Introduction to Vector Calculus

Phillip Zenor
Department of Mathematics
Auburn University

Edward E. Slaminka
Department of Mathematics
Auburn University

Donald Thaxton
Department of Physics
Auburn University

October 15, 2008

4.4 Arc Length for Curves in Other Coordinate Systems

Arc Length for Polar Coordinates

Suppose that $\vec{u}(t) = (r(t), \theta(t)),\ a \le t \le b$, is a parametrization of a path in the plane given by polar coordinates. Then $(x(t), y(t)) = \vec{s}(t) = \vec{P}(\vec{u}(t))$ is a parametrization for the path in rectangular coordinates. The derivative is given by

$$\frac{d}{dt}(x(t), y(t)) = \frac{d\vec{s}(t)}{dt} = \begin{pmatrix} \cos(\theta(t)) & -r(t)\sin(\theta(t)) \\ \sin(\theta(t)) & r(t)\cos(\theta(t)) \end{pmatrix} \begin{pmatrix} r'(t) \\ \theta'(t) \end{pmatrix}.$$

Thus

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \vec{s}\,'(t) = \begin{pmatrix} r'(t)\cos(\theta(t)) - \theta'(t)r(t)\sin(\theta(t)) \\ r'(t)\sin(\theta(t)) + \theta'(t)r(t)\cos(\theta(t)) \end{pmatrix}.$$

We now have

$$\|\vec{s}\,'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$$
$$= \Big(((r'(t)\cos(\theta(t)) - \theta'(t)r(t)\sin(\theta(t)))^2 + (r'(t)\sin(\theta(t)) + \theta'(t)r(t)\cos(\theta(t)))^2\Big)^{1/2}.$$

Suppressing t and expanding, we obtain

$$\|\vec{s}\,'\| = \Big(r'^2\cos^2\theta + \theta'^2 r^2\sin^2\theta - 2r'\theta' r\cos\theta\sin\theta + r'^2\sin^2\theta + \theta'^2r^2\cos^2\theta + 2r'\theta' r\sin\theta\cos\theta\Big)^{1/2}$$

$$= \Big(r'^2(\cos^2\theta + \sin^2\theta) + \theta'^2r^2(\sin^2\theta + \cos^2\theta)\Big)^{1/2}$$
$$= \sqrt{(r'(t))^2 + (\theta'(t)r(t))^2}.$$

Thus the length of the curve, where $\vec{u}(a) = \vec{A},\ \vec{u}(b) = \vec{B}$, and the image of $\vec{u}$ is C, is given by

$$L = \int_{\vec{A}_C}^{\vec{B}} ds = \int_a^b \sqrt{(r'(t))^2 + (\theta'(t)r(t))^2}\ dt. \qquad (1)$$

EXAMPLE 1: Let C be the graph with polar equation $r = k$, for $0 \le \theta \le \pi$. Thus C is a vertical line in $r\theta$–space. The graph $\vec{P}(C)$ is the top half of the circle with radius k. (See Figure 1.)

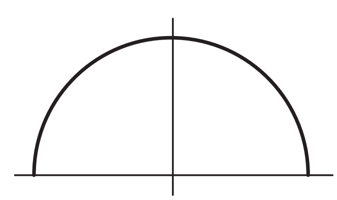

Figure 1. *The graph $\vec{P}(C)$ is the top half of the circle with radius k.*

A parametrization for C in polar coordinates is $\vec{u}(t) = (k, t), 0 \leq t \leq \pi$. The arc length is given by $\int_{\vec{A}_C}^{\vec{B}} ds$, where $\vec{A} = \vec{P}(\vec{u}(0)) = (k, 0)$ and $\vec{B} = \vec{P}(\vec{u}(\pi)) = (-k, 0)$. $r'(t) = 0$ and $\theta'(t) = 1$. Using Equation (1), we obtain

$$\begin{aligned} L &= \int_{(k,0)_C}^{(-k,0)} ds = \int_0^{\pi} \sqrt{(r'(t))^2 + (\theta'(t)r(t))^2}\, dt \\ &= \int_0^{\pi} \sqrt{k^2}\, dt = \pi k. \end{aligned}$$

■

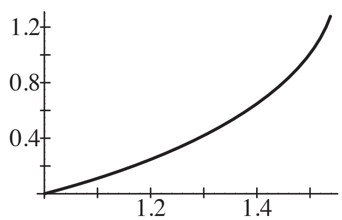

Figure 2. *The graph of the curve in Example 2.*

EXAMPLE 2: Let C be the graph with polar equation $r = e^{\theta}$, for $0 \leq \theta \leq \ln 2$. A parametrization for C in polar coordinates is $\vec{u}(t) = (e^t, t), 0 \leq t \leq \ln 2$. Thus $r'(t) = e^t$ and $\theta'(t) = 1$. The arc length is given by $\int_{\vec{A}_C}^{\vec{B}} ds$, where $\vec{A} = \vec{P}(u(0)) = (1, 0)$ and $\vec{B} = \vec{P}(u(\ln 2)) = (2\cos(\ln 2), 2\sin(\ln 2))$. Using Equation (1), we obtain

$$\begin{aligned} L = \int_{(1,0)}^{(2\cos(\ln 2), 2\sin(\ln 2))} d\vec{s} &= \int_0^{\ln 2} \sqrt{(r'(t))^2 + (\theta'(t)r(t))^2}\, dt \\ &= \int_0^{\ln 2} \sqrt{e^{2t} + e^{2t}}\, dt \\ &= \sqrt{2} \int_0^{\ln 2} e^t\, dt \\ &= 2\sqrt{2} - \sqrt{2} = \sqrt{2}. \end{aligned}$$

■

Arc Length for Cylindrical Coordinates

In a similar fashion, we can compute arc length in cylindrical coordinates in $\mathbb{R}^3$. If $\vec{u}(t) = (r(t), \theta(t), z(t))$ is a parametrization of a path in $\mathbb{R}^3$, then $(x(t), y(t), z(t)) = \vec{s}(t) = \vec{C}(\vec{u}(t))$ is a parametrization of the path in rectangular coordinates. The derivative is given by

$$\begin{aligned} \frac{d}{dt}(x(t), y(t), z(t)) &= \frac{d\vec{s}(t)}{dt} \\ &= \begin{pmatrix} \cos(\theta(t)) & -r(t)\sin(\theta(t)) & 0 \\ \sin(\theta(t)) & r(t)\cos(\theta(t)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r'(t) \\ \theta'(t) \\ z'(t) \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \vec{s}\,'(t) = \begin{pmatrix} r'(t)\cos(\theta(t)) - \theta'(t)r(t)\sin(\theta(t)) \\ r'(t)\sin(\theta(t)) + \theta'(t)r(t)\cos(\theta(t)) \\ z'(t) \end{pmatrix}.$$

