Introduction to Vector Calculus

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Chapter 4

Change of Coordinate Systems

In Section 3.6 we viewed a function from $\mathbb{R}^n$ into $\mathbb{R}^n$ as a vector field. We can also use functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ to transform one coordinate system into another. This is not unlike changing a lens on a camera in order to get a clearer picture. We can sometimes get a better description of a set in $\mathbb{R}^n$ by changing the way we describe the points in $\mathbb{R}^n$. We can also use functions from subsets of $\mathbb{R}^n$ into $\mathbb{R}^m$ to describe interesting geometric objects in $\mathbb{R}^m$, much in the same way as we used functions from intervals into $\mathbb{R}^m$ to describe simple curves. In this chapter we introduce some special functions from $\mathbb{R}^n$ into $\mathbb{R}^m$, and explore some of their properties.

4.1 Translations and Linear Transformations

A *translation* is a function from $\mathbb{R}^n$ onto $\mathbb{R}^n$ defined by adding a constant to every point in $\mathbb{R}^n$. Translations are used to shift objects in $\mathbb{R}^n$ in a rigid fashion without any rotations. The formal definition is

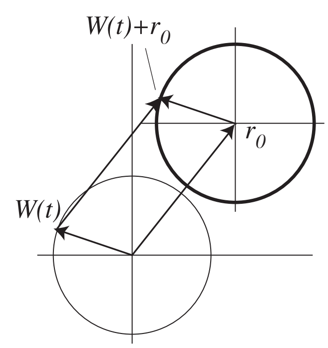


Figure 1. *The unit circle translated by $\vec{r}_0$.*

Definition: Translations

The statement that the function $\vec{T}$ is a translation by $\vec{r}_0$ means that $\vec{r}_0$ is a vector and $\vec{T}(\vec{r}) = \vec{r} + \vec{r}_0$.

EXAMPLE 1: The image of the wrapping function $\vec{W}(t)$ is the unit circle centered at the origin. The composition of $\vec{W}$ followed by a translation by $\vec{r}_0$ yields the function $\vec{f}(t) = \vec{W}(t) + r_0$, the image of which is the unit circle centered at $\vec{r}_0$. See Figure 1. ■

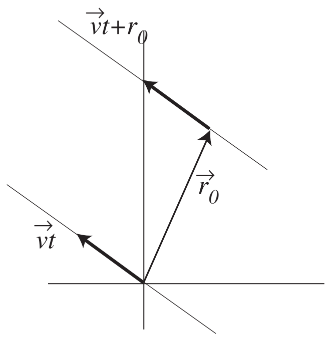


Figure 2. *The line $t\vec{v}$ translated by $\vec{r}_0$.*

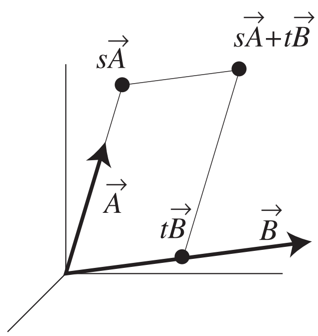


Figure 3. *The plane determined by vectors $\vec{A}$ and $\vec{B}$.*

EXAMPLE 2: The image of the function $\vec{r}(t) = t\vec{v}$ is the line with direction $\vec{v}$ that contains the origin. If we follow $\vec{r}(t)$ with a translation by $\vec{r}_0$, we have $\vec{h}(t) = \vec{r}(t) + \vec{r}_0 = \vec{v}t + \vec{r}_0$, the image of which is the line with direction $\vec{v}$ that contains the point $\vec{r}_0$. See Figure 2. ■

EXAMPLE 3: Consider two vectors $\vec{A}$ and $\vec{B}$ drawn emanating from the origin that do not have the same direction. That is, $\vec{A}\times\vec{B} \neq \vec{0}$. Then these vectors determine a plane containing the origin. In fact, if $\vec{r}$ is any point on that plane, then we can find a pair of numbers (s,t) so that $\vec{r} = s\vec{A} + t\vec{B}$, as illustrated in Figure 3. This defines a function $\vec{h}(s,t) = s\vec{A}+t\vec{B}$ from the $st-$plane onto the plane determined by the vectors $\vec{A}$ and $\vec{B}$. ■

This is one example of a type of function called a linear function or linear transformation that is used extensively in mathematical modeling.

Definition: Linear Functions

A function $\vec{T}$ from $\mathbb{R}^n$ into $\mathbb{R}^m$ is said to be a *linear function* or a *linear transformation* if for each u in $\mathbb{R}$ and each $\vec{r}_1$ and $\vec{r}_2$ in $\mathbb{R}^n$,

- $\vec{T}(\vec{r}_1) + \vec{T}(r_2) = \vec{T}(\vec{r}_1 + \vec{r}_2)$ (Superposition Property)
- $\vec{T}(u\vec{r}_1)) = u\vec{T}(\vec{r}_1)$. (Scalability Property)

EXAMPLE 4: Show that the function $f(x) = 3x$ from $\mathbb{R}$ to $\mathbb{R}$ is linear.

SOLUTION: First, we check for the superposition property.

$$f(x_1 + x_2) = 3(x_1 + x_2) = 3x_1 + 3x_2 = f(x_1) + f(x_2).$$

Next, for scalability.

$$f(ux) = 3ux = u(3x) = uf(x). \quad ■$$

EXAMPLE 5: Even though the graph of the function $f(x) = 3x + 1$ is a line, f is not linear. To show this, let $x_1 = 1$ and $x_2 = 1$. Then $f(1) + f(1) = (3+1) + (3+1) = 8$. However, $f(2) = 7$. Thus $f(1+1) \neq f(1)+f(1)$, and f fails to have the superposition property. ■

We could easily dedicate the rest of this book to the study of linear transformations and still not exhaust the material that is important to the engineer or scientist. However, in this text we only introduce techniques for constructing linear transformations and concentrate on some special properties of these functions. The following theorem is our working characterization of linear transformations.

Theorem 1 *The function $\vec{T} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there are vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n$ in $\mathbb{R}^m$ such that*

$$\vec{T}(x_1, x_2, \ldots, x_n) = x_1\vec{A}_1 + x_2\vec{A}_2 + \ldots + x_n\vec{A}_n.$$

Proof: We argue only the case that $\mathbb{R}^n = \mathbb{R}^m = \mathbb{R}^3$. The general case is similar. First, suppose that $\vec{T}$ is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$. Let $\vec{A}_1 = \vec{T}(1, 0, 0)$, $\vec{A}_2 = \vec{T}(0, 1, 0)$, and $\vec{A}_3 = \vec{T}(0, 0, 1)$. Then from the definition,

$$\begin{aligned}
\vec{T}(x_1, x_2, x_3) &= \vec{T}(x_1, 0, 0) + \vec{T}(0, x_2, x_3) \\
&= \vec{T}(x_1, 0, 0) + (\vec{T}(0, x_2, 0) + \vec{T}(0, 0, x_3)) \\
&= x_1\vec{T}(1, 0, 0) + x_2\vec{T}(0, 1, 0) + x_3\vec{T}(0, 0, 1) \\
&= x_1\vec{A}_1 + x_2\vec{A}_2 + x_3\vec{A}_3.
\end{aligned}$$

Suppose that $\vec{A} = (a_x, a_y, a_z)$, $\vec{B} = (b_x, b_y, b_z)$, and $\vec{C} = (c_x, c_y, c_z)$ are vectors and $\vec{T}$ is defined by $\vec{T}(x, y, z) = x\vec{A} + y\vec{B} + z\vec{C}$. Let u and v be numbers, and let $\vec{r}_1 = (x_1, y_1, z_1)$ and $\vec{r}_2 = (x_2, y_2, z_2)$ be vectors.

$$\begin{aligned}
\vec{T}(u\vec{r}_1 + v\vec{r}_2) &= \vec{T}\left(u(x_1, y_1, z_1) + v(x_2, y_2, z_2)\right) \\
&= \vec{T}\left(ux_1 + vx_2, uy_1 + vy_2, uz_1 + vz_2\right) \\
&= (ux_1 + vx_2)\vec{A} + (uy_1 + vy_2)\vec{B} + (uz_1 + vz_2)\vec{C} \\
&= (ux_1\vec{A} + uy_1\vec{B} + uz_1\vec{C}) + (vx_2\vec{A} + vy_2\vec{B} + vz_2\vec{C}) \\
&= u(x_1\vec{A} + y_1\vec{B} + z_1\vec{C}) + v(x_2\vec{A} + y_2\vec{B} + z_2\vec{C}) \\
&= u\vec{T}(\vec{r}_1) + v\vec{T}(\vec{r}_2).
\end{aligned}$$

This shows that $\vec{T}$ is linear. ■

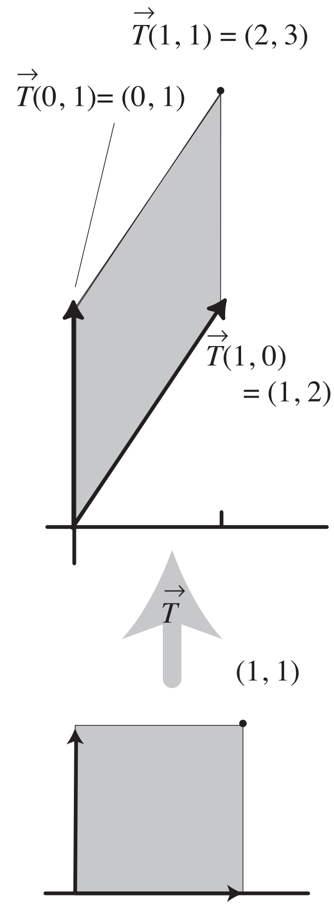


Figure 4.

$$\begin{aligned}
\vec{T}(u, v) &= u(1, 2) + v(0, 1) \\
&= (u, 2u + v).
\end{aligned}$$

EXAMPLE 6: Let $\vec{T}$ be defined by $\vec{T}(u, v) = u(1, 2) + v(0, 1) = (u, 2u + v)$. In this example, $\vec{A} = (1, 2)$ and $\vec{B} = (0, 1)$. It helps to visualize the function by realizing that $\vec{T}$ is that linear transformation that takes $(1, 0)$ onto $(1, 2)$ and $(0, 1)$ onto $(0, 1)$, and it takes the unit square with adjacent edges the vectors $\hat{\imath}$ and $\hat{\jmath}$ onto the parallelogram with adjacent edges $\vec{A}$ and $\vec{B}$. See Figure 4. ∎

EXAMPLE 7:

(a) Let $\vec{T}$ be defined by $\vec{T}(u,v) = u(1,0) + v(0,-1) = (u,-v)$. We could equivalently define $\vec{T}$ with the pair of equations $x = u$ and $y = -v$. The linear transformation $\vec{T}$ reflects the plane across the x–axis. See Figure 5.a.

(b) Let $\vec{T}$ be defined by $T(u,v) = u(0,1) + v(1,0) = (v,u)$. The linear transformation $\vec{T}$ reflects the plane across the line $y = x$. See Figure 5.b. ∎

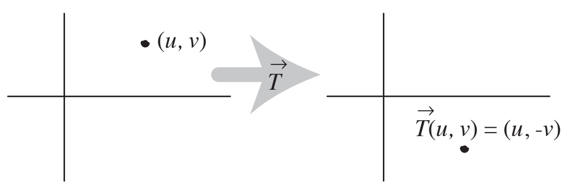


Figure 5.a *The transformation $\vec{T}(u,v) = (u,-v)$.*

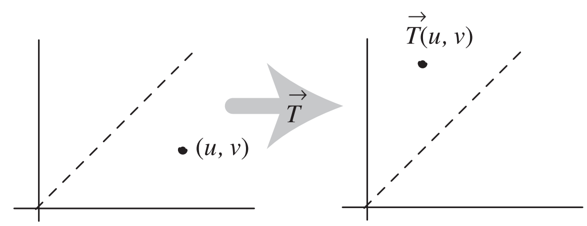


Figure 5.b *The transformation $\vec{T}(u,v) = (v,u)$.*

EXAMPLE 8:

(a) $\vec{T}(u,v,w) = u(1,1,1)+v(1,0,0)+w(1,0,1) = (u+v+w,u,u+w)$ defines a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$.

(b) $\vec{T}(u,v,w) = u(1,2) + v(0,1) + w(1,-1) = (u+w, 2u+v-w)$ defines a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$.

(c) $\vec{T}(u,v) = u(1,0,-1) + v(1,2,1) = (u+v, 2v, -u+v)$ defines a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. ∎

Important Facts about Linear Transformations

(a) If $\vec{T}$ is a linear transformation from $\mathbb{R}$ into $\mathbb{R}^n$ and $\vec{T}(1) = \vec{A}$, then $\vec{T}(u) = u\vec{A}$.

(b) If $\vec{T}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^n$ and $\vec{T}(1,0) = \vec{A}$ and $\vec{T}(0,1) = \vec{B}$, then $\vec{T}(u,v) = u\vec{A} + v\vec{B}$.

(c) If $\vec{T}$ is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^n$ and $\vec{T}(1,0,0) = \vec{A}, \vec{T}(0,1,0) = \vec{B}$ and $\vec{T}(0,0,1) = \vec{C}$, then $\vec{T}(u,v,w) = u\vec{A} + v\vec{B} + w\vec{C}$.

