

Introduction to Vector Calculus

Phillip Zenor
Department of Mathematics
Auburn University

Edward E. Slaminka
Department of Mathematics
Auburn University

Donald Thaxton
Department of Physics
Auburn University

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SECTION 4.3

$$1. \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \quad 3. \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \quad 5. (1 \ 1)$$

$$7. \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & -3 \\ 1 & 1 & 1 \end{pmatrix} \quad 9. \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$11. \begin{pmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$13. \begin{pmatrix} \cos \theta \sin \phi & r \cos \phi \cos \theta & -r \sin \phi \sin \theta \\ \sin \phi \sin \theta & r \cos \phi \sin \theta & r \sin \phi \cos \theta \\ \cos \phi & -r \sin \phi & 0 \end{pmatrix}$$

$$15. n = 2, m = 3 \quad 17. n = 1, m = 2$$

$$19. n = 3, m = 3$$

$$21. \vec{s}_0 = (0, 2) \quad \vec{v}_0 = (-4, -1)$$

$$23. \vec{s}_0 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \\ \vec{v}_0 = \left(\sqrt{2} + \frac{3\sqrt{2}}{2}, -\sqrt{2} + \frac{3\sqrt{2}}{2} \right)$$

$$25. \vec{s}_0 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 2 \right) \\ \vec{v}_0 = \left(\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}, 0 \right)$$

$$27. \vec{s}_0 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -2 \right) \\ \vec{v}_0 = \left(\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 \right)$$

$$29. \vec{s}_0 = \left(0, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \\ \vec{v}_0 = \left(0, -\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$$

$$31. \vec{s}_0 = \left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{2} \right) \\ \vec{v}_0 = \left(\frac{\sqrt{2}-5\sqrt{6}}{4}, -\frac{1+\sqrt{3}}{2\sqrt{2}}, \frac{2+\sqrt{3}}{2} \right)$$

$$33. \begin{pmatrix} 2u \cos v & -u^2 \sin v \\ 2u \sin v & u^2 \cos v \\ 1 & 0 \end{pmatrix}.$$

$$35. \frac{\partial \phi}{\partial u} = (2u \cos v + \sin v)e^{u^2 \cos v + u \sin v} \\ \frac{\partial \phi}{\partial v} = (-u^2 \sin v + u \cos v)e^{u^2 \cos v + u \sin v}$$

$$37. \frac{\partial \phi}{\partial \theta} = r(\cos \theta - \sin \theta)$$

$$39. \frac{\partial \phi}{\partial \theta} = -\tan \theta + 3r \cos \theta$$

$$41. \frac{\partial \phi}{\partial \theta} = 0$$

$$43. (2, 2.9) \quad 45. (1.9, 0.9, 2.9)$$

$$47. \begin{pmatrix} \frac{1}{2} & \sqrt{3} \\ \frac{\sqrt{3}}{2} & -1 \end{pmatrix} \begin{pmatrix} r+2 \\ \theta - \frac{\pi}{3} \end{pmatrix} + \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}$$

$$49. \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} r+1 \\ \theta - \frac{\pi}{6} \end{pmatrix} - \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$51. \begin{pmatrix} 0 & -2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r+2 \\ \theta - \frac{3\pi}{2} \\ z+1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$53. \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r+2 \\ \theta - \frac{\pi}{6} \\ z-5 \end{pmatrix} + \begin{pmatrix} -\sqrt{3} \\ -\frac{1}{2} \\ 5 \end{pmatrix}$$

$$55. \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\sqrt{2} \\ -\frac{1}{\sqrt{2}} & 0 & \sqrt{2} \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} r+2 \\ \phi - \frac{3\pi}{2} \\ \theta - \frac{\pi}{4} \end{pmatrix} \\ + \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$$

$$57. \begin{pmatrix} -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} r+1 \\ \phi - \frac{\pi}{6} \\ \theta - \frac{5\pi}{6} \end{pmatrix} \\ + \begin{pmatrix} \frac{\sqrt{3}}{4} \\ -\frac{1}{4} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$$

$$59. (-0.853, -0.723, 5.1)$$

SECTION 4.4

$$1. 4\sqrt{5} \quad 3. 3 \quad 5. 4$$

$$7. 4\sqrt{4\pi^2 + 1} \quad 9. 4\sqrt{4\pi^2 + 1}$$

SECTION 4.5

$$1. 6 \quad 3. 2\sqrt{3} \quad 5. 0$$

$$7. 2 \quad 9. 0$$

$$11. T(u, v) = (3u, 2v), \text{ Area: } 6\pi$$

$$13. T(u, v, w) = \left(\frac{u}{\sqrt{3}}, \frac{v}{2}, \frac{w}{\sqrt{2}} \right), \text{ Area: } \frac{2\pi}{3\sqrt{6}}$$

$$15. 0 \quad 17. \frac{\sqrt{35}}{12} \quad 19. 680 \quad 21. \frac{952\pi}{15}$$