Suppressing the variable t,

$$\begin{aligned} \|\vec{s}\,'\| &= \sqrt{x'^2 + y'^2 + z'^2} \\ &= \sqrt{(r'\cos\theta - \theta' r\sin\theta)^2 + (r'\sin\theta + \theta' r\cos\theta)^2 + z'^2} \\ &= \sqrt{r'2^2\cos^2\theta + \theta'^2 r^2\sin^2\theta - 2r'\theta' r\cos\theta\sin\theta +} \\ &= \overline{r'^2\sin^2\theta + \theta'^2 r^2\cos^2\theta + 2r'\theta' r\sin\theta\cos\theta + z'^2} \\ &= \sqrt{r'^2(\cos^2\theta + \sin^2\theta) + \theta'^2 r^2(\cos^2\theta + \sin^2\theta) + z'^2} \\ &= \sqrt{r'^2 + r^2\theta'^2 + z'^2}. \end{aligned}$$

Thus the length of the curve is given by

$$L = \int_{\vec{A}_C}^{\vec{B}} dr = \int_a^b \sqrt{r'^2(t) + r^2(t)\theta'^2 + z'^2(t)}\; dt. \tag{2}$$

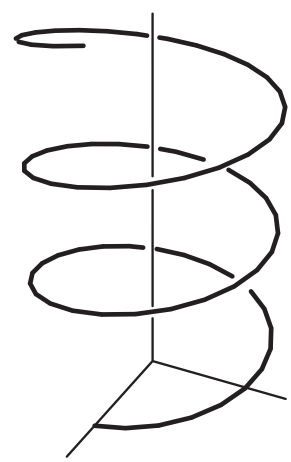

Figure 3. *The helix from Example 3.*

EXAMPLE 3: Let $\vec{u}(t) = (1, 2\pi t, t)$, for $0 \le t \le 3$, be a parametrization of a helix in cylindrical coordinates. Find the length of the helix. See Figure 3.

SOLUTION: $\vec{s}(t) = \vec{C}(\vec{u}(t))$. By Equation (2),

$$\begin{aligned} \|\vec{s}\,'(t)\| &= \sqrt{r'^2(t) + r^2(t)\theta'^2(t) + z'^2(t)} \\ &= \sqrt{(2\pi)^2 + 1} = \sqrt{4\pi^2 + 1} \end{aligned}$$

Thus the arc length is given by

$$L = \int_0^3 \sqrt{4\pi^2 + 1}\; dt = 3\sqrt{4\pi^2 + 1}.$$

■

Arc Length for Spherical Coordinates

For spherical coordinates we follow a similar procedure. Let $\vec{u}(t) = (\rho(t), \phi(t),\ \theta(t))$ be a parametrization of a path in $\mathbb{R}^3$ in spherical coordinates. Then $(x(t), y(t), z(t)) = \vec{s}(t) = \vec{S}(\vec{u}(t))$ will be a parametrization of the path in rectangular coordinates. The derivative is given by

$$\frac{d}{dt}(x(t), y(t), z(t)) = \frac{d\vec{s}(t)}{dt}$$
$$= \begin{pmatrix} \cos\theta(t)\sin\phi(t) & \rho(t)\cos\theta(t)\cos\phi(t) & -\rho(t)\sin\theta(t)\sin\phi(t) \\ \sin\theta(t)\sin\phi(t) & \rho(t)\sin\theta(t)\cos\phi(t) & \rho(t)\cos\theta(t)\sin\phi(t) \\ \cos\phi(t) & -\rho(t)\sin\phi(t) & 0 \end{pmatrix} \begin{pmatrix} \rho'(t) \\ \phi'(t) \\ \theta'(t) \end{pmatrix}.$$

Suppressing the variable t,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \vec{s}\,' = \begin{pmatrix} \rho'\cos\theta\sin\phi + \phi'\rho\cos\theta\cos\phi - \theta'\rho\sin\theta\sin\phi \\ \rho'\sin\theta\sin\phi + \phi'\rho\sin\theta\cos\phi + \theta'\rho\cos\theta\sin\phi \\ \rho'\cos\phi - \phi'\rho\sin\phi \end{pmatrix}.$$

We now have

$$\begin{aligned} \|\vec{s}\,'\| &= \sqrt{x'^2 + y'^2 + z'^2} \\ &= \big((\rho'\cos\theta\sin\phi + \phi'\rho\cos\theta\cos\phi - \theta'\rho\sin\theta\sin\phi)^2 \\ &\quad +(\rho'\sin\theta\sin\phi + \phi'\rho\sin\theta\cos\phi + \theta'\rho\cos\phi\sin\phi)^2 \\ &\quad +(\rho'\cos\phi - \theta'\rho\sin\phi)^2\big) \\ &= \sqrt{\rho'^2 + \rho^2\phi'^2 + \rho^2\theta'^2\sin^2\phi}. \end{aligned}$$

By expanding and simplifying, we obtain

$$L = \int_{\vec{A}_C}^{\vec{B}} dr = \int_a^b \sqrt{\rho'^2(t) + \rho^2(t)\phi'^2(t) + \rho^2(t)\theta'^2(t)\sin^2\phi(t)}\; dt.$$

EXERCISES 4.4

In Exercises 1–5, find the length, in xy–space, of the graphs with the given polar equation.

1. $r = e^{2\theta}, \quad 0 \le \theta \le \ln 3.$
2. $r = 2\cos(\theta), \quad 0 \le \theta \le 2\pi.$
3. $r = 3\sec(\theta), \quad 0 \le \theta \le \frac{\pi}{4}.$
4. $r = \cos^2(\theta/2), \quad 0 \le \theta \le 2\pi.$
5. $r = \sin^2(\theta/2), \quad 0 \le \theta \le 2\pi.$
6. Let $\vec{h}(t) = (\rho(t), \phi(t), \theta(t))$ describe, in spherical coordinates, the location of a particle at time t. Show that the magnitude of the particle's velocity in xyz–space is given by

$$\sqrt{\rho'^2(t) + \rho^2(t)\phi'^2(t) + \rho^2(t)\theta'^2(t)\sin^2\phi(t)}.$$

7. Find the arc length of the helix given (in xyz–coordinates) by

$$\vec{r}(t) = (\cos(2\pi t), \sin(2\pi t), t), 0 \leq t \leq 4,$$

using both the xyz–coordinate version of the arc length integral and the cylindrical coordinate version.

8. The path parametrized by

$$\vec{r}(t) = \begin{pmatrix} \cos(2\pi t)\sin\left(\frac{t\pi}{4}\right) \\ \sin(2\pi t)\sin\left(\frac{t\pi}{4}\right) \\ \cos\left(\frac{t\pi}{4}\right) \end{pmatrix}, \ 0 \leq t \leq 4.$$

in xyz–coordinates describes a "spiral" along the unit sphere centered at the origin, starting at the north pole and descending to the south pole. Compute the arc length of this path using both xyz–coordinates and spherical coordinates.