EXAMPLE 9:

(a) Let $\vec{T}$ be defined by

$$\begin{aligned}\vec{T}(u,v,w) &= \begin{pmatrix} 3u+5v+2w \\ u-2v-w \\ u-v+w \end{pmatrix} \\ &= u\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + v\begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} + w\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.\end{aligned}$$

The domain of $\vec{T}$ is $\mathbb{R}^3$ and the range is $\mathbb{R}^3$. Let $\vec{A} = \vec{T}(1,0,0) = (3,1,1)$, $\vec{B} = \vec{T}(0,1,0) = (5,-2,-1)$, and $\vec{C} = \vec{T}(0,0,1) = (2,-1,1)$. Then

$$\vec{T}(u,v,w) = u\vec{A} + v\vec{B} + w\vec{C}.$$

(b) Let $\vec{T}$ be defined by

$$\vec{T}(u,v) = \begin{pmatrix} -2u+v \\ 3u \\ 5u-v \end{pmatrix} = u\begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + v\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

The domain of $\vec{T}$ is $\mathbb{R}^2$ and the range is $\mathbb{R}^3$. Let $\vec{A} = \vec{T}(1,0) = (-2,3,5)$ and $\vec{B} = \vec{T}(0,1) = (1,0,-1)$. Then

$$\vec{T}(u,v) = u\vec{A} + v\vec{B}.$$

(c) Let $\vec{T}$ be defined by

$$\vec{T}(u,v,w) = (3u+2v-w).$$

The domain of $\vec{T}$ is $\mathbb{R}^3$ and the range is $\mathbb{R}$. Let $\vec{A} = \vec{T}(1,0,0) = (3)$, $\vec{B} = \vec{T}(0,1,0) = (2)$ and $\vec{C} = \vec{T}(0,0,1) = (-1)$ (notice that we are thinking of members of $\mathbb{R}$ as vectors). Then

$$\vec{T}(u,v,w) = u\vec{A} + v\vec{B} + w\vec{C}.$$ ■

Example 9 points to a compact way to describe any given linear transformation. A *matrix* is a rectangular array of numbers. A $n \times m$–matrix is a matrix with n rows and m columns.

EXAMPLE 10:

(a) $A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 0 & -6 \end{pmatrix}$ is a 2×3–matrix.

(b) $A = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$ is a 2×2–matrix.

(c) $A = \begin{pmatrix} 3 & 2 & -1 \end{pmatrix}$ is a 1×3–matrix.

(d) $A = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is a 3×1–matrix. ■

A $1 \times n$–matrix as in Example 10.c is a *row matrix* while a $n \times 1$–matrix is a *column matrix*. If $\vec{A}$ is a vector, then we write $[A]$ to denote the column matrix corresponding to $\vec{A}$. Thus, if $\vec{A} = (-1, 2, 3)$, then $[A] = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$.

If $\vec{T}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^n$, and $\vec{T}(1,0) = \vec{A}$ and $\vec{T}(0,1) = \vec{B}$, then $\vec{T}(u,v) = u\vec{A} + v\vec{B}$. We let $A_{\vec{T}}$ be the matrix with n rows and 2 columns, the first column of which is $[A]$ and the second column of which is $[B]$.

$$A_{\vec{T}} = \begin{pmatrix} [A] & [B] \end{pmatrix}.$$

In Example 9(b), $\vec{T}$ was defined by $\vec{T}(u,v) = u\vec{A} + v\vec{B}$, where $\vec{A} = \vec{T}(1,0) = (-2,3,5)$ and $\vec{B} = \vec{T}(0,1) = (1,0,-1)$. Then $A_{\vec{T}}$ is the matrix

$$A_{\vec{T}} = \begin{pmatrix} [A] & [B] \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & 0 \\ 5 & -1 \end{pmatrix}.$$

Notice that the matrix $A_{\vec{T}}$ contains all of the information necessary to work with the linear transformation $\vec{T}$.

We define multiplication of $A_{\vec{T}}$ with the column vector $\begin{pmatrix} u \\ v \end{pmatrix}$ by

$$A_{\vec{T}} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{T}(u,v) = u\vec{A} + v\vec{B}.$$

In Example 9(a), $\vec{T}$ was defined by $\vec{T}(u,v,w) = u\vec{A} + v\vec{B} + w\vec{C}$, where $\vec{A} = \vec{T}(1,0,0) = (3,1,1)$, $\vec{B} = \vec{T}(0,1,0) = (5,-2,-1)$, and $\vec{C} = \vec{T}(0,0,1) = (2,-1,1)$. The matrix $A_{\vec{T}}$ is

$$A_{\vec{T}} = \begin{pmatrix} [A] & [B] & [C] \end{pmatrix} = \begin{pmatrix} 3 & 5 & 2 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

In this case, we define multiplication of $A_{\vec{T}}$ with the column vector $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ by

$$A_{\vec{T}} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \vec{T}(u, v, w) = u\vec{A} + v\vec{B} + w\vec{C}.$$

EXAMPLE 11: Let $\vec{T}$ be the linear transformation with matrix

$$A_{\vec{T}} = \begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 0 & 3 \end{pmatrix}.$$

Then $\vec{T}(u, v) = A_{\vec{T}} \begin{pmatrix} u \\ v \end{pmatrix} = u(2, 3, 0) + v(0, -1, 3)$. ■

Suppose that $\vec{T}(u, v, w) = u\vec{A} + v\vec{B} + w\vec{C}$ is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^n$, where $\vec{T}(1, 0, 0) = \vec{A}$, $\vec{T}(0, 1, 0) = \vec{B}$ and $\vec{T}(0, 0, 1) = \vec{C}$. As in the case of functions from $\mathbb{R}^2$ into $\mathbb{R}^n$, we can completely describe a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^n$ with a matrix $A_{\vec{T}}$ with n rows and 3 columns, the columns of which are the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$.

$$A_{\vec{T}} = \begin{pmatrix} [A] & [B] & [C] \end{pmatrix}.$$

Exactly as before, we define multiplication of $A_{\vec{T}}$ with the column vector $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ by

$$A_{\vec{T}} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \vec{T}(u, v, w) = u\vec{A} + v\vec{B} + w\vec{C}.$$

EXAMPLE 12: The matrix for the transformation in Example 9(a) is

$$A_{\vec{T}} = \begin{pmatrix} 3 & 5 & 2 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Thus

$$\vec{T}(u, v, w) = A_{\vec{T}} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + v \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} + w \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}. ■$$

EXAMPLE 13: Let $\vec{T}$ be the linear transformation with matrix

$$A_{\vec{T}} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 2 & -3 & 2 \end{pmatrix}.$$

Then

$$\vec{T}(u,v,w) = A_{\vec{T}} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + v \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} + w \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}. \blacksquare$$

EXAMPLE 14: Let $\vec{T}$ be the linear transformation with matrix

$$A_{\vec{T}} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -3 & 2 \end{pmatrix}.$$

Then

$$\vec{T}(u,v,w) = A_{\vec{T}} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u \begin{pmatrix} 1 \\ 2 \end{pmatrix} + v \begin{pmatrix} 0 \\ -3 \end{pmatrix} + w \begin{pmatrix} -1 \\ 2 \end{pmatrix}. \blacksquare$$

Observation: Notice that the number of columns in $A_{\vec{T}}$ is the dimension of the domain of $\vec{T}$, and the number of rows in $A_{\vec{T}}$ is the dimension of the range. Thus, if $A_{\vec{T}}$ has 2 rows and 3 columns, then $\vec{T}$ is a function from $\mathbb{R}^3$ into $\mathbb{R}^2$. If $A_{\vec{T}}$ has 3 rows and 2 columns, then $\vec{T}$ is a function from $\mathbb{R}^2$ into $\mathbb{R}^3$. If $A_{\vec{T}}$ has 2 rows and 2 columns, then $\vec{T}$ is a function from $\mathbb{R}^2$ into $\mathbb{R}^2$.

In Example 3, we used a linear transformation to *parametrize* a plane determined by two vectors emanating from the origin. If $\vec{T}(u,v) = u\vec{A} + v\vec{B}$ is any linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that $\vec{A} \times \vec{B} \neq 0$, then $\vec{T}$ parametrizes the plane determined by $\vec{A}$ and $\vec{B}$, when drawn emanating from the origin. (The test $\vec{A} \times \vec{B} \neq \vec{0}$ is a check to be sure that the two vectors do not have the same direction.) It is also true that any plane containing the origin is the image of a linear transformation.

EXAMPLE 15: Find a linear transformation that takes the $u,v-$plane onto the plane with equation $2x + 3y - 4z = 0$. We proceed by solving for x of the variables in terms of y and z to obtain $x = 2z - \frac{3y}{2}$. We let $y(u,v) = u$ and $z(u,v) = v$, which gives $x(u,v) = 2v - \frac{3u}{2}$, and we have the coordinate functions for

$$\vec{T}(u,v) = \begin{pmatrix} 2v - 3/2u \\ u \\ v \end{pmatrix}.$$

$\blacksquare$

Theorem 2 *If $\mathcal{P}$ is the plane with equation $ax+by+cz=0$, then $\mathcal{P}$ is the image of a linear transformation with domain $\mathbb{R}^2$.*

Proof: At least one of the coordinates of (a,b,c) is not zero. Assume $a \neq 0$. Then we solve $ax+by+cz=0$ for x in terms of y and z to obtain $x = -\left(\frac{b}{a}\right)y - \left(\frac{c}{a}\right)z$. As in Example 13, we let $y(u,v)=u$ and $z(u,v)=v$ and define

$$\vec{T}(u,v) = \begin{pmatrix} -\left(\frac{b}{a}\right)u - \left(\frac{c}{a}\right)v \\ u \\ v \end{pmatrix}.$$

$\mathcal{P}$ is parametrized by $\vec{T}$.

If $a=0$, then we can solve for either y in terms of x and z or z in terms of x and y.

We have described planes in $\mathbb{R}^3$ that contain the origin as the images of linear transformations with domain $\mathbb{R}^2$. Now, suppose that $\vec{A}$ and $\vec{B}$ are vectors in $\mathbb{R}^3$, drawn emanating from $\vec{r}_0$, such that $\vec{A}\times\vec{B} \neq 0$. The vector $\vec{A}\times\vec{B}$ is normal to the plane and $\vec{r}_0$ is a point in the plane. $\vec{T}(u,v) = u\vec{A} + v\vec{B}$ parametrizes a plane containing the origin, which is parallel to $\mathcal{P}$. We can translate the image of $\vec{T}(u,v)$ to $\mathcal{P}$ by composing $\vec{T}$ with the translation by $\vec{r}_0$ to obtain

$$\vec{h}(u,v) = \vec{T}(u,v) + \vec{r}_0.$$

EXAMPLE 16: The vectors $\vec{A} = (1,2,-1)$ and $\vec{B} = (0,1,1)$ are drawn emanating from $\vec{C} = (1,1,2)$ to define a plane $\mathcal{P}$. Find a linear transformation $\vec{T}$ with domain $\mathbb{R}^2$ and a vector $\vec{r}_0$ such that $\vec{h}(u,v) = \vec{T}(u,v) + \vec{r}_0$ parametrizes $\mathcal{P}$.

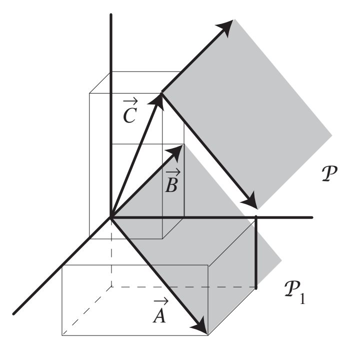


Figure 6. *The plane $\mathcal{P}$ is a translation of $\mathcal{P}_1$ by the vector $\vec{C} = (1,1,2)$.*

SOLUTION: First, we parametrize the plane determined by the vectors $\vec{A}$ and $\vec{B}$ drawn emanating from the origin. Then we translate this plane by the vector $\vec{C}$. See Figure 6. Let

$$A_{\vec{T}} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\vec{T}(u,v) = A_{\vec{T}} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ 2u+v \\ -u+v \end{pmatrix}$$

is a parametrization of the plane $\mathcal{P}_1$ containing the origin that is parallel to $\mathcal{P}$. Now we compose $\vec{T}$ with a translation by $\vec{r}_0 = \vec{C}$ to obtain

$$\begin{aligned}\vec{h}(\vec{r}) &= A_{\vec{T}}\vec{r} + \vec{C} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} u+1 \\ 2u+v+1 \\ -u+v+2 \end{pmatrix}.\end{aligned}$$

■

EXAMPLE 17: Find a linear transformation composed with a translation that parametrizes the plane $x - y + z = 2$.

SOLUTION: First, we parametrize the plane $x - y + z = 0$ as in Example 15 with $\vec{T}(u, v) = (u - v, u, v)$ and then translate the result with any point in the plane, say $(0, 0, 2)$, to obtain $\vec{h}(u, v) = (u - v, u, v + 2)$. ■

Summary

$\vec{T}$ is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ if and only if $A_{\vec{T}}$ is a $m \times n$–matrix (n columns and m rows.)

If $\vec{T}(u_1, \ldots, u_n) = u_1\vec{A}_1 + \cdots + u_n\vec{A}_n$, then $A_{\vec{T}} = \begin{pmatrix} [A_1] & \ldots & [A_n] \end{pmatrix}$.

Any plane in $\mathbb{R}^3$ can be parametrized with a function of the form $\vec{h}(\vec{r}) = A\vec{r} + \vec{r}_0$, where A is a matrix with two columns and three rows and $\vec{r}_0$ is an arbitrary point in the plane.

EXERCISES 4.1

In Exercises 1–6, determine the domain and range of the linear transformation associated with the matrix.

1. $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$.

2. $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix}$.

3. $\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$.

4. $\begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 0 \end{pmatrix}$.

