Introduction to Vector Calculus

Phillip Zenor
Department of Mathematics
Auburn University

Edward E. Slaminka
Department of Mathematics
Auburn University

Donald Thaxton
Department of Physics
Auburn University

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SECTION 4.3

1. $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ 3. $\begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$ 5. $\begin{pmatrix} 1 & 1 \end{pmatrix}$

7. $\begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ 9. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

11. $\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

13. $\begin{pmatrix} \cos\theta\sin\phi & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{pmatrix}$

15. $n = 2,\ m = 3$ 17. $n = 1,\ m = 2$

19. $n = 3,\ m = 3$

21. $\vec{s}_0 = (0, 2)$ $\vec{v}_0 = (-4, -1)$

23. $\vec{s}_0 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\vec{v}_0 = \left(\sqrt{2} + \frac{3\sqrt{2}}{2}, -\sqrt{2} + \frac{3\sqrt{2}}{2}\right)$

25. $\vec{s}_0 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 2\right)$
$\vec{v}_0 = \left(\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}, 0\right)$

27. $\vec{s}_0 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -2\right)$
$\vec{v}_0 = \left(\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$

29. $\vec{s}_0 = \left(0, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\vec{v}_0 = \left(0, -\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

31. $\vec{s}_0 = \left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{2}\right)$
$\vec{v}_0 = \left(\frac{\sqrt{2}-5\sqrt{6}}{4}, -\frac{1+\sqrt{3}}{2\sqrt{2}}, \frac{2+\sqrt{3}}{2}\right)$

33. $\begin{pmatrix} 2u\cos v & -u^2\sin v \\ 2u\sin v & u^2\cos v \\ 1 & 0 \end{pmatrix}.$

35. $\frac{\partial\phi}{\partial u} = (2u\cos v + \sin v)e^{u^2\cos v + u\sin v}$
$\frac{\partial\phi}{\partial v} = (-u^2\sin v + u\cos v)e^{u^2\cos v + u\sin v}$

37. $\frac{\partial\phi}{\partial\theta} = r(\cos\theta - \sin\theta)$

39. $\frac{\partial\phi}{\partial\theta} = -\tan\theta + 3r\cos\theta$

41. $\frac{\partial\phi}{\partial\theta} = 0$

43. $(2, 2.9)$ 45. $(1.9, 0.9, 2.9)$

47. $\begin{pmatrix} \frac{1}{2} & \sqrt{3} \\ \frac{\sqrt{3}}{2} & -1 \end{pmatrix}\begin{pmatrix} r+2 \\ \theta - \frac{\pi}{3} \end{pmatrix} + \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}$

49. $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} r+1 \\ \theta - \frac{\pi}{6} \end{pmatrix} - \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$

51. $\begin{pmatrix} 0 & -2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} r+2 \\ \theta - \frac{3\pi}{2} \\ z+1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

53. $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} r+2 \\ \theta - \frac{\pi}{6} \\ z-5 \end{pmatrix} + \begin{pmatrix} -\sqrt{3} \\ -\frac{1}{2} \\ 5 \end{pmatrix}$

55. $\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\sqrt{2} \\ -\frac{1}{\sqrt{2}} & 0 & \sqrt{2} \\ 0 & -2 & 0 \end{pmatrix}\begin{pmatrix} r+2 \\ \phi - \frac{3\pi}{2} \\ \theta - \frac{\pi}{4} \end{pmatrix} + \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$

57. $\begin{pmatrix} -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} r+1 \\ \phi - \frac{\pi}{6} \\ \theta - \frac{5\pi}{6} \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{3}}{4} \\ -\frac{1}{4} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$

59. $(-0.853, -0.723, 5.1)$

SECTION 4.4

1. $4\sqrt{5}$ 3. 3 5. 4

7. $4\sqrt{4\pi^2+1}$ 9. $4\sqrt{4\pi^2+1}$

SECTION 4.5

1. 6 3. $2\sqrt{3}$ 5. 0

7. 2 9. 0

11. $T(u, v) = (3u, 2v)$, Area: 6π

13. $T(u, v, w) = \left(\frac{u}{\sqrt{3}}, \frac{v}{2}, \frac{w}{\sqrt{2}}\right)$, Area: $\frac{2\pi}{3\sqrt{6}}$.