9. Let $\vec{r}(t) = (1, 2\pi t, t), 0 \leq t \leq 4$, represent the helix of Exercise 7 in cylindrical coordinates.

   a. Show that the helix can be represented as

$$\vec{u}(t) = \left(\sqrt{t^2+1}, \text{Arcsin}\left(\frac{1}{\sqrt{t^2+1}}\right), 2\pi t\right),$$

   $0 \leq t \leq 4$, in spherical $\rho\phi\theta$–coordinates.

   b. Compute the arc length in spherical coordinates, and compare it to Exercise 7.

4.5 Change of Area with Linear Transformations

In this section, we introduce the idea of the rate that a linear transformation changes area or volume. First, it is helpful to recall some facts about areas of parallelograms and volumes of parallelepipeds.

- Suppose that the vectors $\vec{A}$ and $\vec{B}$ are drawn emanating from a common point in $\mathbb{R}^3$ forming adjacent edges of a parallelogram $\mathcal{P}$. Then $\|\vec{A} \times \vec{B}\|$ is the area of $\mathcal{P}$.

- If the vectors $\vec{A} = (a_1, b_1)$ and $\vec{B} = (b_1, b_2)$ are drawn emanating from the origin in $\mathbb{R}^2$ forming adjacent edges of a parallelogram $\mathcal{P}$, then $\|\vec{(a_1, a_2, 0)} \times \vec{(b_1, b_2, 0)}\| = |a_1 b_2 - a_2 b_1|$ is the area of $\mathcal{P}$.

- $\left|\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\right| = |a_1 b_2 - a_2 b_1|$ (which is the area of $\mathcal{P}$).

- If the vectors $\vec{A} = (a_1, a_2, a_3)$, $\vec{B} = (b_1, b_2, b_3)$ and $\vec{C} = (c_1, c_2, c_3)$ are drawn emanating from the origin and they do not lie in a common plane (they are not co-planer), then they form the adjacent edges of a parallelepiped, $\mathcal{P}$. The magnitude of their triple product is the volume of $\mathcal{P}$. Computationally, the volume of $\mathcal{P}$ is

$$|\vec{A} \cdot (\vec{B} \times \vec{C})| = \left| \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right|.$$

The following theorem is be useful in this and subsequent sections.

Theorem 1

$$det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

and

$$det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

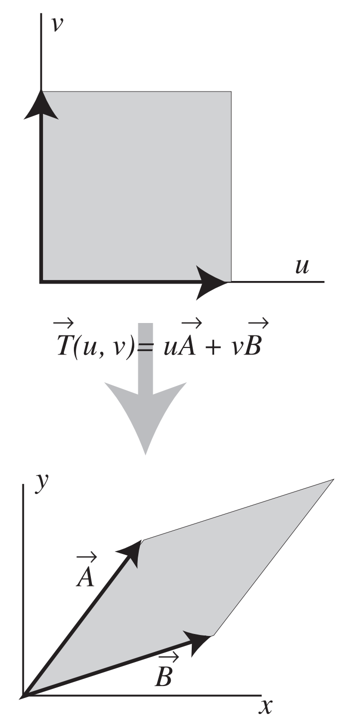

Figure 1. *The image of the unit square* $0 \le u \le 1$, $0 \le v \le 1$ *is the parallelogram with adjacent edges* $\vec{A}$ *and* $\vec{B}$ *drawn emanating from the origin.*

While this theorem is easily proven with direct computation, it seems rather remarkable that the parallelepiped determined by the vectors (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3) has the same volume as the parallelepiped determined by (a_1, b_1, c_1), (a_2, b_2, c_2), and (a_3, b_3, c_3).

Let $\vec{T}$ be defined by

$$\vec{T}(u, v) = u\vec{A} + v\vec{B},$$

where $\vec{A}$ and $\vec{B}$ are vectors in $\mathbb{R}^3$. If $\vec{A}$ and $\vec{B}$ are drawn emanating from the origin, they are adjacent sides of a parallelogram P with area $\|\vec{A} \times \vec{B}\|$. Let R be the unit square in uv–space with adjacent sides the vectors $(1, 0)$ and $(0, 1)$. The area of R is one square unit, and $\vec{T}(R) = P$, which has an area of $\|\vec{A} \times \vec{B}\|$. (See Figure 1.) Thus $\vec{T}$ will take one square unit of area onto a parallelogram having area $\|\vec{A} \times \vec{B}\|$. It turns out that $\|\vec{A} \times \vec{B}\|$ can properly be thought of as *the rate that* T *changes area.* If C is a set in uv–space, then the area of $\vec{T}(C)$ is Area$(C)\|\vec{A} \times \vec{B}\|$.

If $\vec{A} = (a_1, a_2)$ and $\vec{B} = (b_1, b_2)$ are in $\mathbb{R}^2$, then

$$\|(a_1, a_2, 0) \times (b_1, b_2, 0)\| = |a_1 b_2 - a_2 b_1| = \left| \det D\vec{T} \right|$$

gives the area of the parallelogram determined by $\vec{A}$ and $\vec{B}$. Thus, if $\vec{T}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$, then $|\det D(\vec{T})|$ is the rate that $\vec{T}$ changes area.

In the same fashion, if $\vec{T}$ is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$, then $\left|\det(D\vec{T})\right|$ is the rate that $\vec{T}$ changes volume.

To recapitulate:

- If $\vec{T}(r) = r\vec{A}$ is a linear transformation from $\mathbb{R}$ into $\mathbb{R}^n$, then $\|\vec{A}\|$ is the rate that $\vec{T}$ changes length.
- If $\vec{T}(\vec{r}) = A_T\vec{r}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$, then $|\det A_T|$ is the rate that $\vec{T}$ changes area.
- If $\vec{T}(\vec{r}) = A_T\vec{r}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$, then $\left\|\vec{T}(1,0) \times \vec{T}(0,1)\right\|$ is the rate that $\vec{T}$ changes area.
- If $\vec{T}(\vec{r}) = A_T\vec{r}$ is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$, then $|\det A_T|$ is the rate that $\vec{T}$ changes volume.

