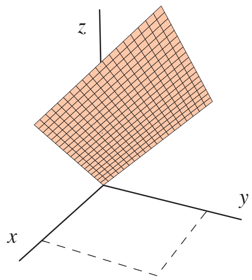
Figure 9
The solid in Exercise 22.



In Exercises 23–26, find the area of the set of points (x,y,z) such that (x,y) is in S and $z = f(x,y)$.

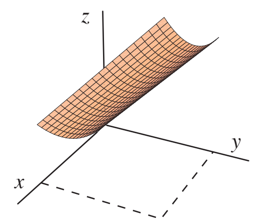
23. S is the square in xy–space bounded by the lines $x = 0, x = 1, y = 0, y = 1$, and $f(x,y) = x+y$.

Figure 10
The surface in Exercise 23.



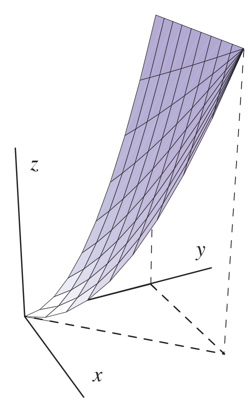
24. S is the square of Exercise 23, and $f(x,y) = y + \frac{x^2}{2}$.

Figure 11
The surface in Exercise 24.



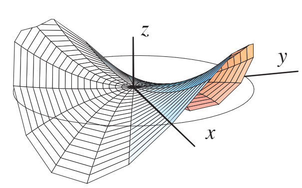
25. S is the triangle with vertices $(0,0)$, $(0,4)$, $(4,4)$, and $f(x,y) = y^2$.

Figure 12
The surface in Exercise 25.



26. S is the disc of radius 4 centered at the origin, and $f(x,y) = xy$.

Figure 13
The surface in Exercise 26.



27. Let S be the part of the graph of $z = x^2+y^2$ lying above the disk $x^2+y^2 \leq 1$. Find the area of S.

28. Let S be the part of the graph of $z = x^2 + y^2$ lying above the disk $x^2 + y^2 \leq 1$. Express $\iint_S x + y^2 + z\,dS$ as a double integral.

29. Let S be the part of the graph of $z = x^2 + y^2$ lying above the disk $(x-1)^2 + (y+2)^2 \leq 3$. Express $\iint_S x + y^2 + z\,dS$ as a double integral.

30. Let S be the part of the graph of $z = x^2 + y^2$ lying above the triangle with verticies $(-1, 0)$,$(1, -2)$, and $(3, 3)$. Express the area of S as a double integral.

31. Let S be the part of the cylinder $x^2+z^2 = 1,\ z \geq 0$ lying above the disc $x^2 + y^2 \leq 1$. Express the area of S as a double integral.

32. Let S be the total surface area of the intersection of $x^2+y^2 \leq 1$ and $x^2+z^2 \leq 1$. Express the area of S as the sum of four double integrals.

5.6 Simple Solids

The concept of parametrization has been a constant theme throughout this text. We have parametrized curves and surfaces in both 2–space and 3–space. For the purpose of integration, we put restrictions on the type of parametrization that we could use. These restrictions gave us the concepts of fundamental curve and simple surfaces. We now complete this discussion by defining simple solids.

Definition: Simple Solids

The set S in 3–space is a *simple solid* if there is a box B bounded by planes parallel to the coordinate planes and a differentiable function $\vec{h}$ from B onto S such that

(a) $\vec{h}$ is one-to-one in the interior of B;

(b) $\vec{h}$ and all of the first partial derivatives of $\vec{h}$ are continuous; and

(c) $J(\vec{h}(u, v, w))$ is not 0 in the interior of B.

The idea of a simple solid is the three dimensional version of the idea of a simple surface in 2– or 3–space, and the idea of a fundamental curve in 1–, 2–, or 3–space. One might think of a piece of malleable plastic or clay in the shape of the box B and use $\vec{h}$ to distort the box into S. Condition (a) means that no two points in the interior of the box B are pinched together in S. The condition that $J(\vec{h}(u, v, w)) \neq 0$ essentially means that no piece of volume is somehow shrunk to a point, a line, or a plane. That $\vec{h}$ is continuous means that the box is not broken when it is distorted by $\vec{h}$. We build solids from boxes just as we build surfaces from rectangles.

First, we extend the idea of parametrizing the surface of a sphere to that of parametrizing the solid ball.

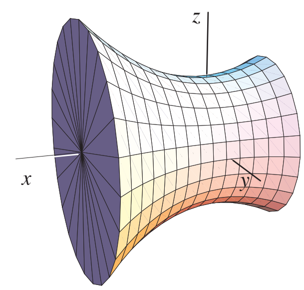


Figure 1.a *The solid obtained by rotating the region bounded between the graph of a function and the x–axis about the x–axis.*

EXAMPLE 1: Let $\vec{h}(r,\phi,\theta)$ be the restriction of the spherical transformation to the box B bounded by the planes $r=0, r=R, \phi=0, \phi=\pi, \theta=0$, and $\theta=2\pi$. Then $\vec{h}$ is one-to-one on the interior of B, and its first partial derivatives are continuous. $J(\vec{h}(r,\phi,\theta)) = r^2\sin\phi \neq 0$ on the interior of B, since $r>0$ and $0<\phi<\pi$ there. Thus $\vec{h}$ is a parametrization of the ball of radius R. ■

The next example is an extension of the idea of a surface of rotation.

EXAMPLE 2: Suppose that f is a function that is continuous and differentiable on the interval $[a,b]$ such that $f(x)\neq 0$ for every x in $[a,b]$. Let R be the region in the plane bounded by the graph of f, the x–axis, and the lines $x=a$ and $x=b$. Let S be the solid obtained by rotating R about the x–axis. (See Figure 1.) Find a parametrization for the solid S.

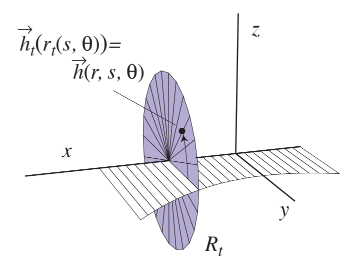


Figure 1.b *The cross section R_t.*

SOLUTION: For each point t in the interval $[a,b]$, let R_t be the surface obtained by intersecting the plane $x=t$ with the solid S. Then R_t is a disc of radius $f(t)$. A parametrization for R_t is given by $\vec{r}_t(s,\theta)=(t, sf(t)\cos\theta, sf(t)\sin\theta)$. (See Figure 1.b.)

Let $\vec{h}$ be defined by

$$\vec{h}(t,s,\theta)=(t,\vec{r}_t(s,\theta))=(t, sf(t)\cos\theta, sf(t)\sin\theta)$$

for

$$a\le t\le b, 0\le s\le 1, 0\le\theta\le 2\pi.$$

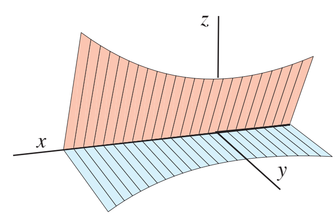


Figure 1.c *The "slice" obtained by holding θ constant.*

Figure 1.c displays the "slice" of the solid obtained by holding θ fixed at $\theta=\frac{\pi}{3}$ radians. In Figure 1.d, the wedge $0\le\theta\le\frac{\pi}{3}$ is removed to reveal the cross sections of the solid.