5. $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix}$.

6. $\begin{pmatrix} 1 & -1 \end{pmatrix}$.

In Exercises 7–14, find vectors $\vec{A}$ and $\vec{B}$, such that $\vec{T}(u,v) = u\vec{A} + v\vec{B}$, and find the associated $n \times 2$ matrix $A_{\vec{T}}$.

7. $\vec{T}(u,v) = (2u, 3v)$.
8. $\vec{T}(u,v) = (u+v, u-v)$.
9. $\vec{T}(u,v) = \left(3u - 6v, \frac{v}{2} - \frac{u}{3}\right)$.
10. $\vec{T}(u,v) = \left(-v, u - \frac{v}{2}\right)$.
11. $\vec{T}(u,v) = (-v, u-v)$.
12. $\vec{T}(u,v) = (u - v, 2u + 6v)$.
13. $\vec{T}(u,v) = \left(v - 6u, -u - v, u + \frac{v}{3}\right)$.
14. $\vec{T}(u,v) = (6u + 3v)$.

In Exercises 15–22, find vectors $\vec{A}, \vec{B}$, and $\vec{C}$ such that $\vec{T}(u,v,w) = u\vec{A} + v\vec{B} + w\vec{C}$, and find the associated $n \times 3$ matrix $A_{\vec{T}}$.

15. $\vec{T}(u,v,w) = (u+v+w, u-v+w, v-w)$.
16. $\vec{T}(u,v,w) = (u - 3v, 3u + v - \pi w)$.
17. $\vec{T}(u,v,w) = (6u + 3v - w, -u - v - w, u - w)$.
18. $\vec{T}(u,v,w) = (5w, v + w, v)$.
19. $\vec{T}(u,v,w) = (u + v + w, u - v)$.
20. $\vec{T}(u,v,w) = (u - v + w, u + v - 6w)$.
21. $\vec{T}(u,v,w) = (u + 6v - 10w)$.
22. $\vec{T}(u,v,w) = (15u - 6v + 2w)$.

In Exercises 23–30, find the matrix for the linear transformation $\vec{T}$.

23. $\vec{T}$ takes $(0,1)$ onto $(1,3)$, and $(1,0)$ onto $(-1,5)$.
24. $\vec{T}$ takes $(1,0,0)$ onto $(1,3)$, $(0,1,0)$ onto $(-1,5)$, and $(0,0,1)$ onto $(-1,0)$.
25. $\vec{T}$ takes $(1,0,0)$ onto $(1,3,2)$, $(0,1,0)$ onto $(-1,5,1)$, and $(0,0,1)$ onto $(-1,0,3)$.
26. $\vec{T}$ takes $(1,0,0)$ onto -1, $(0,1,0)$ onto 1, and $(0,0,1)$ onto 3.
27. $\vec{T}$ takes $(1,0,0)$ onto 1, $(0,1,0)$ onto 2, and $(0,0,1)$ onto 3.
28. $\vec{T}$ takes $\hat{\imath}$ onto $\hat{\jmath}$, $\hat{\jmath}$ onto $-\hat{\imath}$ and $\hat{k}$ onto $-\hat{k}$.
29. $\vec{T}$ reflects the plane over the x–axis.
30. $\vec{T}$ reflects the plane over the line $x = -y$.

In Exercises 31–36, determine which of the planes is the image of a linear transformation. Where appropriate, find a linear transformation that parametrizes the plane.

31. $\mathcal{P}$ is the graph of $x + y + z = 0$.
32. $\mathcal{P}$ is the graph of $x - 2y + 3z = 0$.
33. $\mathcal{P}$ is the graph of $x + 2y + 3z = 3$.
34. The vector $(1,2,3)$ is normal to $\mathcal{P}$, and $\mathcal{P}$ contains the origin.
35. The vector $(1,2,3)$ is normal to $\mathcal{P}$, and $\mathcal{P}$ contains $(1,1,1)$.
36. The vector $(1,2,3)$ is normal to $\mathcal{P}$, and $\mathcal{P}$ contains $(-1,-1,1)$.

In Exercises 37–42, find a function of the form $\vec{h}(s,t) = A\begin{pmatrix} s \\ t \end{pmatrix} + (x_0, y_0, z_0)$, such that the plane $\mathcal{P}$ is the image of $\vec{h}$.

37. $\mathcal{P}$ is the graph of $x + y + z = 3$.
38. $\mathcal{P}$ is the graph of $x + 2y + 3z = -1$.
39. $\mathcal{P}$ is the graph of $x - 2y + 3z = 3.39)$

40. The vector $(1,2,3)$ is normal to $\mathcal{P}$, and $\mathcal{P}$ contains $(1,1,1)$.

41. The vector $(1,2,3)$ is normal to $\mathcal{P}$, and $\mathcal{P}$ contains $(1,-1,2)$.

42. The vector $(1,2,3)$ is normal to $\mathcal{P}$, and $\mathcal{P}$ contains $(-1,-1,1)$.

In Exercises 43–47, prove that $\vec{T}$ is not linear by showing that $\vec{T}(\vec{r}+\vec{s}) \neq \vec{T}(\vec{r})+\vec{T}(\vec{s})$ for some choice of vectors $\vec{r}$ and $\vec{s}$.

43. $\vec{T}(u,v) = (3u - v + 1, 2u + v - 6)$.

44. $\vec{T}(u,v,w) = (u - 2v + 3, u + w)$.

45. $\vec{T}(u,v) = (u^2 + 2u, u - v)$.

46. $\vec{T}(u,v,w) = (u^2 + v^2, u + v - 2, u + v)$.

47. $\vec{T}(u,v) = (\sin u, \cos v)$.

In Exercises 48–52, $\vec{T}_\theta$ is the transformation that rotates the plane θ radians, as in Figure 7.

48. Find the **unit** vectors $\hat{e}_1$ and $\hat{e}_2$ in terms of θ.

49. Find the matrix for $\vec{T}_\theta$.

50. Find $\vec{T}_{\pi/4}(1,1)$.

51. Find $\vec{T}_{\pi/2}(1,1)$.

52. Find $\vec{T}_{-\pi/4}(1,1)$.

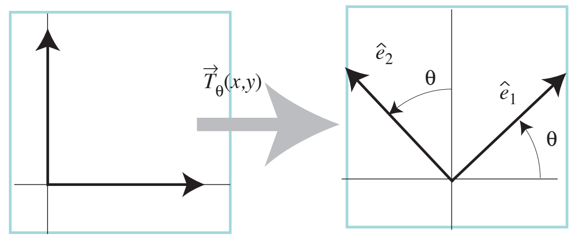


Figure 7.

53. Let $\vec{T}(u,v) = (u,-v)$ be the linear transformation that reflects the plane over the x–axis. Show that $\vec{T} \circ \vec{T}_{\pi/4} \neq \vec{T}_{\pi/4} \circ \vec{T}$ (that is, a rotation followed by a reflection gives a different result than does a reflection followed by a rotation.)

54. Let $\vec{T}$ be a function from $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that if each of its coordinate functions is linear, $\vec{T}$ is linear.

55. Let $\vec{T}$ be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that each of its coordinate functions is linear.

56. Show that if the function $\vec{T}$ from $\mathbb{R}^n$ into $\mathbb{R}^m$ is linear, then $\vec{T}(\vec{0}) = \vec{0}$. (Hint: $\vec{T}(\vec{0}+\vec{0}) = \vec{T}(\vec{0})$.)

4.2 Other Transformations

There are a number of functions or transformations that are not linear but still quite important. Linear transformations have the nice property that they take planes onto single points, lines, or other planes. However, it may be helpful to change our point of view more drastically.

Suppose $\vec{F}$ is a function from uvw–space into xyz–space, given by

$$\vec{F}(u,v,w) \quad = \quad \begin{pmatrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{pmatrix}.$$

We define

$$\frac{\partial \vec{F}}{\partial u}\Big|_{(u,v,w)} = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}\Bigg|_{(u,v,w)}, \quad \frac{\partial \vec{F}}{\partial v}\Big|_{(u,v,w)} = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}\Bigg|_{(u,v,w)},$$

and

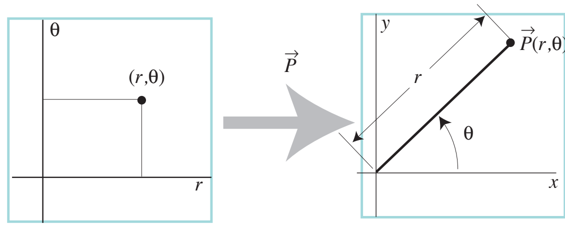
$$\frac{\partial \vec{F}}{\partial w}\Big|_{(u,v,w)} = \begin{pmatrix} \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial w} \end{pmatrix}\Bigg|_{(u,v,w)}.$$

The Polar Transformation

EXAMPLE 1: Let $\vec{P}$ be the function from $r\theta$–space defined by

$$\vec{P}(r,\theta) = (r\cos\theta, r\sin\theta).$$

Figure 1
$\vec{P}(r,\theta)$ is the point r units from the origin on the line with inclination θ.



The function $\vec{P}$ is called the *polar transformation.* If $\vec{P}(r,\theta) = (x,y)$, then (r,θ) are called *polar coordinates* for (x,y), and (x,y) are the *rectangular coordinates* for (r,θ).

The transformation P takes a horizontal line of the form $\theta = \theta_0$ in $r\theta$–space onto a line passing through the origin in xy–space, as illustrated in Figure 2.a. Similarly, P takes a vertical line of the form $r = r_0$ in $r\theta$–space onto a circle of radius r_0 in xy–space, as in Figure 2.b.

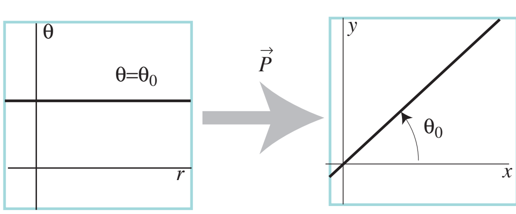


Figure 2.a *The line $\theta = \theta_0$ in $r\theta$–space goes onto the line containing the origin with inclination θ_0 in xy–space.*

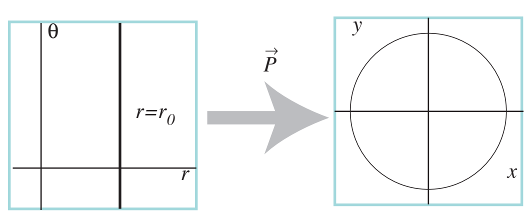


Figure 2.b *The line $r = r_0$ in $r\theta$–space goes onto the circle centered at the origin with radius r_0 in xy–space.*

The partial derivatives of $\vec{P}$ are

$$\frac{\partial \vec{P}}{\partial r} = (\cos\theta, \sin\theta) \text{ and } \frac{\partial \vec{P}}{\partial \theta} = (-r\sin\theta, r\cos\theta).$$

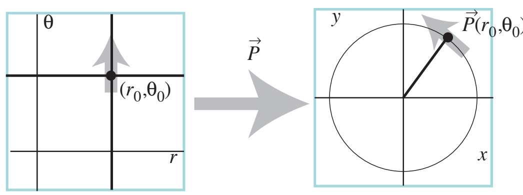


Figure 3.a *As the position in $r\theta$–space moves in the positive direction on the line $r = r_0$, its image in xy–space moves counterclockwise around the circle of radius r_0.*

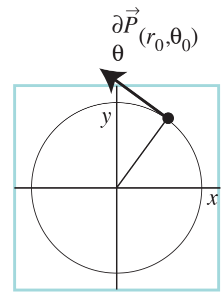


Figure 3.b $\left.\frac{\partial \vec{P}}{\partial \theta}\right|_{(r_0,\theta_0)}$ *is a vector tangent to the circle* $x^2 + y^2 = r_0^2$ *at the point* $\vec{P}(r_0, \theta_0)$.

Consider the geometry associated with the partial derivatives of P. Figure 3.a illustrates that as the position in $r\theta$–space moves in the positive direction on the line $r = r_0$, its image in xy–space moves counterclockwise around the circle of radius r_0. Thus, $\left.\frac{\partial \vec{P}}{\partial \theta}\right|_{(r_0,\theta_0)}$ is a vector tangent to the circle $x^2 + y^2 = r_0^2$ at the point $\vec{P}(r_0, \theta_0) = r_0\left(\cos(\theta_0), r_0\sin(\theta_0)\right)$, as illustrated in Figure 3.b.

As illustrated in Figures 4.a and 4.b, we can discern the geometry of $\frac{\partial P}{\partial r}$ by inspection.

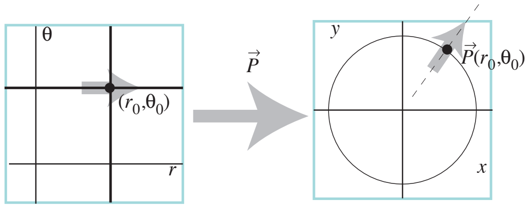


Figure 4.a *As the position in $r\theta$–space moves in the positive direction on the line $\theta = \theta_0$, its image in xy–space moves out the line with inclination θ_0.*

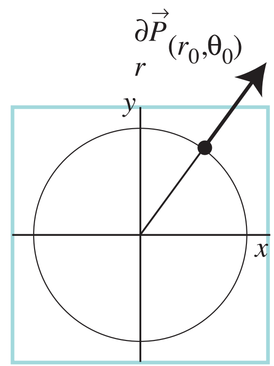


Figure 4.b $\left.\frac{\partial \vec{P}}{\partial r}\right|_{(r_0,\theta_0)}$ *is a vector emanating from the point (r_0, θ_0). The vector points away from the origin.*

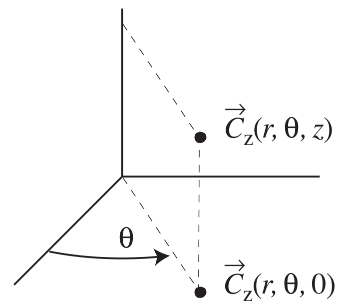


Figure 5.a *The point $\vec{C}_z(r, \theta, z)$.*

The Cylindrical Transformation

EXAMPLE 2: Let $\vec{C}_z$ be the function from $r\theta z$–space into xyz–space, defined by

$$\vec{C}_z(r, \theta, z) = (r\cos\theta, r\sin\theta, z).$$

In Figure 5.a we plot the image of the point (r, θ, z).