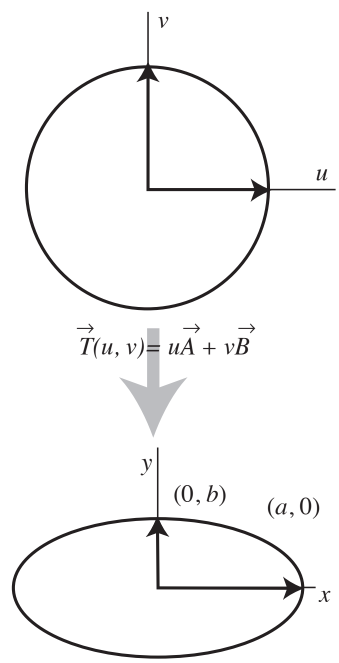

Figure 2. *The image of the unit disc $u^2 + v^2 \le 1$ is the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \le 1$*

EXAMPLE 1: Let $\vec{A} = (a, 0)$ and $\vec{B} = (0, b)$, where a and b are positive numbers. Let $\vec{T}$ be defined by

$$\vec{T}(u, v) = u\vec{A} + v\vec{B} = (au, bv).$$

Let C be the unit circle $u^2 + v^2 = 1$. Then $\vec{T}(C)$ is the ellipse with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. See Figure 2. The area bounded by C is π and the area bounded by $\vec{T}(C)$ is $\|\vec{A} \times \vec{B}\|\pi = ab\pi$. Thus the area bounded by the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is $ab\pi$. This agrees with the value obtained by integration. ■

EXAMPLE 2: Let $\mathcal{C}$ be the unit circle in uv–space.

(a) Let $\vec{T}(u, v) = \begin{pmatrix} 2u + 3v \\ u \\ u + v \end{pmatrix}$. Then $\vec{T}(1, 0) = (2, 1, 1)$ and $\vec{T}(0, 1) = (3, 0, 1)$. The rate that $\vec{T}$ changes area is $\|(2, 1, 1) \times (3, 0, 1)\| = \sqrt{11}$. The area of $\vec{T}(\mathcal{C})$ is $\pi\sqrt{11}$.

(b) Let $\vec{T}(u, v) = \begin{pmatrix} 2u + 3v \\ u \end{pmatrix}$. The rate that $\vec{T}$ changes area is

$$|\det\left(D\vec{T}\right)| = \left|\det\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}\right| = 3.$$

Since the area of the unit circle is π, the area of $\vec{T}(\mathcal{C})$ is 3π. ■

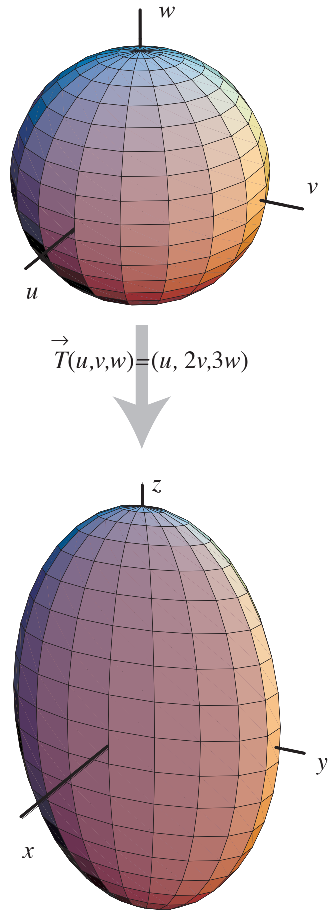

Figure 3.
$\vec{T}(u,v,w) = (u, 2v, 3w)$ takes the unit sphere onto the ellipse $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$.

EXAMPLE 3: Let $\mathcal{S}$ denote the surface bounded between the graphs of $v = u^2$ and $v = u$. Let $\vec{T}(u, v) = (2u + v, v - u)$. Find the area of the image of $\mathcal{S}$.

SOLUTION: The area of $\mathcal{S}$ is $\int_0^1 u - u^2\, du = \frac{1}{6}$. The rate that $\vec{T}$ changes area is $\left|\det\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\right| = 3$. Thus the area of $\vec{T}(\mathcal{S}) = (3) * \left(\frac{1}{6}\right) = \frac{1}{2}$. ■

EXAMPLE 4: Let $\vec{T}$ be the transformation defined by

$$\vec{T}(u, v, w) = (au, bv, cw).$$

Then $\vec{T}$ takes the unit sphere $u^2 + v^2 + w^2 \leq 1$ onto the ellipsoid E with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1$. See Figure 3 for the case that $a = 1$, $b = 2$, and $c = 3$.

$$D\vec{T} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Thus $\vec{T}$ changes volume at a rate of abc, since the unit cube goes onto a box with sides of length a, b, and c. Since the volume bounded by a unit sphere is $\left(\frac{4}{3}\right)\pi$, the volume bounded by the ellipsoid E is $\frac{4}{3}abc\pi$. ■

If $\vec{T}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}$, then $\vec{T}$ takes an object with area onto an object with no area. Therefore, the rate that $\vec{T}$ changes area is zero. This is consistent with the fact that $\vec{T}(1, 0)$ and $\vec{T}(0, 1)$ must point in the same or opposite directions (they are either a positive number, a negative number, or zero.) Indeed, if the domain of $\vec{T}$ is $\mathbb{R}^2$, then:

- If the range of $\vec{T}$ is $\mathbb{R}^1$, then the rate that $\vec{T}$ changes area is zero.

- If the range of $\vec{T}$ is $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\vec{T}(0, 1) \times \vec{T}(1, 0) = \vec{0}$ if and only if the image of $\vec{T}$ is a line or the origin.

Similarly, if the domain of $\vec{T}$ is $\mathbb{R}^3$, then:

- If the range of $\vec{T}$ is either $\mathbb{R}$ or $\mathbb{R}^2$, then the rate that $\vec{T}$ changes volume is zero.

- If the range is $\mathbb{R}^3$ then $\det A_{\vec{T}} = 0$ if and only if the image of $\vec{T}$ is a plane, a line, or the origin.

Summary

If $\vec{T}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$, then the rate that $\vec{T}$ changes area is $|\det(A_{\vec{T}})| = |\det(DT)|$.

If $\vec{T}$ is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$, then the rate that $\vec{T}$ changes volume is $|\det(A_{\vec{T}})| = |\det(DT)|$.

If $\vec{T}(u,v) = u\vec{A} + v\vec{B}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$, then the rate that $\vec{T}$ changes area is $\|\vec{A} \times \vec{B}\|$.

If $\vec{T}$ is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ or $\mathbb{R}$, then the rate that $\vec{T}$ changes volume is 0.

If $\vec{T}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}$, then the rate that $\vec{T}$ changes volume is 0.

EXERCISES 4.5

In Exercises 1–5, determine the rate that each transformation changes area.

1. $\vec{T}(u,v) = (3u, -2v)$.
2. $\vec{T}(u,v) = (u + 3v, v - u)$.
3. $\vec{T}(u,v) = (u - v, 2v, u + v)$.
4. $\vec{T}(s,t) = (s - t, 3t + s, t - s)$.
5. $\vec{T}(u,v) = u + v$.

In Exercises 6–10, determine the rate that each transformation changes volume.