The partial derivatives of $\vec{h}$ are:

$$\begin{aligned}\frac{\partial h(t,s,\theta)}{\partial t} &= (1, sf'(t)\cos\theta, sf'(t)\sin\theta);\\ \frac{\partial h(t,s,\theta)}{\partial s} &= (0, f(t)\cos\theta, f(t)\sin\theta);\end{aligned}$$

and

$$\frac{\partial h(t,s,\theta)}{\partial\theta}=(0,-sf(t)\sin\theta, sf(t)\cos\theta).$$

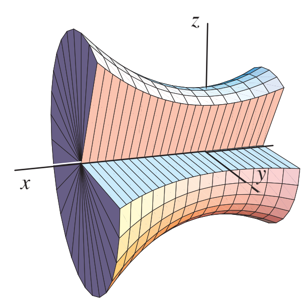


Figure 1.d *The image of $\vec{h}(t,s,\theta)$, $\frac{\pi}{3}\le\theta\le 2\pi$.*

Thus all of the first partial derivatives of $\vec{h}$ are continuous.

$$J(\vec{h}(t,s,\theta)) = s[f(t)]^2, \text{ which is not 0 if } s>0 \text{ and } a \le t \le b.$$

To see that $\vec{h}$ is one-to-one, suppose that $\vec{h}(t_1,s_1,\theta_1) = \vec{h}(t_2,s_2,\theta_2)$. This implies that

$$(t_1, s_1 f(t_1)\cos\theta_1, s_1 f(t_1)\sin\theta_1) = (t_2, s_2 f(t_2)\cos\theta_2, s_2 f(t_2)\sin\theta_2). \tag{1}$$

By comparing the first coordinates of each side of Equation (1), we see that $t_1 = t_2$. Since the sum of the squares of the second and third coordinates of each side of Equation (1) should be equal, we see that

$$(s_1 f(t_1))^2 = (s_2 f(t_2))^2 .$$

However, $t_1 = t_2$ implies that $s_1^2 = s_2^2$. Since $0 < s_i < 1$ for $i = 1,2$, we see that $s_1 = s_2$. By comparing the second and third coordinates of each side of Equation (1) we see that

$$\cos\theta_1 = \cos\theta_2 \quad \text{and} \quad \sin\theta_1 = \sin\theta_2.$$

This is only possible if $\theta_1 = \theta_2$. Thus $\vec{h}$ is one-to-one in the interior of the domain of $\vec{h}$. ■

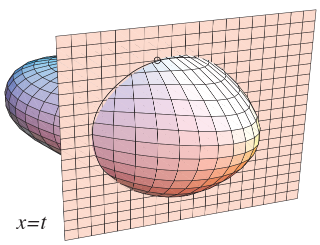


Figure 2.a *The plane $x = t$ intersecting the solid*

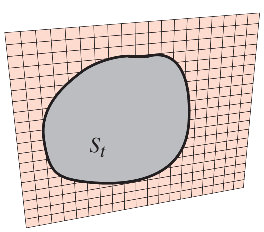


Figure 2.b *S_t is the common part of the solid and the plane $x = t$.*

The solid of revolution in Example 2 is obtained by first parametrizing the cross sections of the solid in a "continuous" manner so that the parametrizing functions can be put together to parametrize the solid. This is an example of a general approach described in the following theorem.

Theorem 1 (Solids with Continuous Cross Sections)
Suppose that S is a solid and $[a,b]$ is an interval such that:

(*a*) *If $a < t < b$, then the cross section of S obtained by intersecting the plane $x = t$ with S is a surface S_t (as in Figure 2);*

(*b*) *S_t is parametrized by the function $\vec{h}_t(u,v) = (t, y_t(u,v), z_t(u,v))$ with domain the rectangle in $uv-$space bounded by the lines $u = c, u = d, v = e$, and $v = f$, where $c < d$, and $e < f$;*

(*c*) *All of S lies between the planes $x = a$ and $x = b$;*

(*d*) *The function* $\vec{h}$, *defined by* $\vec{h}(t,u,v) = (t, \vec{h}_t(u,v)) = (t, y_t(u,v), z_t(u,v))$, *with domain the box* B *bounded by the planes* $t = a, t = b, u = c, u = d, v = e$, *and* $v = f$, *is continuous; and*

(*e*) $\dfrac{\partial \vec{h}(t,u,v)}{\partial t}$ *is continuous;*

then $\vec{h}$ *parametrizes* S, *and* $J(\vec{h}(t,u,v)) = J(\vec{h}_t(u,v))$.

Proof: We will show that $\vec{h}$ is one-to-one, that the partial derivatives of $\vec{h}$ are continuous, and that the Jacobian of $\vec{h}$ is not zero on the interior of B.

First we prove that $\vec{h}$ is one-to-one. Consider (t_1, u_1, v_1) and (t_2, u_2, v_2) in the interior of B such that $\vec{h}(t_1, u_1, v_1) = \vec{h}(t_2, u_2, v_2)$. If $t_1 \neq t_2$, then $\vec{h}(t_1, u_1, v_1)$ and $\vec{h}(t_2, u_2, v_2)$ differ in the first coordinate. Thus $t_1 = t_2$. This implies that $h_{t_1}(u_1, v_1) = h_{t_1}(u_2, v_2)$. However, h_{t_1} is a parametrization for S_{t_1} and is, by definition, one-to-one. Thus $(u_1, v_1) = (u_2, v_2)$, and $\vec{h}$ is one-to-one.

Second, we show that the partial derivatives of $\vec{h}$ are continuous. It is given in the hypothesis that $\frac{\partial \vec{h}}{\partial t}$ is continuous. The partial derivative of $\vec{h}$ with respect to u is given by

$$\frac{\partial \vec{h}(t,u,v)}{\partial u} = \left(0, \frac{\partial y_t(u,v)}{\partial u}, \frac{\partial z_t(u,v)}{\partial u}\right).$$

Since $\vec{h}_t$ is a parametrization for S_t (t is held constant for $\vec{h}_t$), we know that $\frac{\partial y_t(u,v)}{\partial u}$ and $\frac{\partial z_t(u,v)}{\partial u}$ are continuous. Thus $\frac{\partial \vec{h}(t,u,v)}{\partial u}$ is continuous. A similar argument shows that $\frac{\partial \vec{h}}{\partial v}$ is continuous.

Finally, we compute $J(\vec{h}(t,u,v))$. With the information given, it is not possible to compute $\frac{\partial \vec{h}(t,u,v)}{\partial t}$, but we do know that the first coordinate of $\frac{\partial \vec{h}(t,u,v)}{\partial t}$ is 1 so that $\frac{\partial \vec{h}(t,u,v)}{\partial t} = \left(1, \frac{\partial y(t,u,v)}{\partial t}, \frac{\partial z(t,u,v)}{\partial t}\right)$. Thus

$$D\vec{h}|_{(t,u,v)} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial y(t,u,v)}{\partial t} & \frac{\partial y_t(u,v)}{\partial u} & \frac{\partial y_t(u,v)}{\partial v} \\ \frac{\partial z(t,u,v)}{\partial t} & \frac{\partial z_t(u,v)}{\partial u} & \frac{\partial z_t(u,v)}{\partial v} \end{pmatrix}.$$

Expanding by cofactors we obtain:

$$J(\vec{h}(t,u,v)) = \left|\det\left(D\vec{h}|_{(t,u,v)}\right)\right|$$

$$= \left| \det \begin{pmatrix} \frac{\partial y_t(u,v)}{\partial u} & \frac{\partial y_t(u,v)}{\partial v} \\ \frac{\partial z_t(u,v)}{\partial u} & \frac{\partial z_t(u,v)}{\partial v} \end{pmatrix} \right| = J(\vec{h}_t(u,v)).$$

Since $J(\vec{h}_t(u,v)) \neq 0$, we see that $J(\vec{h}(t,u,v)) \neq 0$ in the interior of B. ■

In Example 2, the cross section of the solid in the plane $x = t$ is a circle of radius $f(t)$. We now look at an example where the cross sections are squares.