15. 0 17. $\frac{\sqrt{35}}{12}$ 19. 680 21. $\frac{952\pi}{15}$

SECTION 4.6

1. 2 3. 2 5. 2 7. 2

9. 0 11. $2\sqrt{2}$ 13. $r^2\sin(\phi)$.

15. 7 17. $14\sqrt{2}$ 19. $|-16u+48v+16|$

21. $32(-2u+6v+2w)^2\sin(u+v+2w-1)$

25. Sphere of radius 2 centered at the origin.

29. Nothing. Example: let $g(u,v)=(u,0,v)$ and consider $C_z\circ g$.

SECTION 5.1

1. $\dfrac{e^{2y}-e^{-y}}{y}$ 3. 0 5. $e^2+e^{-2}-2$

7. $\frac{37}{6}$ 9. 272 11. 256

13. $\frac{20}{3}$ 15. $-\frac{1}{8}\ln(3)$ 17. $\frac{4}{\pi}$

19. $\frac{2}{3}$ 21. 1

SECTION 5.2

1. $\vec{h}(s,t)=\begin{pmatrix} s \\ st \end{pmatrix},\ \begin{cases} 0\le t\le 1 \\ 1\le s\le 2 \end{cases}$
$J(\vec{h}(s,t))=s$

3. $\vec{h}(s,t)=\begin{pmatrix} s \\ t(e^s+s)-s \end{pmatrix},\ \begin{cases} 0\le t\le 1 \\ 0\le s\le 1 \end{cases}$
$J(\vec{h}(s,t))=e^s+s$

5. $\vec{h}(s,t)=\begin{pmatrix} s \\ t(s-\ln s)+\ln s \end{pmatrix},\ \begin{cases} 0\le t\le 1 \\ 1\le s\le 2 \end{cases}$
$J(\vec{h}(s,t))=s-\ln s$

7. $\vec{h}(s,t)=\begin{pmatrix} 1+s \\ -s-t \\ 2s+3 \end{pmatrix},\ \begin{cases} 0\le t\le 1 \\ 0\le s\le 1 \end{cases}$
$D(\vec{h}(s,t))=\begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 2 & 0 \end{pmatrix}$
$J(\vec{h}(s,t))=\sqrt{5}$

9. $\vec{h}(s,t)=\begin{pmatrix} 1-2s \\ 2-2s+t \end{pmatrix},\ \begin{cases} 0\le t\le 1 \\ 0\le s\le 1 \end{cases}$
$D(\vec{h}(s,t))=\begin{pmatrix} -2 & 0 \\ -2 & 1 \end{pmatrix}$
$J(\vec{h}(s,t))=2$

11. $\vec{h}(s,t)=\begin{pmatrix} 3s+4t-3 \\ -1+2t \\ s+3t \end{pmatrix},\ \begin{cases} 0\le t\le 1 \\ 0\le s\le 1 \end{cases}$
$D(\vec{h}(s,t))=\begin{pmatrix} 3 & 4 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$
$J(\vec{h}(s,t))=\sqrt{65}$

13. $\vec{h}(\phi,\theta)=\begin{pmatrix} 6\sin\phi\cos\theta-1 \\ 6\sin\phi\sin\theta+3 \\ 6\cos\phi \end{pmatrix},\ \begin{cases} 0\le \theta\le 2\pi \\ 0\le \phi\le \pi \end{cases}$
$D(\vec{h}(\phi,\theta))=\begin{pmatrix} 6\cos\phi\cos\theta & -6\sin\phi\sin\theta \\ 6\cos\phi\sin\theta & 6\sin\phi\cos\theta \\ -6\sin\phi & 0 \end{pmatrix},$
$J(\vec{h}(\phi,\theta))=36|\sin\phi|$

15. $\vec{h}(r,\theta)=\begin{pmatrix} r\sin\theta+1 \\ r\cos\theta+1 \end{pmatrix},\ \begin{cases} 0\le r\le 1 \\ 0\le \theta\le 2\pi \end{cases}$
$D(\vec{h}(s,t))=\begin{pmatrix} \sin\theta & r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix}$
$J(\vec{h}(s,t))=r$