6. $\vec{T}(u,v,w) = (2u - v, u + w, u + v + w)$.
7. $\vec{T}(r,s,t) = (2r + s - t, r - s - 3t, r + s + t)$.
8. $\vec{T}(u,v,w) = (2u - w + v, u + v - 22w)$.
9. $\vec{T}(u,v,w) = (u + v + w, u + v + w)$.
10. $\vec{T}(u,v,w) = u + v + w$.
11. Find a linear transformation from the uv–plane into the xy–plane that takes the circle $u^2 + v^2 = 1$ onto the ellipse E with equation $4x^2 + 9y^2 = 36$. Find the area bounded by E using the techniques of this section.
12. Find the area bounded by the ellipse $2x^2 + 3y^2 = 5$.
13. Find a linear transformation from uvw–space onto xyz–space that takes the unit sphere centered at the origin onto the ellipsoid E with equation $3x^2 + 4y^2 + 2z^2 = 1$. Find the volume bounded by E.
14. Let $\mathcal{R}$ be the rectangle bounded between the lines $u = 2$, $u = -2$, $v = 0$, and $v = 4$ and let $\vec{T}(u,v) = (-u + v, u - 2v, 2u + 4v)$. Find the area of $\vec{T}(\mathcal{R})$.
15. Let $\mathcal{R}$ be the rectangle bounded between the lines $u = 2$, $u = -2$, $v = 2$, and $v = 6$ and let $\vec{T}(u,v) = (-u + v, 2u - 2v)$. Find the area of $\vec{T}(\mathcal{R})$.

16. Let $\mathcal{R}$ be the rectangle bounded between the lines $u = 2$, $u = -2$, $v = 2$, and $v = 6$ and let $\vec{T}(u,v) = (-u+v, u-2v)$. Find the area of $\vec{T}(\mathcal{R})$.

17. Let $\mathcal{R}$ be the region bounded between the graphs of $v = u^2$ and $v = u^3$ and let $\vec{T}(u,v) = (u+v, u-2v, 2u+v)$. Find the area of $\vec{T}(\mathcal{R})$.

18. Let $\mathcal{R}$ be the region bounded between the graphs of $u = v^2$ and $u = v^3$ and let $\vec{T}(u,v) = (3u+v, u-2v)$. Find the area of $\vec{T}(\mathcal{R})$.

19. Let $\mathcal{C}$ be the box bounded between the planes $u = 2$, $u = -2$, $v = 3$, $v = 4$, $w = 0$, and $w = 10$. Let $\vec{T}(u,v,w) = (-u+v+2w, u-2v, 2u+4v+w)$. Find the volume of $\vec{T}(\mathcal{C})$.

20. Let $\mathcal{E}$ be the ellipsoid $\frac{u^2}{4} + \frac{y^2}{9} + z^2 = 1$, and let $\vec{T}(u,v) = (u+v+2w, u-2v, 2u+v+w)$. Find the volume of $\vec{T}(\mathcal{E})$.

21. Let $\mathcal{V}$ be the solid obtained by rotating the region bounded between the u–axis and the graph of $v = u^2+1$, $-1 \le u \le 1$ about the u–axis. Let $\vec{T}(u,v,w) = (u+v+2w, u-2v, 2u+v+w)$. Find the volume of $\vec{T}(\mathcal{E})$.

4.6 The Jacobian

When a function from $\mathbb{R}^n$ into $\mathbb{R}^m$ is not linear, then the rate that the function changes area or volume becomes a local property. In this section, we learn how a function changes area or volume at a point. This idea is not really new. Let $\vec{f}$ be a function from $\mathbb{R}$ into $\mathbb{R}^n$. Recall the geometric interpretation of the derivative, in which $\|\vec{f}\,'(t)\|$ represents the rate that $\vec{f}$ changes arc length at t. To see that this is a reasonable interpretation, let $[t, t+h]$ be an interval in the domain of $\vec{f}$. (See Figure 1.) If h is small, then $\|\vec{f}(t+h) - \vec{f}(t)\|$ is an approximation of the length of $\vec{f}([t, t+h])$, and

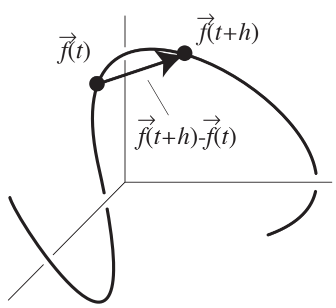

Figure 1. $\|\vec{f}(t+h) - \vec{f}(t)\|$ *approximates the arc length from* $\vec{f}(t)$ *to* $\vec{f}(t+h)$.

$$\begin{aligned} \|\vec{f}\,'(t)\| &= \lim_{h\to 0} \frac{\text{length of } \vec{f}([t,t+h])}{\text{length of } [t,t+h]} \\ &= \lim_{h\to 0} \frac{\|\vec{f}(t+h) - \vec{f}(t)\|}{h} \,. \end{aligned}$$

The rate that $\vec{f}$ changes length at t is called the *Jacobian* of $\vec{f}$ at t. We denote the Jacobian of $\vec{f}$ at t by $J\vec{f}(t)$. Notice that if g is a function from a subset of $\mathbb{R}^n$ into $\mathbb{R}$ and if $\vec{r}$ is a parametrization for a curve C with endpoints $\vec{A}$ and $\vec{B}$ and domain $[a,b]$, then

$$\int_{\vec{A}_C}^{\vec{B}} g\; d\vec{r} = \int_a^b g(\vec{r}(t)) J\vec{r}(t)\; dt.$$

The Jacobian can also be defined for functions with domain in $\mathbb{R}^n$

for any positive integer n. However, we define it only for functions with domain in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Let D be a subset of $\mathbb{R}^2$ (uv–space), and let $\vec{f}$ be a function from D into $\mathbb{R}^2$ or $\mathbb{R}^3$. We want the Jacobian of $\vec{f}$ at (u, v) to be the rate that $\vec{f}$ changes area at (u, v). Let S_h be the square having sides of length h with vertices $(u, v), (u + h, v), (u, v + h)$ and $(u + h, v + h)$. Assume that S_h is a subset of D. We approximate the surface $\vec{f}(S_h)$ with the parallelogram P_h with the adjacent sides the vectors $\vec{f}(u+h, v) - \vec{f}(u, v)$ and $\vec{f}(u, v+h) - \vec{f}(u, v)$, drawn emanating from $\vec{f}(u, v)$. See Figure 2.