EXAMPLE 3: Let $f(t) = t^2, 0 \leq t \leq 1$. Let S be the solid in xyz–space such that the cross section of S lying in the plane $x = t$ is a square with vertices $(t, 0, 0), (t, f(t), 0), (t, 0, f(t))$, and $(t, f(t), f(t))$. Parametrize S. See Figure 3. Let T_t denote the cross section obtained by intersecting the plane $x = t$ with S. Let R be the rectangle in uv–space bounded by the lines $u = 0, u = 1, v = 0$ and $v = 1$. A parametrization for T_t is given by

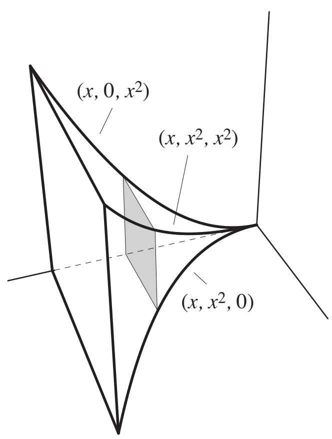


Figure 3.
The cross section T_t.

$$\vec{h}_t(u,v) = (uf(t), vf(t)) = (ut^2, vt^2),\ 0 \leq u \leq 1,\ 0 \leq v \leq 1.$$

Thus a parametrization for S is given by

$$\vec{h}(t,u,v) = (t, uf(t), vf(t)) = (t, ut^2, vt^2),$$

where

$$0 \leq t \leq 1,\ 0 \leq u \leq 1\, 0 \leq v \leq 1.$$

The partial derivatives of $\vec{h}$ are given by

$$\frac{\partial h(t,u,v)}{\partial t} = (1, 2ut, 2vt);$$

$$\frac{\partial h(t,u,v)}{\partial u} = (0, t^2, 0);$$

and

$$\frac{\partial h(t,u,v)}{\partial v} = (0, 0, t^2).$$

Thus

$$J(\vec{h}(t,u,v)) = \left| \det \begin{pmatrix} 1 & 0 & 0 \\ 2ut & t^2 & 0 \\ 2vt & 0 & t^2 \end{pmatrix} \right| = t^4. \quad ■$$

EXAMPLE 4: Let S be the solid in xyz–space bounded by the graphs of $y = x^2$ and $z = x^2$, the xy and xz–coordinate planes, and the plane $x = 1$. Find a parametrization for S.

SOLUTION: The problem here is to visualize what S looks like. A tactic that proves useful is to sketch the bounding surfaces. Figure 4.a illustrates the graph of $y = x^2$, and Figure 4.b is a sketch of the graph of $z = x^2$. Figure 4.c displays the intersecting bounding surfaces, and the solid S is shown in Figure 4.d. This is the same solid as the one in Example 3! ■

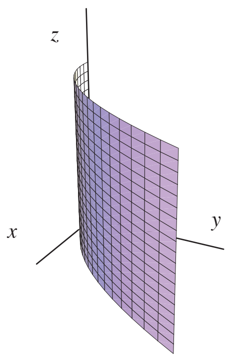


Figure 4.a *The bounding graph* $y = x^2$.

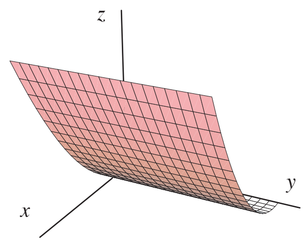


Figure 4.b *The bounding graph* $z = x^2$.

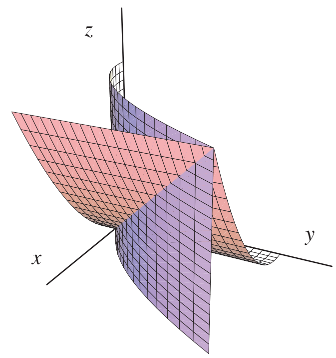


Figure 4.c *The intersection of the bounding graphs.*

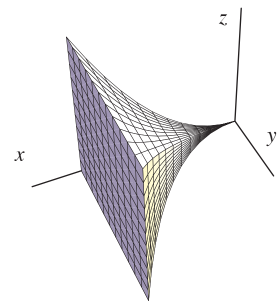


Figure 4.d *The solid* S.

EXAMPLE 5: Let S be the solid in xyz–space bounded by the graph of $z = x^2$ (Figure 5.a), the plane $z-y = 0$ (Figure 5.b), the xy–coordinate plane, the xz–coordinate plane, and the plane $z = 1$ (Figure 5.c). Figure 5.d displays the intersecting planes, and Figure 5.e shows the solid S. Use continuous cross sections to parametrize S.

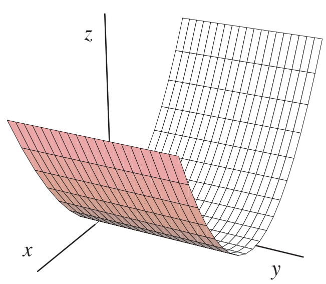


Figure 5.a *The bounding graph* $z = x^2$.

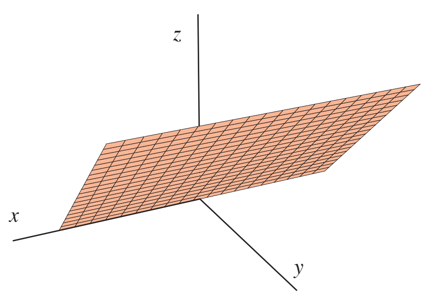


Figure 5.b *The plane* $z - y = 0$.

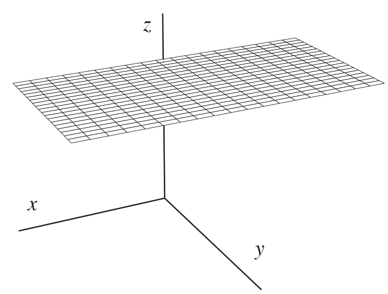


Figure 5.c *The plane* $z = 1$.

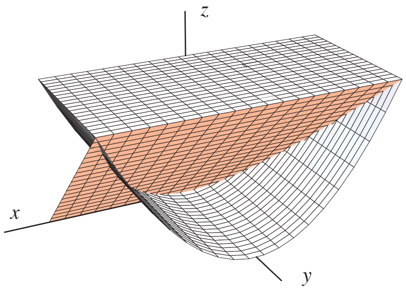


Figure 5.d *The intersecting bounding planes.*

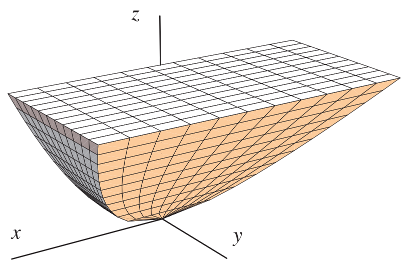


Figure 5.e *The solid S.*

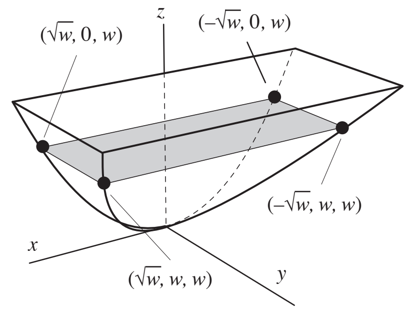


Figure 5.f *A cross section parallel to the xy–coordinate plane.*

Solution: From the diagrams above, we see that the best cross sections are those parallel to the xy–coordinate plane, which are rectangles. (See Figure 5.f.)