The function $\vec{C}_z$ is called the *cylindrical transformation (about the z–axis)* because it takes planes of the form $r = r_0$ onto the lateral surface of the cylinder $x^2 + y^2 = r_0$. Figure 5.b illustrates how planes parallel to the $r\theta$–plane in $r\theta z$–space are carried into xyz–space by $\vec{C}_z$. If $\vec{C}_z(r, \theta, z) = (x, y, z)$, then (r, θ, z) are the *cylindrical coordinates* for (x, y, z), and (x, y, z) are the *rectangular coordinates* for (r, θ, z).

Thus

$$\begin{aligned}
\frac{\partial \vec{C}_z(r, \theta, z)}{\partial r} &= (\cos\theta, \sin\theta, 0),\\
\frac{\partial \vec{C}_z(r, \theta, z)}{\partial \theta} &= (-r\sin\theta, r\cos\theta, 0), \text{ and}\\
\frac{\partial \vec{C}_z(r, \theta, z)}{\partial z} &= (0, 0, 1).
\end{aligned}$$

The geometry associated with the cylindrical transformation and its partial derivatives is illustrated in Figures 6.a, 6.b, and 6.c.

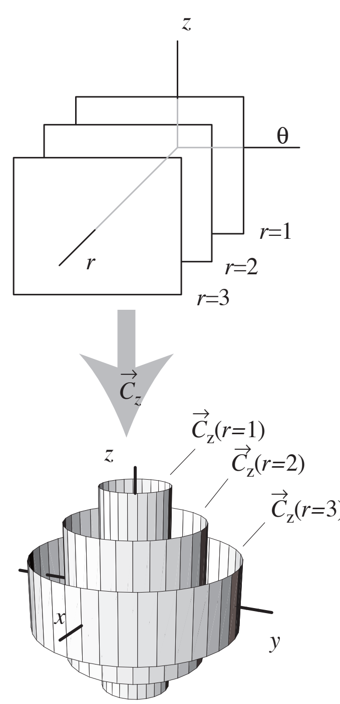


Figure 5.b *Images (in xyz–space) of planes that are parallel to the θz–coordinate plane in $r\theta z$–space.*

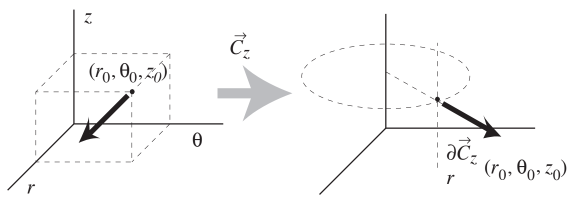


Figure 6.a *As the point moves away from (r_0, θ_0, z_0) in the r direction holding θ and z fixed, $\vec{C}_z(r, \theta, z)$ moves directly away from the z-axis.*

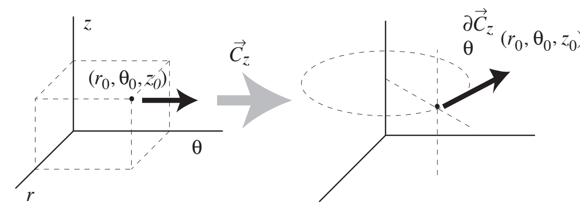


Figure 6.b *As the point moves away from (r_0, θ_0, z_0) in the θ direction holding r and z fixed, $\vec{C}_z(r, \theta, z)$ moves in direction tangent to the circle parametrized by*

$$\vec{h}(\theta) = \begin{pmatrix} r_0\cos(\theta) \\ r_0\sin(\theta) \\ z_0 \end{pmatrix} = \vec{C}_z(r_0, \theta, z_0)$$

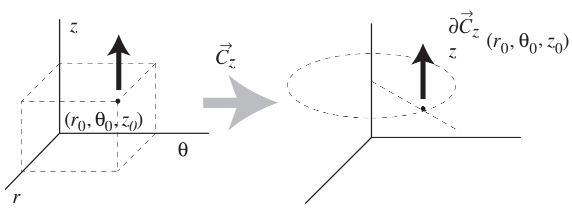


Figure 6.c *As the point moves away from (r_0, θ_0, z_0) in the z direction holding r and θ fixed, $\vec{C}_z(r, \theta, z)$ moves in the z direction in xyz–space.*

The Spherical Transformation

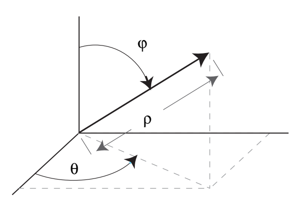
EXAMPLE 3: Let

$$\vec{S}(\rho, \phi, \theta) = \begin{pmatrix} \rho\cos(\theta)\sin(\phi) \\ \rho\sin(\theta)\sin(\phi) \\ \rho\cos(\phi) \end{pmatrix}.$$

$\vec{S}$ is called the *spherical transformation* since $\vec{S}$ takes the plane $\rho = \rho_0$ in $\rho\phi\theta$–space onto a sphere of radius ρ_0. See Figure 7.

Figure 7

The point $\vec{S}(\rho, \phi, \theta) = \begin{pmatrix} \rho\cos(\theta)\sin(\phi) \\ \rho\sin(\theta)\sin(\phi) \\ \rho\cos(\phi) \end{pmatrix}.$



The spherical transformation appears formidable. However, as we illustrate in Figures 8.a and 8.b, the derivation of the function is a straightforward application of elementary trigonometry and geometry. ρ is the length of the hypotenuse of the right triangle $\vec{O}\vec{A}\vec{B}$. The z–coordinate of $\vec{S}(\rho,\phi,\theta)$ is $\rho\cos(\phi)$ since $\vec{O}\vec{A}$ is the side adjacent ϕ in the right triangle $\vec{O}\vec{A}\vec{B}$. The line segment $\vec{O}\vec{C}$ is the hypotenuse of the right triangle $\vec{O}\vec{C}\vec{D}$, and its length is $\rho\sin(\phi)$. $\vec{0}\vec{D}$ is the side adjacent the angle θ. Therefore, the x–coordinate of $\vec{S}(\rho,\phi,\theta)$ is $\cos(\theta)(\rho\sin(\phi))$. In the same way, $\vec{B}\vec{C}$ is the side opposite θ in the triangle $\vec{O}\vec{C}\vec{D}$, and the y–coordinate of $\vec{S}(\rho,\phi,\theta)$ is $\sin(\theta)(\rho\sin(\phi))$.

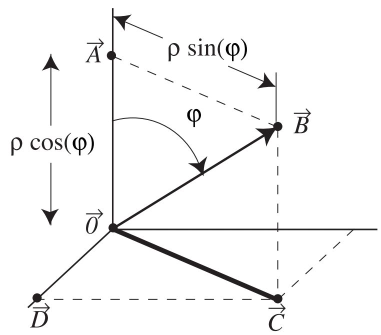


Figure 8.a *The z–coordinate of $\vec{S}(\rho,\phi,\theta)$ is* $\cos(\phi)$.

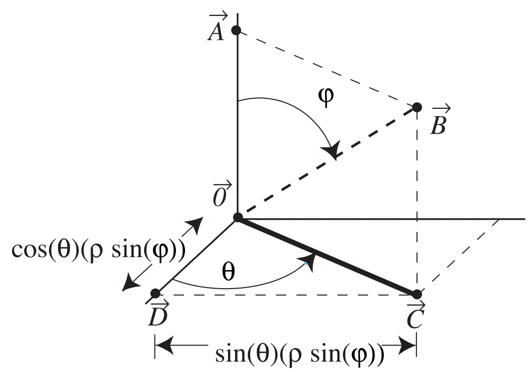


Figure 8.b *The x–coordinate of $\vec{S}(\rho,\phi,\theta)$ is* $\cos(\theta)(\rho\sin(\phi))$, *and the y–coordinate of $\vec{S}(\rho,\phi,\theta)$ is* $\sin(\theta)(\rho\sin(\phi))$.

Figures 9.a–c. illustrate how $\vec{S}$ takes planes parallel to the coordinate planes in uvw–space into xyz–space. If $\vec{S}(\rho,\phi,\theta) = (x,y,z)$, then (ρ,ϕ,θ) are called *spherical coordinates* for (x,y,z), and (x,y,z) are called the *rectangular coordinates* for (ρ,ϕ,θ).

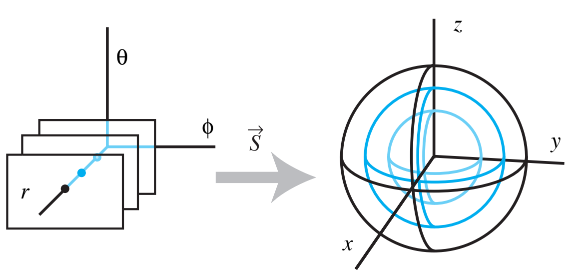


Figure 9.a $\vec{S}$ *takes planes parallel to the $\theta\phi$–coordinate plane onto spheres.*

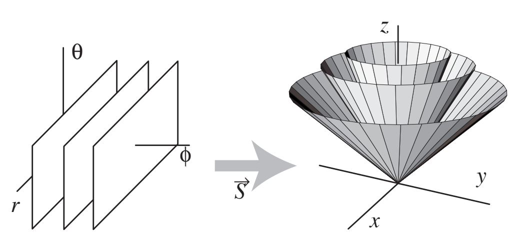


Figure 9.b $\vec{S}$ *takes planes parallel to the $r\theta$–coordinate plane onto cones.*

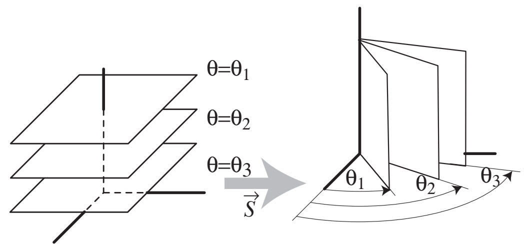


Figure 9.c *$\vec{S}$ takes planes parallel to the $r\phi$–coordinate plane onto planes containing the z–axis, like pages of a book.*

The geometry associated with the partial derivatives of the spherical transformation is illustrated in Figures 10.1, 10.b, and 10.c.

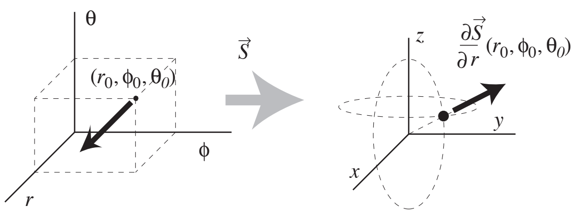


Figure 10.a *As the point moves away from (r_0, ϕ_0, θ_0) in the r direction holding ϕ and θ fixed, $\vec{S}(r, \phi, \theta)$ moves radially away from the origin.*

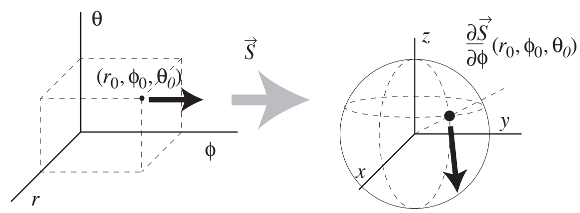


Figure 10.b *As the point moves away from (r_0, ϕ_0, θ_0) in the θ direction holding r and ϕ fixed, $\vec{S}(r, \phi, \theta)$ moves in direction tangent to the latitude parametrized by $\vec{h}(\theta) = (r_0, \phi_0, \theta)$.*

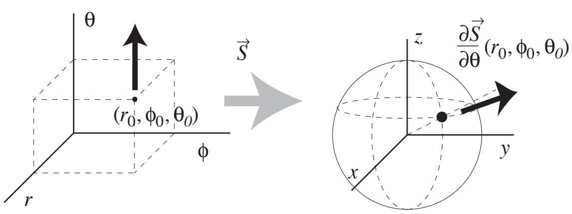


Figure 10.c *As the point moves away from (r_0, ϕ_0, θ_0) in the ϕ direction holding r and θ fixed, $\vec{S}(r, \phi, \theta)$ moves around the longitude parametrized by $\vec{h}(\phi) = (r_0, \phi, \theta_0)$.*

EXERCISES 4.2

In Exercises 1–9, plot the points in xy–space that have the given polar coordinates, and compute the partial derivatives of the polar transformation $\vec{P}$ at the given point. (The angles are measured in radians.)

1. $(2,0)$. 2. $(2,2\pi)$. 3. $(2,-2\pi)$.
4. $(-2,\pi)$. 5. $\left(5,\frac{\pi}{4}\right)$. 6. $\left(-5,\frac{5\pi}{4}\right)$.
7. $\left(5,-\frac{\pi}{4}\right)$. 8. $\left(1,\frac{\pi}{6}\right)$. 9. $\left(-1,\frac{7\pi}{6}\right)$.