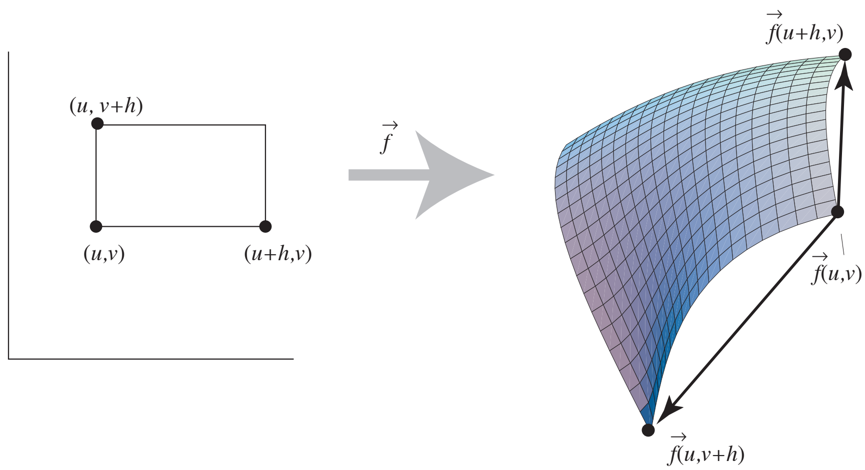

Figure 2. *The area of $\vec{f}(S_h)$ is approximated by the area of P_h.*

The area of P_h is $\|(\vec{f}(u+h, v) - \vec{f}(u, v)) \times (\vec{f}(u, v+h) - \vec{f}(u, v))\|$. The rate that $\vec{f}$ changes area at (u, v) is given by

$$\begin{aligned}
\lim_{h\to 0} \frac{\text{area of } P_h}{\text{area of } S_h} &= \lim_{h\to 0} \frac{\|(\vec{f}(u+h, v) - \vec{f}(u, v)) \times (\vec{f}(u, v+h) - \vec{f}(u, v))\|}{h^2} \\
&= \lim_{h\to 0} \left\| \frac{\vec{f}(u+h, v) - \vec{f}(u, v)}{h} \times \frac{\vec{f}(u, v+h) - \vec{f}(u, v)}{h} \right\| \\
&= \left\| \frac{\partial \vec{f}(u, v)}{\partial u} \times \frac{\partial \vec{f}(u, v)}{\partial v} \right\|.
\end{aligned}$$

Definition: The Jacobian for Functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ or $\mathbb{R}^3$

If f is a differentiable function from $\mathbb{R}^2$ into $\mathbb{R}^2$ or $\mathbb{R}^3$, we define the *Jacobian of* $\vec{f}$, denoted by $\vec{f}(u, v)$, to be

$$\left\|\frac{\partial \vec{f}(u,v)}{\partial u} \times \frac{\partial \vec{f}(u,v)}{\partial v}\right\|.$$

It represents the rate that $\vec{f}$ changes area at (u,v).

Of course, if $\vec{f}$ is a function from a subset of $\mathbb{R}^2$ into $\mathbb{R}$, then $J\vec{f}(u,v) = 0$.

EXAMPLE 1: Let $\vec{P}$ be the polar transformation from $r\theta$–space defined by

$$\vec{P}(r,\theta) = (r\cos\theta, r\sin\theta).$$

The partial derivatives of $\vec{P}$ are

$$\frac{\partial \vec{P}}{\partial r} = (\cos\theta, \sin\theta) \text{ and } \frac{\partial \vec{P}}{\partial \theta} = (-r\sin\theta, r\cos\theta).$$

Thus the Jacobian of $\vec{P}$ is given by

$$J\vec{P}(r,\theta) = \left\|\frac{\partial \vec{P}}{\partial r} \times \frac{\partial \vec{P}}{\partial \theta}\right\| = |r|,$$

and is the rate that $\vec{P}$ changes area at (r,θ). This fits the geometry of the function very well. Recall that the area of a sector of a circle of radius R is $\frac{R^2(\Delta\theta)}{2}$, where $\Delta\theta$ is the angle of the sector. If we consider the square S in $r\theta$–space with sides of length h and one vertex (r,θ) as in Figure 3, then $\vec{P}(S)$ is the portion of a sector as in Figure 3.

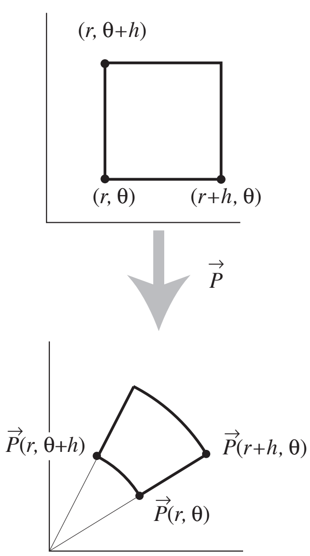

Figure 3. *The square S in $r\theta$–space. The sector $\vec{P}(S)$ in xy–space.*

The area of S is h^2, and the area of $\vec{P}(S)$ is $\frac{(r+h)^2h}{2} - \frac{r^2h}{2} = \frac{(2rh+h^2)h}{2}$. Thus, [area of $\vec{P}(S)$]/[area of S] $= \frac{(2rh+h^2)h/2}{h^2} = r + \frac{h}{2}$. It follows that as h gets close to 0, then [area of $\vec{P}(S)$]/[area of S] gets close to r. ■

EXAMPLE 2: Find the Jacobian of $\vec{h}(u,v) = (u, u^2\cos(v), u^2\sin(v))$.

Solution:

$$\begin{aligned}
\frac{\partial \vec{h}}{\partial u}(u,v) &= (1, 2u\cos(v), 2u\sin(v)).\\
\frac{\partial \vec{h}}{\partial v}(u,v) &= \left(0, -u^2\sin(v), u^2\cos(v)\right).\\
\frac{\partial \vec{h}}{\partial u} \times \frac{\partial \vec{h}}{\partial v} &= \begin{pmatrix} 2u^3\cos^2(v) + 2u^3\sin^2(v) \\ -u^2\cos(v) \\ -u^2\sin(v) \end{pmatrix} = \begin{pmatrix} 2u^3 \\ -u^2\cos(v) \\ -u^2\sin(v) \end{pmatrix}
\end{aligned}$$

$$J\vec{h}(u,v) \quad = \quad \left\| \frac{\partial \vec{h}}{\partial u} \times \frac{\partial \vec{h}}{\partial v} \right\| = \sqrt{4u^6 + u^4}. \qquad \blacksquare$$

Let D be a subset of $\mathbb{R}^3$, and let $\vec{f}$ be a function from D into $\mathbb{R}^3$. We are interested in how $\vec{f}$ changes volume at a point. We proceed exactly as we did with areas and functions from $\mathbb{R}^2$ into $\mathbb{R}^2$.

Let B_h be a square box with three adjacent edges the vectors $[(u+h,v,w)-(u,v,w)], [(u,v+h,w)-(u,v,w)]$, and $[(u,v,w+h)-(u,v,w)]$. Let P_h be the parallelepiped with three adjacent edges $[\vec{f}(u+h,v,w)-\vec{f}(u,v,w)], [\vec{f}(u,v+h,w)-\vec{f}(u,v,w)]$, and $[\vec{f}(u,v,w+h)-\vec{f}(u,v,w)]$. We now approximate the volume of $\vec{f}(B_h)$ with the volume of P_h (see Figure 4).