Let w be a number between 0 and 1 and let R_w be the rectangular cross section obtained by intersecting the plane $z = w$ with the solid S. If (x, y, z) is a point in R_w, then $z = w, 0 \le y \le w$ and $-\sqrt{w} \le x \le \sqrt{w}$. The function

$$\vec{h}_w(u, v) = \begin{pmatrix} u\sqrt{w} \\ wv \\ w \end{pmatrix}, \; -1 \le u \le 1 \text{ and } 0 \le v \le 1,$$

is a parametrization for R_w, and

$$\vec{h}(u, v, w) = \begin{pmatrix} u\sqrt{w} \\ wv \\ w \end{pmatrix}, \; -1 \le u \le 1, \; 0 \le v \le 1, \text{ and } 0 \le w \le 1,$$

parametrizes S. ■

Rotating Surfaces About an Axis

Suppose that $\vec{r}(s, t) = (x(s, t), y(s, t)), \; a \le s \le b, \; c \le t \le d$ parameterizes a surface R in the plane and that R does not intersect the x–axis. Then

$$\vec{h}(\theta, s, t) = \begin{pmatrix} x(s, t) \\ y(s, t)\cos(\theta) \\ y(s, t)\sin(\theta) \end{pmatrix}, \; a \le s \le b, \; c \le t \le d,$$

parametrizes the solid obtained by rotating the surface R about the x–axis. You are asked in the exercises to show that

$$J\vec{h}(\theta, s, t) = |y(s, t)| J\vec{r}(s, t) = |y\det Dh| = |y(s, t)| \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right|.$$

Similarly, if the surface R does not intersect the y–axis, then the solid obtained by rotating R about the y–axis can be parametrized by

$$\vec{h}(\theta, s, t) = \begin{pmatrix} x(s,t)\cos(\theta) \\ y(s,t) \\ x(s,t)\sin(\theta) \end{pmatrix}, \quad a \le s \le b,\ c \le t \le d.$$

$$J\vec{h}(\theta, s, t) = |x(s,t)| J\vec{r}(s,t) = |x \det Dh| = |x(s,t)| \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right|.$$

EXAMPLE 6: Let R be the unit disc centered at $(0,2)$. Parametrize the solid obtained by rotating R about the x–axis, and find the Jacobian of the parametrization.

SOLUTION: $\vec{r}(\rho,\theta) = (\rho\cos(\theta), \rho\sin(\theta)+2)$, $0 \le \theta \le 2\pi$, $0 \le \rho \le 1$, parametrizes the disc. It follows that

$$\vec{h}(\rho, \theta, \phi) = \begin{pmatrix} \rho\cos(\theta) \\ (\rho\sin(\theta)+2)\cos(\phi) \\ (\rho\sin(\theta)+2)\sin(\phi) \end{pmatrix}, \quad \begin{array}{l} 0 \le \rho \le 1, \\ 0 \le \theta \le 2\pi, \\ 0 \le \phi \le 2\pi \end{array}$$

is a parametrization for the solid of rotation. The Jacobian is

$$J\vec{h}(\rho,\theta,\phi) = (\rho\sin(\theta)+2)\rho = \rho^2\sin(\theta) + 2\rho. \qquad \blacksquare$$

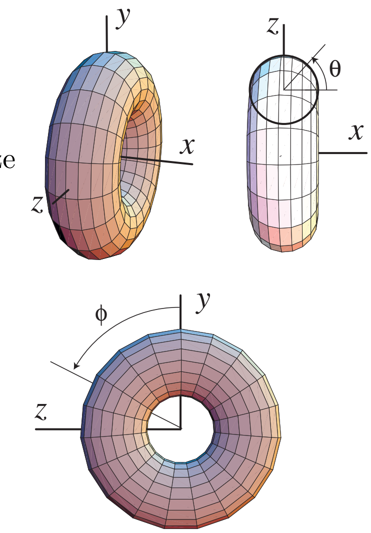


Figure 6.
The solid in Example 6.

EXERCISES 5.6

In Exercises 1–3, the vectors $\vec{A}, \vec{B}$, and $\vec{C}$ are drawn emanating from the position $\vec{D}$ to form adjacent edges of a solid parallelepiped P. Parametrize P and find the Jacobian of your parametrization.

1. $\vec{A} = (0,1,1), \vec{B} = (-1,2,1), \vec{C} = (1,1,1)$, and $\vec{D} = (0,0,0)$.
2. $\vec{A} = (0,2,1), \vec{B} = (-1,2,1), \vec{C} = (1,-1,3)$, and $\vec{D} = (1,1,1)$.
3. $\vec{A} = (0,1,1), \vec{B} = (-1,-1,1), \vec{C} = (1,0,4)$, and $\vec{D} = (-1,2,0)$.

In Exercises 4–8, parametrize the solid S obtained by rotating the region R (lying in the xy–plane) about the x–axis. Determine the rate that your parametrization changes volume (as a function of position.)

4. R is the region bounded by the graph of $y = x$, the x–axis, and the line $x = 1$.
5. R is the region bounded by the graphs of $y = x$ and $y = x^2$.
6. R is the rectangle bounded by the lines $x = 1, x = 3, y = 2$, and $y = 52$.
7. R is the disc of radius 3 centered at the origin.
8. R is the disc of radius 2 centered at the point $(3,4)$.

In Exercises 9–13, the region R lies in the xy–coordinate plane. Parametrize the solid obtained by rotating R about the y–axis, and find the Jacobian of your parametrization.

9. R is the region bounded by the graph of $y = x^2$, the x–axis, and the line $x = 1$.

10. R is the region bounded by the graphs of $y = x^2$ and $y = x^3$.
11. R is the rectangle bounded by the lines $x = 1, x = 3, y = 2$, and $y = 52$.
12. R is the disc of radius 3 centered at the origin.
13. R is the disc of radius 2 centered at the point $(3, 4)$.

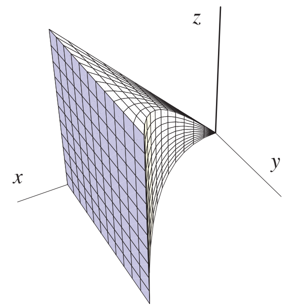
In Exercises 14–21, parametrize the solid S obtained by rotating the region R (lying in the xy–plane) about the line L.

14. R is the region bounded by the graph of $y = x$, the x–axis, and the line $x = 1$. L is the line $x = -1$.
15. R is the region bounded by the graph of $y = x$, the x–axis, and the line $x = 1$. L is the line $x = 3$.
16. R is the region bounded by the graph of $y = x$, the x–axis, and the line $x = 1$. L is the line $y = -1$.
17. R is the region bounded by the graph of $y = x$, the x–axis, and the line $x = 1$. L is the line $y = 6$.
18. R is the region bounded by the graphs of $y = x$ and $y = x^2$. L is the line $x = 4$.
19. R is the rectangle bounded by the lines $x = 1, x = 3, y = 2$, and $y = 52$. L is the line $y = -2$.
20. R is the disc of radius 3 centered at the origin. L is the line $y = -3$.
21. R is the disc of radius 2 centered at the point $(3, 4)$. L is the line $x = -1$.

In Exercises 22–27, S is a solid lying in the upper half of xyz–space, $z \geq 0$, and between the planes $x = 0$ and $x = 1$. The base of S lies in the xy–coordinate plane and is the region bounded by the x–axis, the graph of $y = x^2$, and the line $x = 1$. Let A_t denote the cross section of S lying in the plane $x = t$. Find a parametrization for S for the given shape of A_t.