SECTION 4.6

1. 2 3. 2 5. 2 7. 2
 9. 0 11. $2\sqrt{2}$ 13. $r^2 \sin(\phi)$.
 15. 7 17. $14\sqrt{2}$ 19. $|-16u + 48v + 16|$
 21. $32(-2u + 6v + 2w)^2 \sin(u + v + 2w - 1)$
 25. Sphere of radius 2 centered at the origin.
 29. Nothing. Example: let $g(u, v) = (u, 0, v)$ and consider $C_z \circ g$.

SECTION 5.1

1. $\frac{e^{2y} - e^{-y}}{y}$ 3. 0 5. $e^2 + e^{-2} - 2$
 7. $\frac{37}{6}$ 9. 272 11. 256
 13. $\frac{20}{3}$ 15. $-\frac{1}{8} \ln(3)$ 17. $\frac{4}{\pi}$
 19. $\frac{2}{3}$ 21. 1

SECTION 5.2

1. $\vec{h}(s, t) = \begin{pmatrix} s \\ st \end{pmatrix}, \begin{cases} 0 \leq t \leq 1 \\ 1 \leq s \leq 2 \end{cases}$
 $J(\vec{h}(s, t)) = s$
 3. $\vec{h}(s, t) = \begin{pmatrix} s \\ t(e^s + s) - s \end{pmatrix}, \begin{cases} 0 \leq t \leq 1 \\ 0 \leq s \leq 1 \end{cases}$
 $J(\vec{h}(s, t)) = e^s + s$
 5. $\vec{h}(s, t) = \begin{pmatrix} s \\ t(s - \ln s) + \ln s \end{pmatrix}, \begin{cases} 0 \leq t \leq 1 \\ 1 \leq s \leq 2 \end{cases}$
 $J(\vec{h}(s, t)) = s - \ln s$
 7. $\vec{h}(s, t) = \begin{pmatrix} 1 + s \\ -s - t \\ 2s + 3 \end{pmatrix}, \begin{cases} 0 \leq t \leq 1 \\ 0 \leq s \leq 1 \end{cases}$
 $D(\vec{h}(s, t)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 2 & 0 \end{pmatrix}$
 $J(\vec{h}(s, t)) = \sqrt{5}$
 9. $\vec{h}(s, t) = \begin{pmatrix} 1 - 2s \\ 2 - 2s + t \end{pmatrix}, \begin{cases} 0 \leq t \leq 1 \\ 0 \leq s \leq 1 \end{cases}$
 $D(\vec{h}(s, t)) = \begin{pmatrix} -2 & 0 \\ -2 & 1 \end{pmatrix}$
 $J(\vec{h}(s, t)) = 2$