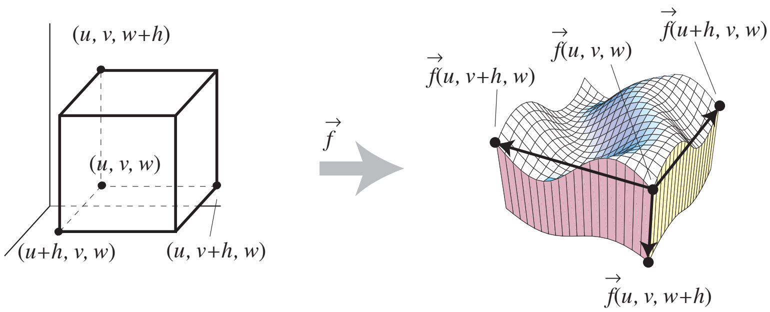

Figure 4. *The volume of $\vec{f}(B_h)$ is approximated by the volume of a parallelepiped.*

The rate that $\vec{f}$ changes volume is given by

$$\lim_{h \to 0} \frac{\text{volume of } P_h}{\text{volume of } B_h}$$

$$= \lim_{h\to 0} \left| \frac{\left(\vec{f}(u+h,v,w)-\vec{f}(u,v,w)\right)\left[\left(\vec{f}(u,v+h,w)-\vec{f}(u,v,w)\right)\times\left(\vec{f}(u,v,w+h)-\vec{f}(u,v,w)\right)\right]}{h^3} \right|$$

$$= \lim_{h\to 0} \left| \frac{\left(\vec{f}(u+h,v,w)-\vec{f}(u,v,w)\right)}{h} \cdot \left[\frac{\left(\vec{f}(u,v+h,w)-\vec{f}(u,v,w)\right)}{h} \times \frac{\left(\vec{f}(u,v,w+h)-\vec{f}(u,v,w)\right)}{h} \right] \right|$$

$$= \left| \frac{\partial \vec{f}}{\partial u} \cdot \left(\frac{\partial \vec{f}}{\partial v} \times \frac{\partial \vec{f}}{\partial w} \right) \right|.$$

Definition: The Jacobian for Functions from $\mathbb{R}^3$ to $\mathbb{R}^3$

If $\vec{f}: \mathbb{R}^3 \to \mathbb{R}^3$ is a differentiable function, we define the *Jacobian of* $\vec{f}$, denoted by $J\vec{f}(u, v, w)$, to be

$$\left| \frac{\partial \vec{f}}{\partial u} \cdot \left(\frac{\partial \vec{f}}{\partial v} \times \frac{\partial \vec{f}}{\partial w} \right) \right|.$$

It represents the rate that $\vec{f}$ changes volume at (u, v, w).

EXAMPLE 3: Let $\vec{C}_z$ be the cylindrical transformation from $r\theta z$–space into xyz–space defined by

$$\vec{C}_z(r, \theta, z) = (r \cos\theta, r \sin\theta, z).$$

Thus

$$\begin{aligned} \frac{\partial \vec{C}_z(r,\theta,z)}{\partial r} &= (\cos\theta, \sin\theta, 0), \\ \frac{\partial \vec{C}_z(r,\theta,z)}{\partial \theta} &= (-r\sin\theta, r\cos\theta, 0), \end{aligned}$$

and

$$\frac{\partial \vec{C}_z(r,\theta,z)}{\partial z} = (0, 0, 1),$$

so

$$J\vec{C}_z(r, \theta, z) = \left| \frac{\partial \vec{C}_z}{\partial r} \cdot \left(\frac{\partial \vec{C}_z}{\partial \theta} \times \frac{\partial \vec{c}}{\partial z} \right) \right| = |r|.$$

EXAMPLE 4: Let S be the spherical transformation defined by

$$\vec{S}(\rho, \phi, \theta) = \begin{pmatrix} \rho\cos(\theta)\sin(\phi) \\ \rho\sin(\theta)\sin(\phi) \\ \rho\cos(\phi) \end{pmatrix}.$$

It is left as an exercise to show that $J\vec{S}(\rho, \phi, \theta) = \rho^2 \sin(\phi)$.

The following theorem follows from direct computation, and its proof is left as an exercise (Exercises 27 and 28).

Theorem 1 *If $\vec{F}$ is a differentiable function from a subset of $\mathbb{R}^2$ into $\mathbb{R}^2$ or from a subset of $\mathbb{R}^3$ into $\mathbb{R}^3$, and $\vec{r}$ is a point in $\mathbb{R}^2$ or $\mathbb{R}^3$, then*

$$J\vec{F}(\vec{r}) = \left| \det\, D(\vec{F})|_{\vec{r}} \right|.$$

If $\vec{F}$ is a differentiable function and $\vec{G}(\vec{r}) = \vec{F}(\vec{r}) + \vec{r}_0$, then $J\vec{F} = J\vec{G}$.

The second part of Theorem 1 is not terribly surprising. A rigid motion such as a translation does not change area or volume. The next example leads to Theorem 2.

EXAMPLE 5: Let $\vec{T_1}(u,v) = \begin{pmatrix} 5u+v \\ -v \end{pmatrix}$, and let $\vec{T_2}(x,y) = \begin{pmatrix} 2x-3y-1 \\ x+y+3 \end{pmatrix}$. Then

$$\begin{aligned} \vec{T_2} \circ \vec{T_1}(u,v) &= \begin{pmatrix} 2(5u+v)-3(-v)-1 \\ (5u+v)+(-v)+3 \end{pmatrix} \\ &= \begin{pmatrix} 10u+5v-1 \\ 5u+3 \end{pmatrix}. \\ J(\vec{T_2} \circ \vec{T_1})(u,v) &= \left| \det D\vec{T_2} \circ \vec{T_1}(u,v) \right| \\ &= \left| \det \begin{pmatrix} 10 & 5 \\ 5 & 0 \end{pmatrix} \right| = 25. \end{aligned}$$

Also,

$$J\vec{T_1}(u,v) = \left| \det \begin{pmatrix} 5 & 1 \\ 0 & -1 \end{pmatrix} \right| = 5$$

and

$$J\vec{T_2}(x,y) = \left| \det \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \right| = 5.$$

Observe that

$$J(\vec{T_2} \circ \vec{T_1})(u,v) = JT_2(x,y)JT_2(u,v).$$ ■

The observation in Example 5 is no coincidence. In general, the following theorem tells us that if $\vec{f} \circ \vec{g}$ is defined, then the rate that $\vec{f} \circ \vec{g}$ changes area (or volume) at $\vec{r}$ is the rate that $\vec{g}$ changes area (or volume) times the rate that $\vec{f}$ changes area (or volume) at $\vec{g}(\vec{r})$.