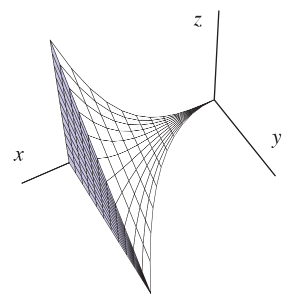
22. A_t is a rectangle with height t.

Figure 7
The solid in Exercise 22.



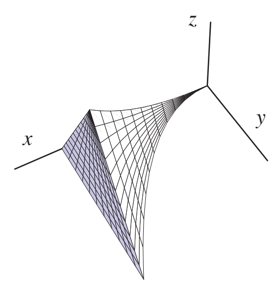
23. A_t is a rectangle with its base in the xy–plane and with height 2.
24. A_t is an isosceles right triangle with one leg in the xy–plane and the other in the xz–plane.

Figure 8
The solid in Exercise 24.



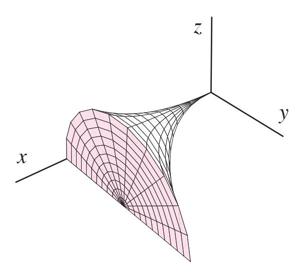
25. A_t is an equilateral triangle.

Figure 9
The solid in Exercise 25.



26. A_t is an isosceles triangle with height t^3.
27. A_t is the upper half of a disk.

Figure 10
The solid in Exercise 27.



In Exercises 28–32, S is a solid lying in the upper half of xyz–space, $z \geq 0$, and between the planes $x = 0$ and $x = 1$. The base of S lies in the xy–coordinate plane and is the region bounded by the graphs of $y = x^2$ and $y = x^3$. Let A_t denote the cross section of S lying in the plane $x = t$. Find a parametrization for S for the given shape of A_t.

28. A_t is a square.

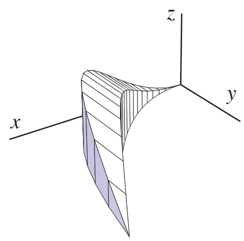


Figure 11
The solid in Exercise 28.

29. A_t is a rectangle with its base in the xy–plane and with height 2.

30. A_t is an isosceles right triangle with one leg in the xy–plane and the other in the xz–plane.

31. A_t is an equilateral triangle.

32. A_t is the upper half of a disk with diameter $t^2 - t^3$.

33. Parametrize the intersection of the cylinders $x^2 + y^2 \leq 1$ and $x^2 + z^2 \leq 1$.

In Exercises 34 and 35, $\vec{r}(s,t) = (x(s,t), y(s,t))$, $a \leq s \leq b$, $c \leq t \leq d$, parameterizes a surface R in the xy–plane .

34. Given that R does not intersect the x–axis, then

$$\vec{h}(\theta, s, t) = \begin{pmatrix} x(x,t) \\ y(s,t)\cos(\theta) \\ y(s,t)\sin(\theta) \end{pmatrix},$$

$$a \leq s \leq b,\ c \leq t \leq d$$

parametrizes the solid obtained by rotating the surface R about the x–axis. Show that $J\vec{h}(\theta, s, t) = |y(s,t)| J\vec{r}(s,t)$. Why are we concerned about the surface intersecting the x–axis?

35. If R does not intersect the y–axis, then the solid obtained by rotating R about the y–axis can be parametrized by

$$\vec{h}(\theta, s, t) = \begin{pmatrix} x(x,t)\cos(\theta) \\ y(s,t) \\ x(s,t)\sin(\theta) \end{pmatrix},$$

$$a \leq s \leq b,\ c \leq t \leq d.$$

Show that $J\vec{h}(\theta, s, t) = |x(s,t)| J\vec{r}(s,t)$. Why are we concerned about the surface intersecting the y–axis?

5.7 Triple Integrals

The development in this section is similar to the development of surface integrals. We start with an example which is a 3–dimensional version of Example 1 of Section 5.1.

EXAMPLE 1: Let B be the solid in xyz–space bounded by the planes $x = 0, x = 1, y = 0, y = 2, z = 0$, and $z = 1$. Suppose that the mass density of B at (x, y, z) is given by $\rho(x,y,z) = x^2 + y^2 + z^2$. Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of the segment $[0,1]$, let $\{y_0, y_1, \ldots, y_m\}$ be a partition of the segment $[0,2]$, and let $\{z_0, z_1, \ldots, z_s\}$ be a partition of the segment $[0,1]$. Then the planes

$$\begin{aligned} &x = x_0,\ x = x_1, \ldots,\ x = x_n; \\ &y = y_0,\ y = y_1, \ldots,\ y = y_m; \\ &z = z_0,\ z = z_1, \ldots,\ z = z_s \end{aligned}$$

partition B into small nonoverlapping boxes. For each $i \leq n, j \leq m$ and $k \leq s$, let $B_{i,j,k}$ be the box bounded by the planes $x = x_{i-1}, x = x_i, y = y_{j-1}, y = y_j, z = z_{k-1}$, and $z = z_k$. See Figures 1 and 2.

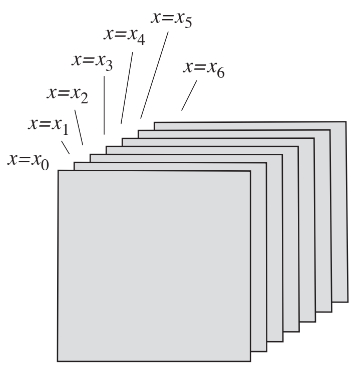


Figure 1.a *Partitioning planes perpendicular to the x–axis.*

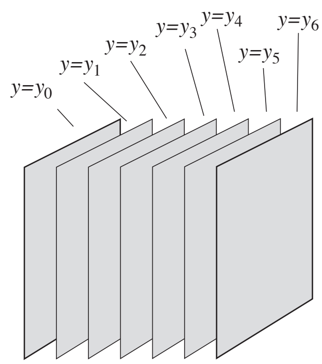


Figure 1.b *Partitioning planes perpendicular to the y–axis.*

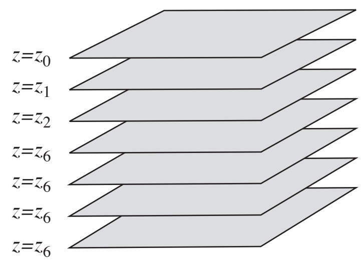


Figure 1.c *Partitioning planes perpendicular to the z–axis.*

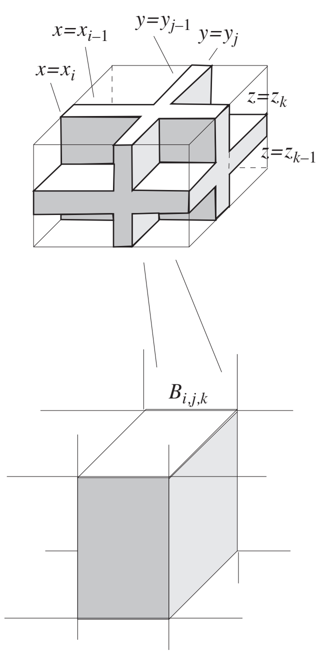


Figure 2. *The box $B_{i,j,k}$.*

$$\begin{aligned} &\rho(x_i, y_j, z_k)\mathrm{Volume}(B_{i,j,k}) \\ &\quad = \quad \rho(x_i, y_j, z_k)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}) \end{aligned}$$

is an approximation of the mass of $B_{i,j,k}$. Thus the sum

$$\sum_{k=1}^{s}\sum_{j=1}^{m}\sum_{i=1}^{n} \rho(x_i, y_j, z_k)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}) \tag{2}$$

is an approximation of the mass of B. However,

$$\sum_{i=1}^{n}(x_i^2 + y_j^2 + z_k^2)(x_i - x_{i-1})$$

is an approximation of

$$\int_0^1 (x^2 + y_j^2 + z_k^2)\ dx,$$

and so Equation (2) is an approximation of

$$\left[\sum_{k=1}^{s}\left[\sum_{j=1}^{m}\left[\int_0^1 (x^2 + y_j^2 + z_k^2)\ dx\right](y_j - y_{j-1})\right](z_k - z_{k-1})\right].$$