11. $\vec{h}(s, t) = \begin{pmatrix} 3s + 4t - 3 \\ -1 + 2t \\ s + 3t \end{pmatrix}, \begin{cases} 0 \leq t \leq 1 \\ 0 \leq s \leq 1 \end{cases}$
 $D(\vec{h}(s, t)) = \begin{pmatrix} 3 & 4 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$
 $J(\vec{h}(s, t)) = \sqrt{65}$
 13. $\vec{h}(\phi, \theta) = \begin{pmatrix} 6 \sin \phi \cos \theta - 1 \\ 6 \sin \phi \sin \theta + 3 \\ 6 \cos \phi \end{pmatrix}, \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases}$
 $D(\vec{h}(\phi, \theta)) = \begin{pmatrix} 6 \cos \phi \cos \theta & -6 \sin \phi \sin \theta \\ 6 \cos \phi \sin \theta & 6 \sin \phi \cos \theta \\ -6 \sin \phi & 0 \end{pmatrix},$
 $J(\vec{h}(\phi, \theta)) = 36|\sin \phi|$
 15. $\vec{h}(r, \theta) = \begin{pmatrix} r \sin \theta + 1 \\ r \cos \theta + 1 \end{pmatrix}, \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$
 $D(\vec{h}(s, t)) = \begin{pmatrix} \sin \theta & r \cos \theta \\ \cos \theta & -r \sin \theta \end{pmatrix}$
 $J(\vec{h}(s, t)) = r$
 17. $\vec{h}(r, \theta) = \begin{pmatrix} r \sin \theta + 1 \\ r \cos \theta - 3 \end{pmatrix}, \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{cases}$
 $D(\vec{h}(r, \theta)) = \begin{pmatrix} \sin \theta & r \cos \theta \\ \cos \theta & -r \sin \theta \end{pmatrix}$
 $J(\vec{h}(r, \theta)) = r$
 19. $\vec{h}(t, \theta) = \begin{pmatrix} t \\ \sin t \cos \theta \\ \sin t \sin \theta \end{pmatrix}, \begin{cases} 0 \leq t \leq \pi \\ 0 \leq \theta \leq 2\pi \end{cases}$
 $D(\vec{h}(t, \theta)) = \begin{pmatrix} 1 & 0 \\ \cos t \cos \theta & -\sin t \sin \theta \\ \cos t \sin \theta & \sin t \cos \theta \end{pmatrix},$
 $J(\vec{h}(t, \theta)) = \sin t \sqrt{\cos^2(t) + 1}$
 21. $\vec{h}(t, \theta) = \begin{pmatrix} \cos t \\ 2 \sin t \cos \theta \\ 2 \sin t \sin \theta \end{pmatrix}, \begin{cases} 0 \leq t \leq \pi \\ 0 \leq \theta \leq 2\pi \end{cases}$
 $D(\vec{h}(t, \theta)) = \begin{pmatrix} 0 & -\sin t \\ -2 \sin t \sin \theta & 2 \cos t \cos \theta \\ 2 \sin t \cos \theta & 2 \cos t \sin \theta \end{pmatrix},$
 $J(\vec{h}(t, \theta)) = 2 \sin t \sqrt{4 \cos^2 t + \sin^2 t}$

$$23. \vec{h}(t, \theta) = \begin{pmatrix} t \cos t \\ t \sin t \cos \theta \\ t \sin t \sin \theta \end{pmatrix}, \begin{cases} 0 \leq t \leq \pi \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$D(\vec{h}(t, \theta)) = \begin{pmatrix} 0 & \cos t - t \sin t \\ -t \sin t \sin \theta & (\sin t + t \cos t) \cos \theta \\ t \sin t \cos \theta & (\sin t + t \cos t) \sin \theta \end{pmatrix},$$

$$J(\vec{h}(t, \theta)) = t \sin t \sqrt{1 + t^2}$$

$$25. \vec{h}(t, \theta) = \begin{pmatrix} \cos(t) - 1 \\ (\sin(t) + 2) \cos \theta \\ (\sin(t) + 2) \sin \theta \end{pmatrix}, \begin{cases} 0 \leq t \leq 2\pi \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$D(\vec{h}(t, \theta)) = \begin{pmatrix} 0 & -\sin t \\ -(\sin(t) + 2) \sin \theta & \cos t \cos \theta \\ (\sin(t) + 2) \cos \theta & \cos t \sin \theta \end{pmatrix},$$

$$J(\vec{h}(t, \theta)) = 2 + \sin t$$

$$27. \vec{h}(t, \theta) = \begin{pmatrix} t \cos \theta \\ \cos t \\ t \sin \theta \end{pmatrix}, \begin{cases} 0 \leq t \leq 2\pi \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$D(\vec{h}(t, \theta)) = \begin{pmatrix} -t \sin \theta & \cos \theta \\ 0 & -\sin t \\ t \cos \theta & \sin \theta \end{pmatrix},$$

$$J(\vec{h}(t, \theta)) = t \sqrt{1 + \sin^2 t}$$

$$29. \vec{h}(t, \theta) = \begin{pmatrix} \cos t \cos \theta \\ 2 \sin t \\ \cos t \sin \theta \end{pmatrix}, \begin{cases} 0 \leq t \leq \pi \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$D(\vec{h}(t, \theta)) = \begin{pmatrix} -\cos t \sin \theta & -\sin t \cos \theta \\ 0 & 2 \cos t \\ \cos t \cos \theta & -\sin t \sin \theta \end{pmatrix},$$