17. $\vec{h}(r,\theta)=\begin{pmatrix} r\sin\theta+1 \\ r\cos\theta-3 \end{pmatrix},\ \begin{cases} 0\le r\le 2 \\ 0\le \theta\le 2\pi \end{cases}$
$D(\vec{h}(r,\theta))=\begin{pmatrix} \sin\theta & r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix}$
$J(\vec{h}(r,\theta))=r$

19. $\vec{h}(t,\theta)=\begin{pmatrix} t \\ \sin t\cos\theta \\ \sin t\sin\theta \end{pmatrix},\ \begin{cases} 0\le t\le \pi \\ 0\le \theta\le 2\pi \end{cases}$
$D(\vec{h}(t,\theta))=\begin{pmatrix} 1 & 0 \\ \cos t\cos\theta & -\sin t\sin\theta \\ \cos t\sin\theta & \sin t\cos\theta \end{pmatrix},$
$J(\vec{h}(t,\theta))=\sin t\sqrt{\cos^2(t)+1}$

21. $\vec{h}(t,\theta)=\begin{pmatrix} \cos t \\ 2\sin t\cos\theta \\ 2\sin t\sin\theta \end{pmatrix},\ \begin{cases} 0\le t\le \pi \\ 0\le \theta\le 2\pi \end{cases}$
$D(\vec{h}(t,\theta))=\begin{pmatrix} 0 & -\sin t \\ -2\sin t\sin\theta & 2\cos t\cos\theta \\ 2\sin t\cos\theta & 2\cos t\sin\theta \end{pmatrix},$
$J(\vec{h}(t,\theta))=2\sin t\sqrt{4\cos^2 t+\sin^2 t}$

23. $\vec{h}(t,\theta) = \begin{pmatrix} t\cos t \\ t\sin t\cos\theta \\ t\sin t\sin\theta \end{pmatrix}, \begin{cases} 0 \le t \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$

$D(\vec{h}(t,\theta)) =$
$\begin{pmatrix} 0 & \cos t - t\sin t \\ -t\sin t\sin\theta & (\sin t + t\cos t)\cos\theta \\ t\sin t\cos\theta & (\sin t + t\cos t)\sin\theta \end{pmatrix},$
$J(\vec{h}(t,\theta)) = t\sin t\sqrt{1+t^2}$

25. $\vec{h}(t,\theta) = \begin{pmatrix} \cos(t)-1 \\ (\sin(t)+2)\cos\theta \\ (\sin(t)+2)\sin\theta \end{pmatrix}, \begin{cases} 0 \le t \le 2\pi \\ 0 \le \theta \le 2\pi \end{cases}$

$D(\vec{h}(t,\theta)) =$
$\begin{pmatrix} 0 & -\sin t \\ -(\sin(t)+2)\sin\theta & \cos t\cos\theta \\ (\sin(t)+2)\cos\theta & \cos t\sin\theta \end{pmatrix},$
$J(\vec{h}(t,\theta)) = 2+\sin t$

27. $\vec{h}(t,\theta) = \begin{pmatrix} t\cos\theta \\ \cos t \\ t\sin\theta \end{pmatrix}, \begin{cases} 0 \le t \le 2\pi \\ 0 \le \theta \le 2\pi \end{cases}$

$D(\vec{h}(t,\theta)) = \begin{pmatrix} -t\sin\theta & \cos\theta \\ 0 & -\sin t \\ t\cos\theta & \sin\theta \end{pmatrix},$
$J(\vec{h}(t,\theta)) = t\sqrt{1+\sin^2 t}$

29. $\vec{h}(t,\theta) = \begin{pmatrix} \cos t\cos\theta \\ 2\sin t \\ \cos t\sin\theta \end{pmatrix}, \begin{cases} 0 \le t \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$