This in turn is an approximation of

$$\left[\sum_{k=1}^{s}\left[\int_0^2\left[\int_0^1 (x^2+y^2+z_k^2)\ dx\right]\ dy\right](z_k-z_{k-1})\right],$$

which approximates

$$\left[\int_0^1\left[\int_0^2\left[\int_0^1 (x^2+y^2+z^2)\ dx\right]\ dy\right]\ dz\right].$$

Thus Equation (2) approximates

$$\left[\int_0^1\left[\int_0^2\left[\int_0^1 (x^2+y^2+z^2)\ dx\right]\ dy\right]\ dz\right],$$

or

$$\begin{aligned}
\int_0^1\int_0^2\int_0^1 (x^2+y^2+z^2)\ dx\,dy\,dz &= \int_0^1\int_0^2\left(\frac{1}{3}+y^2+z^2\right)\ dy\,dz \\
&= \int_0^1\left(\frac{2}{3}+\frac{8}{3}+2z^2\right)\ dz \\
&= \frac{2}{3}+\frac{8}{3}+\frac{2}{3}=4.
\end{aligned}$$

So the mass of B is 4. ■

As in Section 14.1, Example 1 leads us to terminology similar to that for integrals over rectangular boxes. We used points to partition an interval when we were defining an integral on a segment on the line in Chapter 6. We used lines to partiton a rectangle in the plane in Section 1 of this chapter. In the following definition, we use planes to partition a rectangular box in 3–space. As you read this definition, keep in mind that we are just extending to three dimensions what we have already done on the line and in the plane. It might be helpful to refer to Figures 1 and 2.

Definition: $\iiint_B \rho\ dV$

Let B be the solid box in xyz–space bounded by the planes $x = a, x = b, y = c, y = d, z = e$, and $z = f$. Suppose that $\rho(x, y, z)$ is a continuous function defined on B. Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of the segment $[a, b]$, let $\{y_0, y_1, \ldots, y_m\}$ be a partition of the segment $[c, d]$, and let $\{z_0, z_1, \ldots, z_s\}$ be a partition of the segment $[e, f]$ so that the planes

$$\begin{aligned} x &= x_0,\ x = x_1,\ \ldots,\ x = x_n; \\ y &= y_0,\ y = y_1,\ \ldots,\ y = y_m; \\ z &= z_0,\ z = z_1,\ \ldots,\ z = z_s \end{aligned}$$

partition B into small nonoverlapping boxes. This set of planes is called a *partition of* B. For each $i \leq n, j \leq m$ and $k \leq s$, let $B_{i,j,k}$ be the box bounded by the planes $x = x_{i-1}, x = x_i, y = y_{j-1}, y = y_j, z = z_{k-1}$, and $z = z_k$, and let $\vec{s}_{i,j,k}$ be a point in $B_{i,j,k}$. The set $\{\vec{s}_{i,j,k}\}$ is called a *selection for the partition of* B. If L is a number that can be approximated within any specified tolorance by sums of the type

$$\left[\sum_{k=1}^{s}\left[\sum_{j=1}^{m}\left[\sum_{i=1}^{n}\rho(\vec{s}_{i,j,k})(x_i - x_{i-1})\right](y_j - y_{j-1})\right](z_k - z_{k-1})\right]$$

simply by insuring the partition divides the box B into "small enough" pieces, then the number L is called the integral of ρ over the solid box B. We write

$$L = \iiint_B \rho\ dV.$$

Computationally,

$$\begin{aligned} \iiint_B \rho\ dV &= \int_e^f \int_c^d \int_a^b \rho(x,y,z)\ dx\,dy\,dz \\ &= \left[\int_e^f \left[\int_c^d \left[\int_a^s \rho(x,y,z)\ dx\right]\ dy\right]\ dz\right]. \end{aligned}$$

EXAMPLE 2: Returning to Example 1, B is the solid in xyz–space bounded by the planes $x = 0, x = 1, y = 0, y = 2, z = 0$, and $z = 1$ and the mass density of B at (x,y,z) is given by $\rho(x,y,z) = x^2 + y^2 + z^2$. The mass of B is $\iiint_B (x^2 + y^2 + z^2)\ dV$. ■

As with double integrals, the notation

$$\iiint_B \rho\ dV = \int_e^f \int_c^d \int_a^b \rho(x,y,z)\ dx\,dy\,dz$$

means that we first integrate with respect to x, then with respect to y, and finally with respect to z. It can be shown that if ρ is continuous, then the order of integration does not affect the value obtained. Thus

$$\begin{aligned}\int_e^f \int_c^d \int_a^b \rho(x,y,z)\ dx\,dy\,dz &= \int_e^f \int_a^b \int_c^d \rho(x,y,z)\ dy\,dx\,dz \\ &= \int_a^b \int_c^d \int_e^f \rho(x,y,z)\ dz\,dy\,dx, \text{ etc.}\end{aligned}$$

Using triple integrals is quite similar to using double integrals, only now we are dividing a box into small nonoverlapping boxes, while with double integrals we divided a rectangle into small nonoverlapping rectangles. We have

$$\underbrace{\sum\sum\sum}_{\substack{\text{Sum of}\\ \text{many}\\ \text{pieces of}\\ \text{the box}}} \quad \underbrace{\rho(\vec{s}_{i,j,k})}_{\substack{\rho \text{ evaluated}\\ \text{at a point}\\ \text{in a piece of}\\ \text{the box}}} \quad \underbrace{(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})}_{\substack{\text{Volume of a small}\\ \text{piece of the box}}}$$

$$\approx \quad \underbrace{\iiint_B}_{\substack{\text{Sum of}\\ \text{many}\\ \text{pieces of}\\ \text{the box}}} \qquad \underbrace{\rho}_{\substack{\rho \text{ evaluated}\\ \text{at a point}\\ \text{in a piece of}\\ \text{the box}}} \qquad \underbrace{dV}_{\substack{\text{Volume of a}\\ \text{small piece}\\ \text{of the box}}} \qquad \blacksquare$$

Now that we have the notion of the integral over a box, we can extend the notion to integrals over simple solids.

Definition: $\iiint_S f\ dV$

Suppose that f is a continuous function with domain the simple solid S, and L is a number. If, whenever $\vec{h}$ is a parametrization for the solid S with domain B,

$$L = \iiint_B f\vec{h}((x,y,z))\, J(\vec{h}(x,y,z))\ dV,$$

then the number L is called the *integral of f over S*, and it is denoted by $\iiint_S f\ dV$.

Notice that implicit in the above definition is the idea of breaking the solid S into nonoverlapping pieces. Let

$$\mathcal{P} = \begin{array}{l} x = x_0,\ x = x_1,\ \ldots,\ x = x_n \\ y = y_0,\ y = y_1, \ldots,\ y = y_m \\ z = z_0,\ z = z_1, \ldots,\ z = z_s \end{array}$$

be a partition of B. Then

$$\vec{h}(\mathcal{P}) = \begin{array}{l} \vec{h}(x=x_0),\ \vec{h}(x=x_1),\ \ldots,\ \vec{h}(x=x_n) \\ \vec{h}(y=y_0),\ \vec{h}(y=y_1),\ldots,\ \vec{h}(y=y_m) \\ \vec{h}(z=z_0),\ \vec{h}(z=z_1),\ldots,\ \vec{h}(z=z_s) \end{array}$$

"partitions" the solid S as illustrated below.