$$J(\vec{h}(t, \theta)) = |\cos(t)| \sqrt{\sin^2 t + 4 \cos^2 t}$$

$$31. \vec{h}(t, \theta) = \begin{pmatrix} t \cos t \cos \theta \\ t \sin t \\ t \cos t \sin \theta \end{pmatrix}, \begin{cases} \frac{\pi}{2} \leq t \leq \frac{3\pi}{2} \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$D(\vec{h}(t, \theta)) = \begin{pmatrix} -t \cos t \sin \theta & (\cos t - t \sin t) \cos \theta \\ 0 & \sin t + t \cos t \\ t \cos t \cos \theta & (\cos t - t \sin t) \sin \theta \end{pmatrix},$$

$$J(\vec{h}(t, \theta)) = t |\cos t| \sqrt{1 + t^2}$$

$$37. \vec{h}(s, \theta) = \begin{pmatrix} (\cos(s) + 2) \cos \theta \\ \sin(s) - 3 \\ (\cos(s) + 2) \sin \theta \end{pmatrix},$$

$$0 \leq s \leq 2\pi, \quad 0 \leq \theta \leq 2\pi,$$

$$J(\vec{h}(s, \theta)) = 2 + \cos s$$

$$39. \vec{h}(s, \theta) = \begin{pmatrix} \cos(s) + 2 \\ (\sin(s) - 8) \cos(\theta) + 5 \\ (\sin(s) - 8) \sin \theta \end{pmatrix},$$

$$0 \leq s \leq 2\pi, \quad 0 \leq \theta \leq 2\pi,$$

$$J(\vec{h}(s, \theta)) = 8 - \sin s$$

$$41. \vec{h}(s, \theta) = \begin{pmatrix} \cos(s) + 2 \\ (\sin(s) + 2) \cos(\theta) - 5 \\ (\sin(s) + 2) \sin \theta \end{pmatrix},$$

$$0 \leq s \leq 2\pi, \quad 0 \leq \theta \leq 2\pi,$$

$$J(\vec{h}(s, \theta)) = 2 + \sin s$$

$$43. \vec{h}(s, \theta) = \begin{pmatrix} \cos \theta \sin \phi \\ \sin \theta \sin \phi \\ \cos \phi \end{pmatrix},$$

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}$$

$$45. \frac{\sin(\phi) \sqrt{b^2 c^2 \sin^2 \phi \cos^2 \theta + a^2 c^2 \sin^2 \phi \sin^2 \theta}}{+ a^2 b^2 \cos^2 \phi}$$

$$47. (3\sqrt{2}, 2\sqrt{2}, 3)$$

$$49. \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\sqrt{3}}{4} & \frac{\pi\sqrt{3}}{6} \end{pmatrix} \begin{pmatrix} x - \frac{\pi}{3} \\ y - \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{\pi}{3} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$51. y + \sqrt{3}z - 2 = 0$$

SECTION 5.3

$$1. \frac{2\sqrt{6}}{3} \quad 3. \frac{\pi^2}{2} + \pi - \frac{2}{3} \quad 5. \frac{2048\pi}{5}$$

$$7. \frac{21\pi}{2} \quad 9. \frac{7}{12} \quad 11. \pi\sqrt{5}$$

$$13. -\frac{421}{40} \quad 15. \frac{11}{140}$$

$$17. \int_{-2}^2 \int_{-2}^2 \sqrt{1 + 4x^2 + 4y^2} dx dy$$

$$19. \int_{-2}^2 \int_{-2}^2 \sqrt{1 + (4x^2 + 4y^2) \cos^2(x^2 + y^2)} dx dy.$$

$$21. \int_0^2 \int_{-2}^1 \sqrt{1 + (x^2 + y^2) e^{2xy}} dx dy.$$

$$25. \int_0^{2\pi} \int_{-1}^1 (x^2 + 1) \sqrt{1 + 4x^2} dx d\theta$$

$$27. \int_0^{2\pi} \int_0^{2\pi} (\sin(t) + 3) dt d\theta = 12\pi^2.$$

$$33. \int_0^{2\pi} \int_1^3 (x \cos(\theta) + x^3 \sin(\theta)) x \sqrt{1 + 4x^2} dx d\theta.$$