Theorem 2
Suppose that $\vec{F}$ and $\vec{H}$ are functions from $\mathbb{R}^3$ into $\mathbb{R}^3$. Then $J(\vec{F} \circ \vec{H})(\vec{r}) = J\vec{F}(\vec{H}(\vec{r}))J\vec{H}(r)$.

*Suppose that $\vec{F}$ is a function from $\mathbb{R}^2$ into $\mathbb{R}^2$ and $\vec{H}$ is a function from $\mathbb{R}^2$ into **either** $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $J(\vec{F} \circ \vec{H})(\vec{r}) = J\vec{F}(\vec{H}(\vec{r}))J\vec{H}(r)$.*

EXAMPLE 6: The polar function $\vec{P}$ changes area at a rate of r at the point (r, θ). The linear transformation $\vec{T}(x, y) = (2x + 3y, -x + y, x)$ changes area at the rate of $\sqrt{35}$. Then $J(\vec{T} \circ \vec{P})(r, \theta) = \sqrt{35}|r|$. If we compute $J(\vec{T} \circ \vec{P})(r, \theta)$ directly, we have

$$(\vec{T} \circ \vec{P})(r, \theta) = \begin{pmatrix} 2r\cos(\theta) + 3r\sin(\theta) \\ -r\cos(\theta) + r\sin(\theta) \\ r\cos(\theta) \end{pmatrix},$$

and

$$\begin{aligned} & J(\vec{T} \circ \vec{P})(r, \theta) \\ = & \left\| \frac{\partial}{\partial r}(\vec{T} \circ \vec{P})(r, \theta) \times \frac{\partial}{\partial \theta}(\vec{T} \circ \vec{P})(r, \theta) \right\| \\ = & \left\| \begin{pmatrix} 2\cos(\theta) + 3\sin(\theta) \\ -\cos(\theta) + \sin(\theta) \\ \cos(\theta) \end{pmatrix} \times \begin{pmatrix} -2r\sin(\theta) + 3r\cos(\theta) \\ r\sin(\theta) + r\cos(\theta) \\ -r\sin(\theta) \end{pmatrix} \right\| \\ = & \|(-r, 3r, -5r)\| = \sqrt{35r^2} = \sqrt{35}|r|. \end{aligned}$$

■

EXERCISES 4.6

In Exercises 1–4, find the Jacobian of the polar transformation $\vec{P}$ at the given point. (The angles are measured in radians.)

1. $(2, 0)$.
2. $(2, 2\pi)$.
3. $(2, -2\pi)$.
4. $(-2, \pi)$.

In Exercises 5–8, find the Jacobian of the cylindrical transformation $\vec{C}_z$ at the given point. (The angles are measured in radians.)

5. $(2, 0, 2)$.
6. $(2, 2\pi, 2)$.
7. $(2, -2\pi, 2)$.
8. $(-2, \pi, -1)$.

In Exercises 9–12, find the Jacobian of the spherical transformation $S(\rho, \phi, \theta)$ at the given point. (The angles are measured in radians.)

9. $\left(2, 0, \frac{\pi}{4}\right)$.
10. $\left(-2, \pi, \frac{\pi}{4}\right)$.
11. $\left(-2, \frac{\pi}{4}, \frac{7\pi}{4}\right)$.
12. $\left(2, \frac{\pi}{3}, \frac{7\pi}{4}\right)$.

In Exercises 13–14, use Theorem 1 to find the Jacobian for the given function

13. $\vec{S}(r,\phi,\theta)+(2,3,4)$.

14. $\vec{P}(r,\theta)+(-1,4)$.

In Exercises 15–17, you are given $J\vec{f}$ and $J\vec{g}$, find $J(\vec{f}\circ\vec{g})$ at $\vec{r}_0$.

15. $J\vec{g}(u,v)=|3u-2v|$ and $J\vec{f}(x,y)=|x^2(y+2)-x|$. $\vec{r}_0=(1,2)$ and $\vec{g}(\vec{r}_0)=(-1,4)$.

16. $J\vec{g}(u,v)=\sqrt{u^2+v^2}$ and $J\vec{f}(x,y)=|x^2-xy|$. $\vec{r}_0=(-1,2)$ and $\vec{g}(\vec{r}_0)=(2,-1)$.

17. $J\vec{g}(u,v,w)=|w|\sqrt{u^2+v^2}$ and $J\vec{f}(x,y,z)=|x^2-xy+z|$. $\vec{r}_0=(1,-1,2)$ and $\vec{g}(\vec{r}_0)=(2,-1,1)$.

Use Theorem 2 in Exercises 18–22 to find the Jacobian for $\vec{f}\circ\vec{g}$. $\vec{P}$ is the polar function and $\vec{S}$ is the spherical transformation.

18. $\vec{f}=\vec{P}$ and $\vec{g}(u,v)=(2u+6v+2,-u)$.

19. $\vec{f}=\vec{P}$ and $\vec{g}(u,v)=(-2u+6v+2,u+v-1)$.

20. $\vec{f}=\vec{S}$ and $\vec{g}(u,v,w)=(-2u+6v+2w+1,u+v+2w-1,u+v-2w)$.

21. $\vec{f}(u,v)=(-2u+6v+2,u+v-1)$ and $\vec{g}=\vec{P}$.

22. $\vec{f}(u,v,w)=(-2u+6v+2w+1,u+v+2w-1,u+v-2w)$ and $\vec{f}=\vec{S}$.

23. Show that $J(\vec{S}(\rho,\phi,\theta))=\rho^2\sin(\phi)$.

24. Calculate $J(\vec{f}(u,v))$ for $\vec{f}(u,v)=(2\cos(u)\sin(v),2\sin(u)\sin(v),2\cos(v))$.

25. What is the image of $\vec{f}$ of Exercise 24?

26. Show that $J\vec{F}=J(\vec{F}+\vec{r}_0)$.

27. Let $\vec{F}$ be a differentiable function from a subset of $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that
$$J(\vec{F}(u,v))=|\det D(F)|\Big|_{(u,v)}.$$

28. Let $\vec{F}$ be a differentiable function from a subset of $\mathbb{R}^3$ into $\mathbb{R}^3$. Show that
$$J(\vec{F}(u,v,w))=|\det D(F)|\Big|_{(u,v,w)}.$$

29. Suppose that $\vec{f}$ is a function from $\mathbb{R}^3$ into $\mathbb{R}^3$ and $\vec{g}$ is a function from $\mathbb{R}^2$ into $\mathbb{R}^3$. What can be said about the relationship between $J(\vec{f}\circ\vec{g})$ and the product $J(\vec{f})J(\vec{g})$?