In Exercises 10–14, find all polar coordinates for the given point in xy–space. ($r^2=x^2+y^2$, and, if $x\neq 0$, $\tan(\theta)=\frac{y}{x}$.)

10. $\left(0,\frac{1}{2}\right)$. 11. $\left(\frac{1}{2},0\right)$. 12. $(1,1)$.
13. $(-1,\sqrt{3})$. 14. $(0,0)$.

In Exercises 15–23, find the point in xyz–space that has the given cylindrical coordinates, and compute partial derivatives of the cylindrical transformation $\vec{C}_z$ at the given point. (The angles are measured in radians.)

15. $(2,0,2)$. 16. $(2,2\pi,2)$. 17. $(2,-2\pi,2)$.
18. $(-2,\pi,-1)$. 19. $\left(5,\frac{\pi}{4},2\right)$. 20. $\left(-5,\frac{5\pi}{4},2\right)$.
21. $\left(5,-\frac{\pi}{4},3\right)$. 22. $\left(1,\frac{\pi}{6},-1\right)$ 23. $\left(-1,\frac{7\pi}{6},-1\right)$

In Exercises 24–28, find the cylindrical coordinates for the given point in xyz–space. ($r^2=x^2+y^2$, and,if $x\neq 0$, $\tan(\theta)=\frac{y}{x}$.)

24. $\left(0,\frac{1}{2},2\right)$. 25. $\left(\frac{1}{2},0,2\right)$. 26. $(1,1,1)$.
27. $(-1,\sqrt{3},-1)$. 28. $(0,0,2)$.

In Exercises 29–36, the points are given in spherical coordinates. Find the corresponding rectangular coordinates, and the partial derivatives of the spherical transformation $\vec{S}$ at the given point.

29. $\left(2,0,\frac{\pi}{4}\right)$. 30. $\left(-2,\pi,\frac{\pi}{4}\right)$.
31. $\left(-2,0,\frac{7\pi}{4}\right)$. 32. $\left(2,\pi,\frac{7\pi}{4}\right)$.
33. $\left(2,\pi,-\frac{\pi}{4}\right)$. 34. $\left(1,\frac{\pi}{4},\frac{\pi}{4}\right)$.
35. $\left(-1,\frac{5\pi}{4},\frac{\pi}{4}\right)$. 36. $\left(1,\frac{5\pi}{4},-\frac{\pi}{4}\right)$.

In Exercises 37–40, find values for (ρ,ϕ,θ) such that (i) $\rho\geq 0$, (ii) $0\leq\theta<2\pi$, and (iii) $0\leq\phi\leq\pi$ and such that $\vec{S}(\rho,\phi,\theta)$ is the given point. ($\rho^2=x^2+y^2+z^2, z=\rho\cos(\phi)$, and $\tan(\theta)=\frac{y}{x}$.)

37. $(1,0,1)$. 38. $(1,1,1)$.
39. $(1,\sqrt{2},1)$. 40. $(\sqrt{6},2,2\sqrt{2})$

41. Show that if $\vec{p}$ is a point on the plane $\rho=\rho_0$ in $\rho\phi\theta$–space, then $\vec{S}$ takes $\vec{p}$ onto a point in the sphere of radius ρ_0 centered at the origin in xyz–space.

42. Show that if $\vec{q}$ is on the sphere of radius ρ_0, centered at the origin, then there is a point $\vec{p}$ in the plane $\rho=\rho_0$ in $\rho\phi\theta$–space such that $\vec{S}(\vec{p})=\vec{q}$.

43. Show that the partial derivatives of the polar function are mutually perpendicular at (r,θ) provided that $r\neq 0$.

44. Show that the partial derivatives of the cylider-ical transformation are mutually perpendicular at (r,θ,z) provided that $r\neq 0$.

45. Show that the partial derivatives of the spher-ical transformation are mutually perpendicular at (r,ϕ,θ) provided that $r\neq 0$.

In Exercises 46–57, sketch the set satisfying the given conditions.

46. $\vec{P}(A)$, where A is the line $r=2$ in $r\theta$–space.

47. $\vec{P}(A)$, where A is the line $r=-2$ in $r\theta$–space.

48. $\vec{P}(A)$, where A is the line $\theta=\frac{\pi}{2}$ in $r\theta$–space.

49. $\vec{P}(A)$, where A is the line $\theta=\frac{\pi}{4}$ in $r\theta$–space.

50. $\vec{P}(A)$, where A is the line $\theta=\frac{3\pi}{4}$ in $r\theta$–space.

51. $\vec{P}(A)+(1,1)$, where A is the line $r=2$ in $r\theta$–space.

52. $\vec{P}(A)+(-1,2)$, where A is the line $r=4$ in $r\theta$–space.

53. $\vec{P}(A)$, where A is the rectangle in $r\theta$–space $0\leq r\leq 1,\ 0\leq\theta,\leq 2\pi$.

54. $\vec{P}(A)$, where A is the rectangle in $r\theta$–space $0\leq r\leq 1,\ 0\leq\theta\leq\frac{\pi}{4}$.

55. $\vec{P}(A)$, where A is the rectangle in $r\theta$–space $1\leq r\leq 2,\ 0\leq\theta\leq 2\pi$.

56. $\vec{P}(A)$, where A is the rectangle in $r\theta$–space $1 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{4}$.

57. $\vec{P}(A)$, where A is the rectangle in $r\theta$–space $1 \le r \le 2,\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

In Exercises 58–81, describe the set satisfying the given conditions.

58. $\vec{C}_z(A)$, where A is the plane $r = 3$ in $r\theta z$–space.

59. $\vec{C}_z(A)$, where A is the plane $r = -2$ in $r\theta z$–space.

60. $\vec{C}_z(A)$, where A is the plane $\theta = \frac{\pi}{2}$ in $r\theta z$–space.

61. $\vec{C}_z(A)$, where A is the plane $\theta = \frac{\pi}{4}$ in $r\theta z$–space.

62. $\vec{C}_z(A)$, where A is the plane $\theta = \frac{3\pi}{4}$ in $r\theta z$–space.

63. $\vec{C}_z(A) + (1, 2, 3)$, where A is the plane $r = 2$ in $r\theta z$–space.

64. $\vec{C}_z(A)$, where A is the rectangle in $r\theta z$–space $0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ z = 0$.

65. $\vec{C}_z(A)$, where A is the rectangle in $r\theta z$–space $0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{4},\ z = 2$.

66. $\vec{C}_z(A)$, where A is the rectangle in $r\theta z$–space $1 \le r \le 2,\ 0 \le \theta \le 2\pi,\ z = -3$.

67. $\vec{C}_z(A)$, where A is the rectangular box in $r\theta z$–space $0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1$.

68. $\vec{C}_z(A)$, where A is the rectangular box in $r\theta z$–space $1 \le r \le 2,\ 0 \le \theta \le \pi,\ 1 \le z \le 2$.

69. $\vec{C}_z(A)$, where A is the rectangular box in $r\theta z$–space $1 \le r \le 2,\ -\pi \le \theta \le \pi,\ 1 \le z \le 2$.

70. $\vec{S}(A)$, where A is the plane in $r\phi\theta$–space $r = 3$.

71. $\vec{S}(A)$, where A is the plane in $r\phi\theta$–space $r = -2$.

72. $\vec{S}(A)$, where A is the plane in $r\phi\theta$–space $\theta = \frac{\pi}{2}$.

73. $\vec{S}(A)$, where A is the plane in $r\phi\theta$–space $\theta = \frac{\pi}{4}$.

74. $\vec{S}(A)$, where A is the plane in $r\phi\theta$–space $\phi = \frac{3\pi}{4}$.

75. $\vec{S}(A) + (1, 2, 3)$, where A is the plane in $r\phi\theta$–space $r = 2$.

76. $\vec{S}(A)$, where A is the rectangle in $r\phi\theta$–space $0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ \phi = 0$.

77. $\vec{S}(A)$, where A is the rectangle in $r\phi\theta$–space $0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{4},\ \phi = \frac{\pi}{3}$.

78. $\vec{S}(A)$, where A is the rectangle in $r\phi\theta$–space $1 \le r \le 2,\ 0 \le \theta \le 2\pi,\ \phi = \frac{\pi}{3}$.

79. $\vec{S}(A)$, where A is the rectangular box in $r\phi\theta$–space $0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4}$.

80. $\vec{S}(A)$, where A is the rectangular box in $r\phi\theta$–space $1 \le r \le 2,\ 0 \le \theta \le \pi,\ 0 \le \phi \le \pi$.

81. $\vec{S}(A)$, where A is the rectangular box in $r\phi\theta$–space $1 \le r \le 2,\ -\pi \le \theta \le \pi,\ 0 \le \phi \le \frac{\pi}{2}$.

In Exercises 82–85, we define $\vec{C}_x$ similarly to $\vec{C}_z$ except that the x–axis is the axis of symmetry. That is $\vec{C}_x(x, r, \theta) = (x, r\cos\theta, r\sin\theta)$.

82. Show that the partial derivatives of $\vec{C}_x$ are mutually perpendicular, except at points where $r = 0$.

83. Describe $\vec{C}_x(A)$, where A is the plane $x = 2$.

84. Describe $\vec{C}_x(A)$, where A is the plane $r = 2$.

85. Describe $\vec{C}_x(A)$, where A is the plane $\theta = \frac{\pi}{2}$.

In Exercises 86–89, we define $\vec{C}_y(r, y, \theta) = (r\cos\theta, y, r\sin\theta)$.

86. Show that the partial derivatives of $\vec{C}_y$ are mutually perpendicular, except at points where $r = 0$.

87. Describe $\vec{C}_y(A)$, where A is the plane $y = 2$.

88. Describe $\vec{C}_y(A)$, where A is the plane $r = 2$.

89. Describe $\vec{C}_y(A)$, where A is the plane $\theta = \frac{\pi}{2}$.

90. Show that if $0 < \phi < \pi$, then $\vec{S}$ satisfies the right hand rule. That is, the partial derivative with respect to r crossed with the partial derivative with respect to ϕ has the same direction as the partial derivative with respect to the θ.

91. Calculate the partial derivatives of $\vec{f}(u,v) = (2\cos(u)\sin(v), 2\sin(u)\sin(v), 2\cos(v))$.

92. What is the image of $\vec{f}$ of Exercise 91?

4.3 The Derivative

Let $\vec{F}(u,v)$ be a function from $\mathbb{R}^2$ into $\mathbb{R}^2$. Let F_x denote the x–coordinate function, and let F_y denote the y–coordinate function, so that

$$\vec{F}(u,v) = (F_x(u,v), F_y(u,v)).$$

Let $\vec{r}(t)$ be a parametrization for a curve in $\mathbb{R}^2$, and let

$$\vec{g}(t) = (F_x(\vec{r}(t)), F_y(\vec{r}(t))).$$

Now,

$$\frac{d\vec{g}(t)}{dt} = \left(\frac{dF_x(\vec{r}(t))}{dt}, \frac{dF_y(\vec{r}(t))}{dt}\right).$$

By Theorem 1 of Section 11.8, we see that

$$\begin{aligned}\frac{dF_x(\vec{r}(t))}{dt} &= \nabla F_x(\vec{r}(t)) \cdot \vec{r}\,'(t) \\ &= \left(\frac{\partial F_x}{\partial u}(\vec{r}(t))\right) u'(t) + \left(\frac{\partial F_x}{\partial v}(\vec{r}(t))\right) v'(t) \\ &= \frac{\partial F_x}{\partial u}\frac{du}{dt} + \frac{\partial F_x}{\partial v}\frac{dv}{dt},\end{aligned}$$

and

$$\begin{aligned}\frac{dF_y(\vec{r}(t))}{dt} &= \nabla F_y(\vec{r}(t)) \cdot \vec{r}\,'(t) \\ &= \left(\frac{\partial F_y}{\partial u}(\vec{r}(t))\right) u'(t) + \left(\frac{\partial F_y}{\partial v}(\vec{r}(t))\right) v'(t) \\ &= \frac{\partial F_y}{\partial u}\frac{du}{dt} + \frac{\partial F_y}{\partial v}\frac{dv}{dt}.\end{aligned}$$

We can express the above two equations in matrix notation in the following way:

$$\begin{aligned}\frac{d\vec{g}(t)}{dt} &= \begin{pmatrix} \frac{dF_x(\vec{r}(t))}{dt} \\ \frac{dF_y(\vec{r}(t))}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_x}{\partial u}\frac{du}{dt} + \frac{\partial F_x}{\partial v}\frac{dv}{dt} \\ \frac{\partial F_y}{\partial u}\frac{du}{dt} + \frac{\partial F_y}{\partial v}\frac{dv}{dt} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial F_x}{\partial u} & \frac{\partial F_x}{\partial v} \\ \frac{\partial F_y}{\partial u} & \frac{\partial F_y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix}.\end{aligned}$$

Definition: The Derivative of $\vec{F}$ at (u,v)

Let $\vec{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function

$$\vec{F}(u,v) = (F_x(u,v), F_y(u,v)).$$

If $\frac{\partial F_x}{\partial u}$, $\frac{\partial F_x}{\partial v}$, $\frac{\partial F_y}{\partial u}$, and $\frac{\partial F_y}{\partial v}$ exist, we define

$$D\vec{F}|_{(u,v)} = \begin{pmatrix} \frac{\partial F_x}{\partial u} & \frac{\partial F_x}{\partial v} \\ \frac{\partial F_y}{\partial u} & \frac{\partial F_y}{\partial v} \end{pmatrix}$$

to be the *derivative of $\vec{F}$ evaluated at* (u,v).