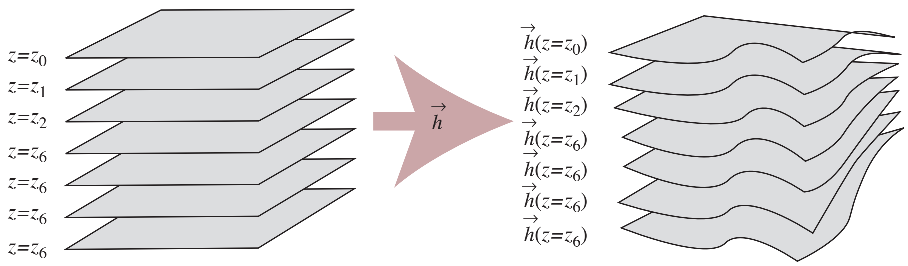


Figure 3. *The planes partitioning the box B are "distorted" by $\vec{h}$ to partition the solid S.*

The ijk^{th} box $B_{i,j,k}$ is distorted by $\vec{h}$ to the piece of the solid S, $\vec{h}(B_{i,j,k})$ as illustrated in Figure 4.

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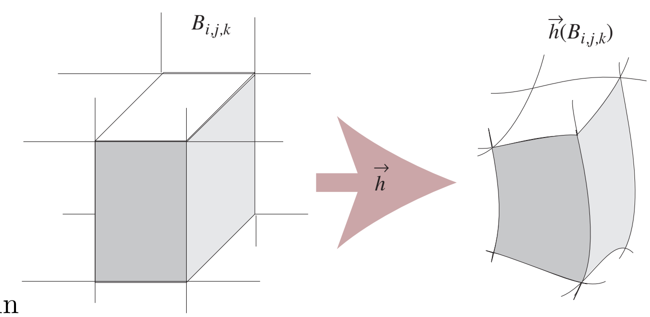

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Figure 4. *The piece of B, $B_{i,j,k}$, is distorted by $\vec{h}$ to the* inin*piece of the solid S, $S_{i,j,k} = \vec{h}(B_{i,j,k})$.*

If $s_{i,j,k}$ is a point in $B_{i,j,k}$, then the volume of $S_{i,j,k} = \vec{h}(B_{i,j,k})$ is approximated by $\mathrm{vol}(B_{i,j,k})J(\vec{h}(\vec{s}_{i,j,k}))$, and $\vec{h}(s_{i,j,k})$ is a point in $\vec{h}(B_{i,j,k})$. The sum

$$\underbrace{\sum_{k=1}^{s}\sum_{j=1}^{m}\sum_{i=1}^{n}}_{\substack{\text{Sum over}\\ \text{all the}\\ \text{pieces of}\\ \text{the box}}} \underbrace{\rho(\vec{h}(\vec{s}_{i,j,k}))}_{\substack{\rho\text{ evaluated}\\ \text{at a point}\\ \text{in a piece of}\\ \text{the solid}}} \underbrace{\mathrm{vol}(B_{i,j,k})}_{\substack{\text{The volume}\\ \text{of the box}\\ B_{i,j,k}.}} \underbrace{J(\vec{h}(\vec{s}_{i,j,k}))}_{\substack{\text{The rate}\\ \text{that } \vec{h}\\ \text{changes}\\ \text{volume.}}}$$

approximates both $\iiint_S \rho \, dV$ and $\iiint_B \rho(\vec{h}(\vec{s}))J(\vec{h}(\vec{s})) \, dV$. The following captures the geometry of the integral over a solid.

$$\underbrace{\iiint_S}_{\substack{\text{Sum over}\\ \text{all the}\\ \text{pieces of}\\ \text{the solid}}} \quad \underbrace{\rho}_{\substack{\rho \text{ evaluated}\\ \text{at a point}\\ \text{in a piece of}\\ \text{the solid}}} \quad \underbrace{dV}_{\substack{\text{Volume of}\\ \text{a small}\\ \text{piece of}\\ \text{the solid}}}$$

$$= \underbrace{\iiint_B}_{\substack{\text{Sum over}\\ \text{all the}\\ \text{pieces of}\\ \text{the box}}} \quad \underbrace{\rho(\vec{h}(\vec{s}))}_{\substack{\rho \circ \vec{h}\\ \text{evaluated at}\\ \text{a point in a}\\ \text{piece of the}\\ \text{box}}} \quad \underbrace{J(\vec{h}(\vec{s}))}_{\substack{\text{Rate that}\\ \vec{h} \text{ changes}\\ \text{volume}}} \quad \underbrace{dV}_{\substack{\text{Volume of}\\ \text{a small}\\ \text{piece of}\\ \text{the box}}} .$$

If $\rho(x, y, z)$ is a mass density function for a solid S, then the mass of S is

$$M = \iiint_S \rho \, dS.$$

We calculate the center of mass of a volume S with density function $\rho(x, y, z)$ by the now familiar formulas

$$X = \frac{\iiint_S x\rho \, dV}{\iiint_S \rho \, dV}, \quad Y = \frac{\iiint_S y\rho \, dV}{\iiint_S \rho \, dV}, \quad \text{and} \quad Z = \frac{\iiint_S z\rho \, dV}{\iiint_S \rho \, dV}.$$

Similarly, if a solid is rotating about an axis, then we can define the moments of inertia.

$I_x = \iiint_S (y^2 + z^2)\rho(x, y, z) \, dV$ = moment of intertia about the x–axis.

$I_y = \iiint_S (x^2 + z^2)\rho(x, y, z) \, dV$ = moment of intertia about the y–axis.

$I_z = \iiint_S (x^2 + y^2)\rho(x, y, z) \, dV$ = moment of intertia about the z–axis.

The kinetic energy of a solid rotating about an axis with angular speed ω is given by $\frac{1}{2}\omega^2 I_a$, where $a = x, y$, or z indicates the axis of rotation.

EXAMPLE 3: A ball of radius 2 m is rotating about a line passing through its center at a rate of 2 rotations/sec. The mass density at a point of the ball is the square of the distance from the point to the center of the ball. Find the kinetic energy of the rotating ball.

SOLUTION: Assume that the ball is situated in xyz–space so that its center is at the origin and so that it is rotating about the z–axis.

Recall that the rotational kinetic energy is given by the formula $ke = \frac{1}{2} I_z \omega^2$, where ω is the angular speed, which is $(2\pi)2 = 4\pi$ radians/sec (2π times the frequency). Let S denote the ball. The kinetic energy of S is

$$\begin{aligned}
&\frac{1}{2}(4\pi)^2 \iiint_S (x^2+y^2)\rho(x,y,z)\ dV \\
&= 8\pi^2 \iiint_S \left(x^2+y^2\right)\left(x^2+y^2+z^2\right)\ dV.
\end{aligned}$$

A convenient parametrization for the ball is obtained by using spherical coordinates. Let $\vec{S}(r,\phi,\theta) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi), 0 \le r \le 2, 0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$. Let B be the box in $r\phi\theta$–space bounded by the planes $r=0,\ r=2,\ \theta=0,\ \theta=2\pi, \phi=0$, and $\phi=\pi$.

$$\begin{aligned}
&8\pi^2 \iiint_S \left(x^2+y^2\right)\left(x^2+y^2+z^2\right)\ dV \\
&= 8\pi^2 \iiint_B \Big((r\cos(\theta)\sin(\phi))^2 + (r\sin(\theta)\sin(\phi))^2\Big)\cdot \\
&\qquad \Big((r\cos(\theta)\sin(\phi))^2 + (r\sin(\theta)\sin(\phi))^2 + r^2\cos^2(\phi)\Big)\cdot \\
&\qquad J(\vec{S}(r,\phi,\theta)\ dV \\
&= 8\pi^2 \iiint_B r^6 \sin^3\phi\ dV \\
&= 8\pi^2 \int_0^{\pi}\int_0^{2\pi}\int_0^2 r^6\sin^3\phi\ dr\,d\theta\,d\phi \\
&= 8\pi^2 \frac{2^7}{7}\int_0^{\pi}\int_0^{2\pi} \sin^3\phi\ d\theta\,d\phi \\
&= \frac{2^{11}\pi^3}{7}\int_0^{\pi}\sin^3\phi\ d\phi = \frac{2^{13}\pi^3}{21}. \qquad \blacksquare
\end{aligned}$$

EXAMPLE 4: Let S be the solid of Example 3 of Section 5.6.