$D(\vec{h}(t,\theta)) = \begin{pmatrix} -\cos t\sin\theta & -\sin t\cos\theta \\ 0 & 2\cos t \\ \cos t\cos\theta & -\sin t\sin\theta \end{pmatrix},$
$J(\vec{h}(t,\theta)) = |\cos(t)|\sqrt{\sin^2 t + 4\cos^2 t}$

31. $\vec{h}(t,\theta) = \begin{pmatrix} t\cos t\cos\theta \\ t\sin t \\ t\cos t\sin\theta \end{pmatrix}, \begin{cases} \frac{\pi}{2} \le t \le \frac{3\pi}{2} \\ 0 \le \theta \le 2\pi \end{cases}$

$D(\vec{h}(t,\theta)) =$
$\begin{pmatrix} -t\cos t\sin\theta & (\cos t - t\sin t)\cos\theta \\ 0 & \sin t + t\cos t \\ t\cos t\cos\theta & (\cos t - t\sin t)\sin\theta \end{pmatrix},$
$J(\vec{h}(t,\theta)) = t|\cos t|\sqrt{1+t^2}$

37. $\vec{h}(s,\theta) = \begin{pmatrix} (\cos(s)+2)\cos\theta \\ \sin(s)-3) \\ (\cos(s)+2)\sin\theta \end{pmatrix},$

$0 \le s \le 2\pi, \quad 0 \le \theta \le 2\pi,$
$J(\vec{h}(s,\theta)) = 2+\cos s$

39. $\vec{h}(s,\theta) = \begin{pmatrix} \cos(s)+2 \\ (\sin(s)-8)\cos(\theta)+5 \\ (\sin(s)-8)\sin\theta \end{pmatrix},$

$0 \le s \le 2\pi, \quad 0 \le \theta \le 2\pi,$
$J(\vec{h}(s,\theta)) = 8-\sin s$

41. $\vec{h}(s,\theta) = \begin{pmatrix} \cos(s)+2 \\ (\sin(s)+2)\cos(\theta)-5 \\ (\sin(s)+2)\sin\theta \end{pmatrix},$

$0 \le s \le 2\pi, \quad 0 \le \theta \le 2\pi,$
$J(\vec{h}(s,\theta)) = 2+\sin s$

43. $\vec{h}(s,\theta) = \begin{pmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{pmatrix},$

$0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \theta \le \frac{\pi}{2}$

45. $\sin(\phi)\sqrt{b^2c^2\sin^2\phi\cos^2\theta + a^2c^2\sin^2\phi\sin^2\theta + a^2b^2\cos^2\phi}$

47. $(3\sqrt{2}, 2\sqrt{2}, 3)$

49. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\sqrt{3}}{4} & \frac{\pi\sqrt{3}}{6} \end{pmatrix} \begin{pmatrix} x-\frac{\pi}{3} \\ y-\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{\pi}{3} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

51. $y+\sqrt{3}z-2=0$

SECTION 5.3

1. $\frac{2\sqrt{6}}{3}$ 3. $\frac{\pi^2}{2}+\pi-\frac{2}{3}$ 5. $\frac{2048\pi}{5}$

7. $\frac{21\pi}{2}$ 9. $\frac{7}{12}$ 11. $\pi\sqrt{5}$

13. $-\frac{421}{40}$ 15. $\frac{11}{140}$

17. $\int_{-2}^{2}\int_{-2}^{2}\sqrt{1+4x^2+4y^2}\,dx\,dy$

19. $\int_{-2}^{2}\int_{-2}^{2}\sqrt{1+(4x^2+4y^2)\cos^2(x^2+y^2)}dx\,dy.$

21. $\int_{0}^{2}\int_{-2}^{1}\sqrt{1+(x^2+y^2)e^{2xy}}dx\,dy.$

25. $\int_{0}^{2\pi}\int_{-1}^{1}(x^2+1)\sqrt{1+4x^2}dx\,d\theta$

27. $\int_{0}^{2\pi}\int_{0}^{2\pi}(\sin(t)+3)\,dt\,d\theta = 12\pi^2.$

33. $\int_{0}^{2\pi}\int_{1}^{3}(x\cos(\theta)+x^3\sin(\theta))x\sqrt{1+4x^2}\,dx\,d\theta.$