Notice that the first row of $D\vec{F}$ is simply ∇F_x, the second row is ∇F_y, and the derivative of $\vec{F}$ is defined so that we have the chain rule for functions from $\mathbb{R}^2$ into $\mathbb{R}^2$.

$$\frac{d}{dt}\vec{F}(\vec{r}(t)) = D\vec{F}|_{\vec{r}(t)}\vec{r}\,'(t).$$

EXAMPLE 1: Consider the polar transformation $\vec{P}(r,\theta) = (r\cos\theta, r\sin\theta)$. Then

$$D\vec{P}|_{(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

If $\vec{r}(t) = (1, 2\pi t)$, then $\vec{P}(\vec{r}(t)) = (\cos(2\pi t), \sin(2\pi t))$ describes a particle moving around a circle in xy–space at a rate of one rotation/sec. We can either calculate the derivative directly as

$$\frac{d\vec{P}(\vec{r}(t))}{dt} = 2\pi(-\sin(2\pi t), \cos(2\pi t)),$$

or use the above to obtain

$$D\vec{P}|_{(r,\theta)} = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix},$$

and

$$\vec{r}\,'(t) = (0, 2\pi).$$

Thus

$$\begin{aligned}\frac{d\vec{P}(\vec{r}(t))}{dt} &= D\vec{P}|_{\vec{r}(t)}\vec{r}\,'(t) \\ &= \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \begin{pmatrix} 0 \\ 2\pi \end{pmatrix} \\ &= 2\pi(-\sin(2\pi t), \cos(2\pi t)).\end{aligned}$$

■

In general, if $\vec{s}(t) = (r(t), \theta(t))$ is an expression for the polar coordinates of a particle at time t, then $\vec{P}(\vec{s}(t))$ will give the rectangular or Cartesian coordinates of the particle. The derivative $\vec{s}\,'(t)$ will then denote the polar coordinates of the velocity vector and

$$\begin{aligned}\frac{d}{dt}\vec{P}(\vec{s}(t)) &= D\vec{P}|_{\vec{s}(t)}\vec{s}\,'(t) \\ &= \begin{pmatrix} \cos(\theta(t)) & -r(t)\sin(\theta(t)) \\ \sin(\theta(t)) & r(t)\cos(\theta(t)) \end{pmatrix} \begin{pmatrix} r'(t) \\ \theta'(t) \end{pmatrix}\end{aligned}$$

will give the Cartesian coordinates of the velocity vector.

Note: It is not uncommon to encounter the notation $d\vec{s}/dt$, which is meant to represent the rate of change of position in rectangular coordinates even though $\vec{s}$ represents the polar coordinates of the point.

EXAMPLE 2: Suppose that a particle is moving in the plane so that when the polar coordinates are $\left(2, \frac{\pi}{4}\right)$, its velocity, in polar coordinates, is $(1, \pi)$. What is the velocity in rectangular coordinates?

SOLUTION: If t_0 denotes the time at which the particle is at $\left(2, \frac{\pi}{4}\right)$, and if $\vec{s}$ is the parametrization giving the position of the particle at time t in polar coordinates, then $\vec{s}(t_0) = \left(2, \frac{\pi}{4}\right)$ and $\vec{s}\,'(t_0) = (1, \pi)$.

$$\begin{aligned}D\vec{P}|_{\vec{s}(t_0)}\vec{s}\,'(t_0) &= \begin{pmatrix} \cos(\theta(t_0)) & -r(t_0)\sin(\theta(t_0)) \\ \sin(\theta(t_0)) & r(t_0)\cos(\theta(t_0)) \end{pmatrix} \begin{pmatrix} r'(t_0) \\ \theta'(t_0) \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -2\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & 2\cos\left(\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} 1 \\ \pi \end{pmatrix}\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\sqrt{2} \\ \frac{\sqrt{2}}{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ \pi \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{2}}{2} - \pi\sqrt{2} \\ \frac{\sqrt{2}}{2} + \pi\sqrt{2} \end{pmatrix}.
\end{aligned}
$$

so the particle's x–coordinate is decreasing at the rate of $\left(\frac{\sqrt{2}}{2}\right) - \pi\sqrt{2}$, while the y–coordinate is increasing at the rate of $\left(\frac{\sqrt{2}}{2}\right) + \pi\sqrt{2}$. ■

We have defined the derivative of a function $\vec{F}$ from uv–space into xy–space (evaluated at (u, v)) to be the 2×2 matrix whose first row is $\nabla F_x|_{(u,v)}$, and whose second row is $\nabla F_y|_{(u,v)}$. Notice also that the first column of $D\vec{F}|_{(u,v)}$ is $\frac{\partial \vec{F}}{\partial u}$, and the second column is $\frac{\partial \vec{F}}{\partial v}$.

$$
\begin{array}{cc}
\frac{\partial \vec{F}}{\partial u} & \\
\downarrow & \\
\left(\boxed{\begin{matrix} \frac{\partial F_x}{\partial u} \\ \frac{\partial F_y}{\partial u} \end{matrix}} \quad \begin{matrix} \frac{\partial F_x}{\partial v} \\ \frac{\partial F_y}{\partial v} \end{matrix} \right)
\end{array}
\qquad
\begin{pmatrix} \boxed{\frac{\partial F_x}{\partial u} \quad \frac{\partial F_x}{\partial v}} \\ \frac{\partial F_y}{\partial u} \quad \frac{\partial F_y}{\partial v} \end{pmatrix} \quad \leftarrow \nabla F_x
$$

$$
\begin{array}{cc}
\left(\begin{matrix} \frac{\partial F_x}{\partial u} \\ \frac{\partial F_y}{\partial u} \end{matrix} \quad \boxed{\begin{matrix} \frac{\partial F_x}{\partial v} \\ \frac{\partial F_y}{\partial v} \end{matrix}} \right) \\
\uparrow \\
\frac{\partial \vec{F}}{\partial v}
\end{array}
\qquad
\begin{pmatrix} \frac{\partial F_x}{\partial u} \quad \frac{\partial F_x}{\partial v} \\ \boxed{\frac{\partial F_y}{\partial u} \quad \frac{\partial F_y}{\partial v}} \end{pmatrix} \quad \leftarrow \nabla F_y
$$

We define the derivative from $\mathbb{R}^n$ into $\mathbb{R}^m$ in a similar manner. If $\vec{F}$ is a function from uv–space into xyz–space, then we define the derivative of $\vec{F}$ to be the 3×2 matrix whose first row is the gradient of F_x, whose second row is the gradient of F_y, and whose third row is the gradient of F_z.

$$
D\vec{F}|_{(u,v)} = \begin{pmatrix} \frac{\partial F_x}{\partial u} & \frac{\partial F_x}{\partial v} \\ \frac{\partial F_y}{\partial u} & \frac{\partial F_y}{\partial v} \\ \frac{\partial F_z}{\partial u} & \frac{\partial F_z}{\partial v} \end{pmatrix}_{\text{evaluated at } (u,v)}
$$

If $\vec{F}$ is a function from uvw–space into xyz–space, then the derivative of $\vec{F}$ is a 3×3 matrix, the rows again being the gradients of the coordinate functions.

$$D\vec{F}|_{(u,v,w)} = \begin{pmatrix} \frac{\partial F_x}{\partial u} & \frac{\partial F_x}{\partial v} & \frac{\partial F_x}{\partial w} \\ \frac{\partial F_y}{\partial u} & \frac{\partial F_y}{\partial v} & \frac{\partial F_y}{\partial w} \\ \frac{\partial F_z}{\partial u} & \frac{\partial F_z}{\partial v} & \frac{\partial F_z}{\partial w} \end{pmatrix}_{\text{evaluated at } (u,v,w)}$$

Finally, if $\vec{F}$ is a function from uvw–space into xy–space, then the derivative of $\vec{F}$ is a 2×3 matrix, again with the rows being the gradients of the coordinate functions.

$$D\vec{F}|_{(u,v,w)} = \begin{pmatrix} \frac{\partial F_x}{\partial u} & \frac{\partial F_x}{\partial v} & \frac{\partial F_x}{\partial w} \\ \frac{\partial F_y}{\partial u} & \frac{\partial F_y}{\partial v} & \frac{\partial F_y}{\partial w} \end{pmatrix}_{\text{evaluated at } (u,v,w)}$$

In general, if $\vec{F}$ is a function from $\mathbb{R}^n$ into $\mathbb{R}^m$, then the derivative of $\vec{F}$ is an $m \times n$ matrix, with the rows being the gradients of the coordinate functions.

EXAMPLE 3: Let $\vec{T}$ be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ with the associated 3×3 matrix $A_{\vec{T}}$, so that $\vec{T}(\vec{p}) = A_{\vec{T}}\vec{p}$. Thus, if

$$\vec{p} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \text{ and } A_{\vec{T}} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

then

$$A_{\vec{T}}\vec{p} = u \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + v \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + w \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

We see that

$$\begin{aligned} T_x(u,v,w) &= (ua_1 + vb_1 + wc_1), \text{ and } \nabla T_x(u,v,w) = (a_1, b_1, c_1) \\ T_y(u,v,w) &= (ua_2 + vb_2 + wc_2), \text{ and } \nabla T_x(u,v,w) = (a_2, b_2, c_2) \\ T_z(u,v,w) &= (ua_3 + vb_3 + wc_3), \text{ and } \nabla T_x(u,v,w) = (a_3, b_3, c_3), \end{aligned}$$

so $D\vec{T}|_{\vec{v}} = A_{\vec{T}}$. ■

Theorem 1 *If $\vec{T}$ is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ and $A_{\vec{T}}$ is the matrix associated with $\vec{T}$, then the derivative of $\vec{T}$ evaluated at any point in $\mathbb{R}^n$ is $A_{\vec{T}}$.*

EXAMPLE 4: Let $\vec{T}$ be defined by $\vec{T}(u,v) = (2u+v, v)$. Then

$$D\vec{T}|_{(u,v)} = A_{\vec{T}} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

If a particle moves with velocity $(1,2)$ in uv–space, then its velocity viewed in xy–space is

$$A_{\vec{T}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}. \qquad \blacksquare$$

The derivative of a function from a subset of $\mathbb{R}^n$ into $\mathbb{R}^m$ satisfies the chain rule.

Theorem 2 (The Chain Rule: Functions from $\mathbb{R}^n$ into $\mathbb{R}^m$) *Suppose that S_1 is a subset of the reals, S_2 is a subset of $\mathbb{R}^n$, $\vec{r} : S_1 \to S_2$ and $\vec{F} : S_2 \to \mathbb{R}^m$ are differentiable. Let*

$$\vec{g}(t) = \vec{F} \circ \vec{r}(t) = \vec{F}(\vec{r}(t)).$$

Then

$$\frac{d\vec{g}}{dt} = D\vec{F}|_{\vec{r}(t)} \vec{r}\,'(t).$$

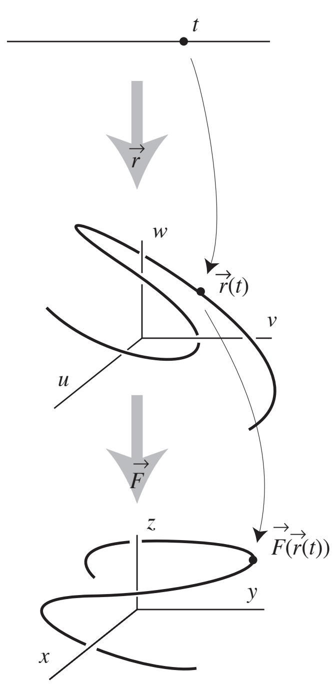


Figure 1. *The composite of $\vec{F}$ following $\vec{r}$.*

EXAMPLE 5: If

$$\vec{r}(t) = (u(t), v(t), w(t))$$

is a differentiable function from a subset of the reals into uvw–space, and

$$\vec{F}(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$$

is a differentiable function from a subset of uvw–space into xyz–space as in Figure 1, then

$$\begin{aligned}
\frac{d\vec{g}}{dt} &= D\vec{F}|_{\vec{r}(t)} \vec{r}\,'(t) \\
&= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \begin{pmatrix} u'(t) \\ v'(t) \\ w'(t) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial x}{\partial w}\frac{dw}{dt} \\ \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt} + \frac{\partial y}{\partial w}\frac{dw}{dt} \\ \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt} \end{pmatrix}
\end{aligned}$$

It is critical to remember that all of the entries in

$$D\vec{F} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

are evaluated at $\vec{r}(t)$. Often, you will see the coordinate functions written out separately as follows.

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial x}{\partial w}\frac{dw}{dt} \\ \frac{dy}{dt} &= \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt} + \frac{\partial y}{\partial w}\frac{dw}{dt} \\ \frac{dz}{dt} &= \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt}, \end{aligned}$$

where all of the terms $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial x}{\partial w}$, $\frac{\partial y}{\partial u}$, etc. are evaluated at $\vec{r}(t) = (u(t), v(t), w(t))$. ■

EXAMPLE 6: Suppose that

$$\vec{r}(t) = (u(t), v(t))$$

is a differentiable function from a subset of the reals into uv–space, and

$$\vec{F}(u, v) = (x(u, v), y(u, v), z(u, v))$$

is a differentiable function from a subset of uv–space into xyz–space (as in Figure 1, except that the intermediate space is a plane rather than a three dimensional space.) Then

$$\frac{d\vec{g}}{dt} = D\vec{F}|_{\vec{r}(t)}\vec{r}\,'(t) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt} \\ \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt} \\ \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} \end{pmatrix}.$$

The coordinate functions of $\frac{d\vec{g}}{dt}$ written separately give

$$\frac{dx}{dt} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt} \\ \frac{dz}{dt} &= \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}. \end{aligned} \qquad \blacksquare$$

The Chain Rule for Partial Derivatives

Suppose that $\vec{F}$ is a differentiable function from uv–space into xy–space, and that ϕ is a real-valued differentiable function defined on a subset of xy–space that contains the range of $\vec{F}$. Then $\phi \circ \vec{F} = \phi(\vec{F})$ is a real-valued function defined on uv–space. We want to find $\frac{\partial(\phi\circ\vec{F})}{\partial u} =$ the rate that ϕ changes as u changes.