(a) Find the volume of S; and

(b) Calculate $\iiint_S (x+y+z)\ dV$.

SOLUTION: In the solution of Example 3 of Section 5.6, we obtained the parametrization of S and the Jacobian.

$$\begin{aligned} \vec{h}(t,u,v) &= (t, ut^2, vt^2), 0 \le t \le 1, 0 \le u \le 1, \text{ and } 0 \le v \le 1 \\ J(\vec{h}(t,u,v)) &= t^4. \end{aligned}$$

Letting B be the box in tuv–space bounded by the planes $t = 0, t = 1, u = 0, u = 1, v = 0$, and $v = 1$, we have

(a) The volume of S is given by

$$\iiint_S dV = \iiint_B J(\vec{h}(t,u,v))\ dV = \int_0^1 \int_0^1 \int_0^1 t^4\, dt\, du\, dv = \frac{1}{5};$$

and

(b)

$$\begin{aligned} \iiint_S (x+y+z)\ dV &= \iiint_B \left(t + ut^2 + vt^2\right) J(\vec{h}(t,u,v))\ dV \\ &= \int_0^1 \int_0^1 \int_0^1 \left(t + ut^2 + vt^2\right) t^4\ dt\, du\, dv \\ &= \int_0^1 \int_0^1 \int_0^1 \left(t^5 + ut^6 + vt^6\right)\ dt\, du\, dv \\ &= \int_0^1 \int_0^1 \left(\frac{1}{6} + \frac{u}{7} + \frac{v}{7}\right)\ dt\, du\, dv \\ &= \frac{13}{42}. \end{aligned}$$

■

EXAMPLE 5: Let S be the solid of Example 5 of the previous section. Find

(a) The volume of S; and

(b) $\iiint_S (xyz)\ dV$.

SOLUTION: From Example 5 of Section 5.6,

$$\vec{h}(u,v,w) = \begin{pmatrix} u\sqrt{w} \\ wv \\ w \end{pmatrix}; \ -1 \le u \le 1, 0 \le v \le 1, \text{ and } 0 \le w \le 1.$$

Let B be the box in uvw–space defined by $-1 \le u \le 1, 0 \le v \le 1$ and $0 \le w \le 1$. Direct computation shows that $J(\vec{h}(u,v,w)) = w^{3/2}$. Thus

(a) The volume of S is

$$\begin{aligned} \iiint_S dV &= \iiint_B J(\vec{h}(u,v,w))\ dV \\ &= \int_0^1 \int_0^1 \int_{-1}^1 w^{3/2}\ du\, dv\, dw = \frac{4}{5}; \end{aligned}$$

and

(b)

$$\iiint_S (xyz)\ dV = \int_0^1 \int_0^1 \int_{-1}^1 uvw^4\ du\, dv\, dw = 0.$$

■

EXERCISES 5.7

In Exercises 1–6, evaluate $\iiint_B f(x,y,z)\ dV$.

1. B is the box bounded by the planes $x = -1, x = 1,\ y = 0,\ y = 1,\ z = 2,\ z = 3$, and $f(x,y,z) = xyz$.

2. B is the box bounded by the planes $x = 1, x = 2,\ y = 0,\ y = 1,\ z = -1,\ z = 3$, and $f(x,y,z) = x + xy + xyz$.

3. B is the box bounded by the planes $x = -1, x = 2, y = -1,\ y = 1,\ z = -1,\ z = 3$, and $f(x,y,z) = x + xz\cos(\pi xy)$.

4. B is the box bounded by the planes $x = 0, x = 2,\ y = -2,\ y = 1,\ z = -1,\ z = 0$, and $f(x,y,z) = e^{x+y+z}$.

5. B is the box bounded by the planes $x = 0, x = 2,\ y = 1,\ y = 2,\ z = \pi,\ z = 2\pi$, and $f(x,y,z) = \dfrac{e^x \ln y \sin(z)}{y}$.

6. B is the box bounded by the planes $x = 0, x = 2,\ y = 1,\ y = 2, z = 0,\ z = 3$, and $f(x,y,z) = \dfrac{e^x \ln y \sin(z)}{y}$.

In Exercises 7 and 8, the vectors $\vec{A}$, $\vec{B}$, *and* $\vec{C}$ *are drawn emanating from the position* $\vec{D}$ *to form adjacent edges of a solid parallelepiped* P *(See Exercise 1 of the previous section). Calculate* $\iiint_P f(x,y,z)\ dV$.

7. $\vec{A} = (0,1,1), \vec{B} = (-1,2,1), \vec{C} = (1,1,1), \vec{D} = (0,0,0)$, and $f(x,y,z) = x + y + z$

8. $\vec{A} = (0,2,1),\ \vec{B} = (-1,2,1),\ \vec{C} = (1,-1,3),\ \vec{D} = (1,1,1)$, and $f(x,y,z) = xy$

9. The box B bounded by the planes $x = -1$ m, $x = 1$ m, $y = 0$ m, $y = 1$ m, $z = 2$ m, and $z = 3$ m, has constant mass density ρ kg/m^3 and it is rotating about the x–axis at a rate of f rotations/sec. Find its kinetic energy.

In Exercises 10–12, the mass density of a sphere of radius 1 m centered at the origin is given by $\rho(x, y, z) = \sqrt{x^2 + y^2}$ *kg/m*^{3}.

10. Find the mass of the sphere.

11. Find the center of mass of the sphere.

12. Find the kinetic energy of the sphere if it is rotating about the z–axis at a rate of f rotations/sec.

13. Find the volume of the solid of Exercise 4 of Section 14.6.

14. Find the volume of the solid of Exercise 5 of Section 14.6.

15. Find the volume of the solid of Exercise 6 of Section 14.6.

16. Find the volume of the solid of Exercise 7 of Section 14.6.

17. Find the volume of the solid of Exercise 8 of Section 14.6.

18. Find the volume of the solid of Exercise 22 of Section 14.6.

19. Find the volume of the solid of Exercise 23 of Section 14.6.

20. Calculate $\iiint_S (xyz)\ dV$, where S is the solid of Exercise 28 of Section 14.6.

21. Calculate $\iiint_S (x+y+z)\ dV$, where S is the solid of Exercise 23 of Section 14.6.

22. Calculate $\iiint_S x\ dV$, where S is the solid of Exercise 28 of Section 14.6.

23. Calculate $\iiint_S (x+y+z)\ dV$, where S is the solid of Exercise 29 of Section 14.6.

24. Let S be the solid consisting of the points that satisfy both of the inequalities $x^2 + y^2 \leq 1$ and $x^2 + z^2 \leq 1$. Find the volume of S.