To set this up, assume that

$$\vec{F}(u,v) = (x(u,v), y(u,v))$$

and let $\vec{r}(t) = (u_1 + t, v_1)$, for some point (u_1, v_1), and $\vec{g}(t) = \vec{F}(\vec{r}(t))$. By the definition of the partial derivative for real-valued differentiable functions, we see that

$$\left.\frac{\partial \phi \circ \vec{F}(u,v)}{\partial u}\right|_{(u_1,v_1)} = \left.\frac{d\phi(\vec{g}(t))}{dt}\right|_{t=0} = \nabla\phi|_{\vec{g}(0)} \cdot \left.\frac{d\vec{g}(t)}{dt}\right|_{t=0}.$$

We know that

$$\nabla\phi|_{\vec{g}(0)} = \left.\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)\right|_{\vec{g}(0)} = \left.\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)\right|_{\vec{F}(u_1,v_1)}.$$

We need to calculate $\frac{d\vec{g}(t)}{dt}|_{t=0}$.

$$\begin{aligned} \left.\frac{d\vec{g}(t)}{dt}\right|_{t=0} &= D\vec{F}|_{\vec{r}(0)}\vec{r}\,'(0) \\ &= \left.\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right|_{\vec{r}(0)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \left.\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right)\right|_{\vec{r}(0)} \\ &= \left.\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right)\right|_{(u_1,v_1)}. \end{aligned}$$

Putting it together we obtain

$$\left.\frac{\partial \phi \circ \vec{F}(u,v)}{\partial u}\right|_{(u_1,v_1)} = \left.\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)\right|_{\vec{F}(u_1,v_1)} \cdot \left.\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right)\right|_{(u_1,v_1)}.$$

More simply, we write

$$\frac{\partial \phi \circ \vec{F}}{\partial u} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) \cdot \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u}.$$

Similarly we can use $\vec{r}(t) = (u_1, v_1 + t)$ and obtain

$$\frac{\partial \phi \circ \vec{F}}{\partial v} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) \cdot \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right) = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v}.$$

It is understood that $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ are evaluated at $\vec{F}(u,v) = (x(u,v), y(u,v))$.

In much the same way, we obtain the three dimensional case, which we present as the following chain rule for partial derivatives without proof.

Theorem 3 (The Chain Rule for Partial Derivatives)
Suppose

$$\vec{F}(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$$

is a differentiable function from uvw–space into xyz–space, and ϕ *is a real valued differentiable function defined on a subset of xyz–space that contains the image of* $\vec{F}$*. Then*

$$\begin{aligned}
\frac{\partial(\phi \circ \vec{F})}{\partial u}(u,v,w) &= \nabla\phi \cdot \frac{\partial \vec{F}}{\partial u} \\
&= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial u}, \\
\frac{\partial(\phi \circ \vec{F})}{\partial v}(u,v,w) &= \nabla\phi \cdot \frac{\partial \vec{F}}{\partial v} \\
&= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial v},
\end{aligned}$$

and

$$\begin{aligned} \frac{\partial(\phi \circ \vec{F})}{\partial w}(u,v,w) &= \nabla\phi \cdot \frac{\partial \vec{F}}{\partial w} \\ &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial w} \end{aligned}$$

$\nabla\phi$, $\frac{\partial\phi}{\partial x}$, $\frac{\partial\phi}{\partial x}$, *and* $\frac{\partial\phi}{\partial x}$ *are evaluated at* $\vec{F}(u,v,w)$. *The partial derivatives of* x, y, *and* z *are evaluated at* (u,v,w). *It is common in the literature to write* $\frac{\partial\phi}{\partial u}$ *to mean* $\frac{\partial(\phi\circ\vec{F})}{\partial u}$.

There are, of course, similar results if, for example,

$$\vec{F}(u,v,w) = \begin{pmatrix} x(u,v,w) \\ y(u,v,w) \end{pmatrix}$$

is a differentiable function from uvw–space into xy–space, and ϕ is a real-valued differentiable function defined on a subset of xy–space that contains the image of $\vec{F}$, or if

$$\vec{F}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

is a differentiable function from uv–space into xyz–space, and ϕ is a real-valued differentiable function defined on a subset of xyz–space that contains the domain of $\vec{F}$.[1]

[1]The general statement of the theorem is as follows.
If $\vec{F}(u_1,\ldots,u_n) = (x_1(u_1,\ldots,u_n),\cdots,x_m(u_1,\ldots,u_n))$ is a differentiable function from a subset of $\mathbb{R}^n$ into $\mathbb{R}^m$, and $\phi(x_1,\cdots,x_m)$ is a real-valued differentiable function defined on a subset of $\mathbb{R}^m$ that contains the image of $\vec{F}$, then

$$\begin{aligned} \frac{\partial(\phi \circ \vec{F})}{\partial u_i}(u_1,\ldots,u_n) &= \nabla\phi\Big|_{\vec{F}(u_1,\ldots,u_n)} \cdot \frac{\partial \vec{F}}{\partial u_i}(u_1,\ldots,u_n) \\ &= \frac{\partial \phi}{\partial x_1}\frac{\partial x_1}{\partial u_i} + \cdots + \frac{\partial \phi}{\partial x_m}\frac{\partial x_m}{\partial u_i} \end{aligned}$$

EXAMPLE 7: Let $\vec{P}(r,\theta) = (r\cos\theta, r\sin\theta)$ and let $\phi(x,y) = x^2 + y^2$. Then

$$\nabla\phi(x,y) = (2x, 2y), \text{ so } \nabla\phi(P(r,\theta)) = (2r\cos\theta, 2r\sin\theta).$$

We also have

$$\frac{\partial P}{\partial r}(r,\theta) = (\cos\theta, \sin\theta) \text{ and } \frac{\partial P}{\partial \theta}(r,\theta) = (-r\sin\theta, r\cos\theta).$$

Thus

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= \nabla\phi(P(r,\theta)) \cdot \frac{\partial P}{\partial r}(r,\theta) \\ &= (2r\cos\theta, 2r\sin\theta) \cdot (\cos\theta, \sin\theta) \\ &= 2r\cos^2\theta + 2r\sin^2\theta = 2r, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \phi}{\partial \theta} &= \nabla\phi(P(r,\theta)) \cdot \frac{\partial P}{\partial \theta}(r,\theta) \\ &= (2r\cos\theta, 2r\sin\theta) \cdot (-r\sin\theta, r\cos\theta)\,. \\ &= -2r^2\cos\theta\sin\theta + 2r^2\cos\theta\sin\theta = 0. \end{aligned}$$

As we would expect, a direct computation produces the same results. $\phi(P(r,\theta)) = r^2$. Thus

$$\frac{\partial}{\partial r}(r^2) = 2r \text{ and } \frac{\partial}{\partial \theta}(r^2) = 0. \qquad \blacksquare$$

Linear Approximations

Suppose that $\vec{h}(u,v) = (x(u,v), y(u,v), z(u,v))$ is a function from $\mathbb{R}^2$ into $\mathbb{R}^3$. Let $\vec{r}_0 = (u_0, v_0)$ be a fixed point. Let $\vec{r}(t)$ be defined by $\vec{r}_1(t) = \vec{h}(t, v_0)$ and $\vec{r}_2(t) = \vec{h}(u_0, t)$. Then

$$\frac{\partial h}{\partial u}(u_0, r_0) = \vec{r}\,'_1(u_0) \quad \text{ and } \quad \frac{\partial h}{\partial v}(u_0, r_0) = \vec{r}\,'_2(v_0).$$

$\vec{r}\,'_1(u_0)$ is tangent to the curve parametrized by it, which in turn lines in the image of $\vec{h}$. It follows that $\vec{U} = \dfrac{\partial \vec{h}}{\partial u}(u_0, v_0)$ is tangent to the image of $\vec{h}$. Similarly, $\vec{V} = \dfrac{\partial \vec{h}}{\partial v}(u_0, v_0)$ is tangent to the image of $\vec{h}$.

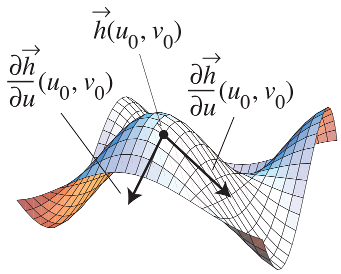


Figure 2. *The vectors* $\frac{\partial \vec{h}}{\partial u}(u_0, v_0)$ *and* $\frac{\partial \vec{h}}{\partial v}(u_0, u_0)$ *determine the plane tangent to the surface at* $\vec{h}(u_0, v_0)$

Thus, as long as $\vec{U} \times \vec{V} \neq \vec{0}$, they determine the plane tangent to the surface at $\vec{h}(u_0, v_0)$. See Figure 2. This plane is the image of

$$\begin{aligned}
\vec{T}(s,t) &= s\vec{U} + t\vec{V} + \vec{r}_0 \\
&= D(\vec{h})\big|_{(u_0,v_0)} \begin{pmatrix} s \\ t \end{pmatrix} + \vec{h}(u_0, v_0) \\
&= \begin{pmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial v}(u_0,v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) \\ \frac{\partial z}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} x(u_0,v_0) \\ y(u_0,v_0) \\ z(u_0,v_0) \end{pmatrix}.
\end{aligned}$$

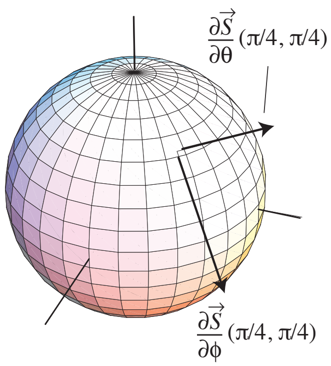


Figure 3.a *The vectors* $\frac{\partial \vec{S}}{\partial \theta}$ *and* $\frac{\partial \vec{S}}{\partial \phi}$ *determine the plane to the sphere at* $\vec{S}\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

Note: $\vec{T}$ is a linear transformation translated to the point of tangency.

EXAMPLE 8: Let $\vec{S}(\phi, \theta) = (\sin(\phi)\cos(\theta), \sin(\phi)(\sin(\theta), \cos(\phi))$. Then the image of $\vec{S}$ is the unit sphere centered at the origin. As illustrated in Figures 3.a and 3.b, $\dfrac{\partial \vec{S}}{\partial \theta}(\phi_0, \theta_0)$ is tangent to a latitude circle and $\dfrac{\partial \vec{S}}{\partial \phi}(\phi_0, \theta_0)$ is tangent to a longitudinal circle. We leave it as an exercise to show that their cross product points away from the origin (its radial), so the plane determined by these vectors drawn emanating from the origin is tangent to the sphere. To find the plane tangent to the sphere at $\vec{S}\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, we find the derivative of $\vec{S}$ at that point.

$$x(\phi, \theta) = \sin(\phi)\cos(\theta),$$

$$\frac{\partial x}{\partial \phi} = \cos(\phi)\cos(\theta), \quad \frac{\partial x}{\partial \phi}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \frac{1}{2},$$

and

$$\frac{\partial x}{\partial \theta} = -\sin(\phi)\sin(\theta), \quad \frac{\partial x}{\partial \theta}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = -\frac{1}{2}.$$

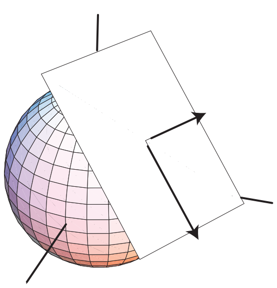


Figure 3.b *The approximating tangent plane to the sphere.*

Similarly, we get

$$\frac{\partial y}{\partial \phi}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \frac{1}{2}, \quad \frac{\partial z}{\partial \phi}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}},$$

and

$$\frac{\partial y}{\partial \theta}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \frac{1}{2}, \quad \frac{\partial z}{\partial \theta}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = 0.$$

The tangent plane is the image of

$$\vec{T}(s,t) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{s}{2} - \frac{t}{2} + \frac{1}{2} \\ \frac{s}{2} + \frac{t}{2} + \frac{1}{2} \\ -\frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{pmatrix}. \quad \blacksquare$$

We have been focusing on tangent planes because they are relatively easy to visualize and, inspecting the basic formula, they closely parallel the earlier work we have done with tangent lines. The parallel does not stop there. Recall that the first order Taylor approximation for a real valued function is given by

$$p_1(x) = f'(x_0)(x - x_0) + f(x_0).$$

In exactly the same way, we can approximate functions from $\mathbb{R}^n$ into $\mathbb{R}^m$. Let $\vec{h}$ be a function from $\mathbb{R}^n$ into $\mathbb{R}^m$. The function

$$\vec{p}(\vec{r}) = D(\vec{h})|_{\vec{r}_0}[\vec{r} - \vec{r}_0] + \vec{h}(\vec{r}_0)$$

is called the *first order Taylor approximation* for $\vec{h}$ at $\vec{r}_0$. Just as in the first order Taylor polynomials for real valued functions, if $\|\vec{r} - \vec{r}_0\|$ is small, then $\|\vec{p}(\vec{r}) - \vec{h}(\vec{r})\|$ is small. In fact, if all of the partial derivatives are continuous, then

$$\lim_{\|\vec{r}-\vec{r}_0\| \to 0} \frac{1}{\|\vec{r} - \vec{r}_0\|} \|\vec{p}(\vec{r}) - \vec{h}(\vec{r})\| = 0.$$

EXAMPLE 9: The first order Taylor polynomial for the function $\vec{S}$ from Example 8 at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ is

$$\vec{p}(\phi,\theta) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \phi - \frac{\pi}{4} \\ \theta - \frac{\pi}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

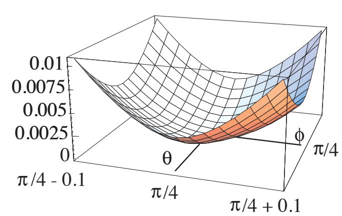


Figure 4. *The error function over the interval* $\frac{\pi}{4} - 0.1 \le \phi \le \frac{\pi}{4} + 0.1$ *and* $\frac{\pi}{4} - 0.1 \le \theta \le \frac{\pi}{4} + 0.1$

The approximating plane is displayed in Figure 3.b. In Figure 4, we sketch the error function $\|\vec{p}(\phi,\theta) - \vec{s}(\phi,\theta)\|$ over the rectangle $\frac{\pi}{4} - 0.1 \le \phi \le \frac{\pi}{4} + 0.1$ and $\frac{\pi}{4} - 0.5 \le \theta \le \frac{\pi}{4} + 0.5$. Inspection of the figure reveals that in this region, the error on the approximating function is small, near $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$. ■

Summary

(a) If $\vec{F}$ is a function from $\mathbb{R}^n$ into $\mathbb{R}^m$, then the derivative of $\vec{F}$ evaluated at $\vec{r}$ is defined to be the $m \times n$ matrix such that for each i, with $1 \leq i \leq m$, the i^{th} row of $D(\vec{F})|_{\vec{r}}$ is the gradient of the i^{th} coordinate function of $\vec{F}$ evaluated at $\vec{r}$.

(b) If $\vec{r}$ is a function from $\mathbb{R}$ into $\mathbb{R}^n$ and $\vec{F}$ is a function from $\mathbb{R}^n$ into $\mathbb{R}^m$, then

$$\frac{d}{dt}\vec{F}(\vec{r}(t)) = D\vec{F}|_{\vec{r}(t)}\vec{r}\,'(t).$$

If

$$\vec{F}(u,v,w) = \begin{pmatrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{pmatrix} \text{ and } \vec{r}(t) = \begin{pmatrix} u(t) \\ v(t) \\ z(t) \end{pmatrix},$$

then

$$\begin{aligned}
\frac{dx}{dt} &= \nabla x(u,v,w)|_{\vec{r}(t)} \cdot \vec{r}\,'(t) = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial x}{\partial w}\frac{dw}{dt} \\
\frac{dy}{dt} &= \nabla y(u,v,w)|_{\vec{r}(t)} \cdot \vec{r}\,'(t) = \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt} + \frac{\partial y}{\partial w}\frac{dw}{dt} \\
\frac{dz}{dt} &= \nabla z(u,v,w)|_{\vec{r}(t)} \cdot \vec{r}\,'(t) = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt}.
\end{aligned}$$

(c) If $\vec{T}$ is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$, and if $A_{\vec{T}}$ is the $m \times n$ matrix associated to $\vec{T}$, then the derivative of $\vec{T}$ is the constant matrix $A_{\vec{T}}$.

(d) If $\vec{F}$ is a differentiable function from a subset of uvw–space into xyz–space, and ϕ is a real-valued differentiable function from a subset of xyz–space containing the image of $\vec{F}$, then

$$\frac{\partial(\phi \circ \vec{F})}{\partial u}(u,v,w) = \nabla\phi|_{\vec{F}(u,v,w)} \cdot \frac{\partial \vec{F}}{\partial u}(u,v,w),$$

which is usually shortened to

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial u}.$$

(e)

$$\vec{p}(\vec{r}) = D(\vec{h})\big|_{\vec{r}_0} \cdot (\vec{r} - \vec{r}_0) + \vec{h}(\vec{r}_0)$$

is the *first order Taylor approximation* for $\vec{h}$ at $\vec{r}_0$.

EXERCISES 4.3

In Exercises 1–7, find the derivative of the linear transformations.

1. $\vec{T}(u,v) = (3u, -2v)$.
2. $\vec{T}(u,v) = (u + 3v, v - u)$.
3. $\vec{T}(u,v) = (u - v, 2v, u + v)$.
4. $\vec{T}(s,t) = (s - t, 3t + s, t - s)$.
5. $\vec{T}(u,v) = u + v$.
6. $\vec{T}(u,v,w) = (2u - v, u + w, u + v + w)$.
7. $\vec{T}(r,s,t) = (2r + s - t, r - s - 3t, r + s + t)$.
8. $\vec{T}(u,v,w) = (2u - w + v, u + v - 22w)$.
9. $\vec{T}(u,v,w) = (u + v + w, u + v + w)$.
10. $\vec{T}(u,v,w) = u + v + w$.
11. Find the derivative of the cylindrical transformation $\vec{C}_z$.
12. Find the derivative of the cylindrical transformation $\vec{C}_x$.
13. Find the derivative of the spherical transformation.

In Exercises 14–19, $\vec{F}$ is a function from $\mathbb{R}^n$ into $\mathbb{R}^m$. Given the derivative of $\vec{F}$ at $\vec{r}_0$, determine n and m.

14. $D\vec{F}\big|_{\vec{r}_0} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$.
15. $D\vec{F}\big|_{\vec{r}_0} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \\ 1 & 0 \end{pmatrix}$.
16. $D\vec{F}\big|_{\vec{r}_0} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & -1 \end{pmatrix}$.
17. $D\vec{F}\big|_{\vec{r}_0} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
18. $D\vec{F}\big|_{\vec{r}_0} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$.
19. $D\vec{F}\big|_{\vec{r}_0} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$.

In Exercises 20–23, a particle's position $\vec{s}_0$ and velocity $\vec{v}_0$ are given in polar coordinates. Find the position and velocity in rectangular coordinates.

20. $\vec{s}_0 = (1, \pi)$, $\quad \vec{v}_0 = (2, 3)$.
21. $\vec{s}_0 = \left(2, \frac{\pi}{2}\right)$, $\quad \vec{v}_0 = (-1, 2)$.
22. $\vec{s}_0 = \left(1, \frac{\pi}{3}\right)$, $\quad \vec{v}_0 = (1, -2)$.
23. $\vec{s}_0 = \left(1, \frac{7\pi}{4}\right)$, $\quad \vec{v}_0 = (2, 3)$.

In Exercises 24–27, a particle's position $\vec{s}_0$ and velocity $\vec{v}_0$ are given in cylindrical coordinates. Find the position and velocity in rectangular coordinates.

24. $\vec{s}_0 = (1, \pi, 1)$, $\quad \vec{v}_0 = (2, 3, 1)$.
25. $\vec{s}_0 = \left(-1, \frac{\pi}{4}, 2\right)$, $\quad \vec{v}_0 = (-1, 2, 0)$.
26. $\vec{s}_0 = \left(1, \frac{\pi}{3}, 1\right)$, $\quad \vec{v}_0 = (1, -2, -2)$.
27. $\vec{s}_0 = \left(1, \frac{7\pi}{4}, -2\right)$, $\quad \vec{v}_0 = (2, 3, 1)$.

In Exercises 28–31, a particle's position $\vec{s}_0$ and velocity $\vec{v}_0$ are given in spherical coordinates. Find the position and velocity in rectangular coordinates.

28. $\vec{s}_0 = \left(1, \pi, \frac{\pi}{4}\right)$, $\quad \vec{v}_0 = (2, 3, 1)$.
29. $\vec{s}_0 = \left(-1, \frac{\pi}{4}, \frac{\pi}{2}\right)$, $\quad \vec{v}_0 = (-1, 2, 0)$.

30. $\vec{s}_0 = \left(1, \frac{\pi}{3}, \frac{\pi}{3}\right), \quad \vec{v}_0 = (1, -2, -2)$.

31. $\vec{s}_0 = \left(1, \frac{7\pi}{4}, -\frac{\pi}{3}\right), \quad \vec{v}_0 = (2, 3, 1)$.

32. Find the derivative of the function $\vec{h}(u, v) = (u\cos(v), u\sin(v), u)$.

33. Find the derivative of the function $\vec{h}(u, v) = (u^2\cos(v), u^2\sin(v), u)$.

In Exercises 34–36, $\phi(x, y, z) = e^{xy+z}$. Use the chain rule for partial derivatives to calculate $\frac{\partial\phi}{\partial u} = \frac{\partial(\phi\circ\vec{F})}{\partial u}$ and $\frac{\partial\phi}{\partial v} = \frac{\partial(\phi\circ\vec{F})}{\partial v}$ for the given function $\vec{F}$.

34. $\vec{F}(u, v) = (u + v, v, uv)$.

35. $\vec{F}(u, v) = (u, u\cos v, u\sin v)$.

36. $\vec{F}(u, v) = (\cos v \sin u, \sin v \sin u, \cos u)$.

In Exercises 37–39, $\vec{P}(r, \theta) = (r\cos\theta, r\sin\theta)$ is the polar transformation. Use the chain rule to calculate $\frac{\partial(\phi\circ P)}{\partial\theta} = \frac{\partial(\phi)}{\partial\theta}$ for the given function ϕ.

37. $\phi(x, y) = x + y$.

38. $\phi(x, y) = xy^2$.

39. $\phi(x, y) = \ln x + 3y$.

In Exercises 40–42, $\vec{S}(\rho, \phi, \theta) = (\rho\cos\theta\sin\phi, \rho\sin\theta \sin\phi, \rho\cos\phi)$ is the spherical transformation. Use the chain rule to calculate $\frac{\partial(\phi\circ\vec{S})}{\partial\theta} = \frac{\partial(\phi)}{\partial\theta}$ for the given function ϕ.

40. $\phi(x, y) = x + y + z$.

41. $\phi(x, y) = x^2 + y^2 + z^2$.

42. $\phi(x, y) = xyz$.

In Exercises 43–45, you are given the derivative of $\vec{F}$, $\vec{r}_0, \vec{F}(\vec{r}_0)$, and $\Delta\vec{r}$. Use Taylor's first order approximation for $\vec{F}$ at $\vec{r}_0$ to approximate $\vec{F}(\vec{r}_0 + \Delta\vec{r})$.

43. $D\vec{F}\big|_{\vec{r}_0} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, \quad \vec{r}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$

$$\vec{F}(\vec{r}_0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \Delta\vec{r} = \begin{pmatrix} 0.1 \\ -0.05 \end{pmatrix}.$$

44. $D\vec{F}\big|_{\vec{r}_0} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \vec{r}_0 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$

$$\vec{F}(\vec{r}_0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ and } \Delta\vec{r} = \begin{pmatrix} 0.1 \\ -0.1 \\ 0.02 \end{pmatrix}.$$

45. $D\vec{F}\big|_{\vec{r}_0} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \vec{r}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$

$$\vec{F}(\vec{r}_0) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \text{ and } \Delta\vec{r} = \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix}.$$

In Exercises 46–49, find the first order Taylor polynomial for the polar transformation $\vec{P}$ at $\vec{r}_0$.

46. $\vec{r}_0 = \left(2, \frac{\pi}{2}\right)$

47. $\vec{r}_0 = \left(-2, \frac{\pi}{3}\right)$

48. $\vec{r}_0 = \left(1, \frac{5\pi}{3}\right)$

49. $\vec{r}_0 = \left(-1, \frac{\pi}{6}\right)$

In Exercises 50–53, find the first order Taylor polynomial for the cylindrical transformation $\vec{C}_z$ at $\vec{r}_0$.

50. $\vec{r}_0 = \left(2, \frac{\pi}{2}, 3\right)$

51. $\vec{r}_0 = \left(-2, \frac{3\pi}{2}, -1\right)$

52. $\vec{r}_0 = \left(1, \frac{5\pi}{3}, 0\right)$

53. $\vec{r}_0 = \left(-1, \frac{\pi}{6}, 5\right)$

In Exercises 54–57, find the first order Taylor polynomial for the spherical transformation $\vec{S}$ at $\vec{r}_0$.

54. $\vec{r}_0 = \left(2, \frac{\pi}{2}, \frac{\pi}{4}\right)$

55. $\vec{r}_0 = \left(-2, \frac{3\pi}{2}, \frac{\pi}{4}\right)$

56. $\vec{r}_0 = \left(1, \frac{5\pi}{3}, \frac{\pi}{3}\right)$

57. $\vec{r}_0 = \left(-1, \frac{\pi}{6}, \frac{5\pi}{6}\right)$

58. Use the Taylor polynomial from Exercise 46 to approximate $\vec{P}(1.9, \frac{\pi}{2} + 0.2)$.

59. Use the Taylor polynomial from Exercise 53 to approximate $\vec{C}_z(\vec{r}_0 + \Delta\vec{r})$, where $\Delta\vec{r} = (-0.1, 0.2, 0.1)$.

60. Use the Taylor polynomial from Exercise 55 to approximate $\vec{S}(\vec{r}_0 + \Delta\vec{r})$, where $\Delta\vec{r} = (-0.1, 0.2, 0.1)$.