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ON THE COSET PATTERN MATRICES AND MINIMUM M-ABERRATION OF 2^{n-p} DESIGNS

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Abstract: The coset pattern matrix (CPM) is formally defined as an elaborate characterization of the aliasing patterns of a fractional factorial design. The possibility of using CPM to check design isomorphism is investigated. Despite containing much information about effect aliasing, the CPM fails to determine a design uniquely. We report and discuss small nonisomorphic designs that have equivalent coset pattern matrices. These examples imply that the aliasing property and the combinatorial structure of a design depend on each other in a complex manner. Based on CPM, a new optimality criterion called the minimum M-aberration criterion is proposed to rank-order designs. Its connections with other existing optimality criteria are discussed.

Key words and phrases: Coset pattern matrix, Fractional factorial design, Isomorphism, Letter pattern matrix, Minimum M-aberration.

1. Introduction

The 2^{n-p} fractional factorial designs are among the most popular experimental plans in practice. For given n and p, how to construct/select the "best" 2^{n-p} design has been the central issue in the study of these designs. It is known that a 2^{n-p} design d is determined by its defining contrast subgroup, denoted by G, which consists of factorial effects aliased with the grand mean. Traditionally, the wordlength pattern $W_0 = (A_{01}, \ldots, A_{0n})$ is used to characterize the aliasing patterns of d, where A_{0i} is the number of effects of order i in G. Minimum aberration (MA) designs, which sequentially minimize A_{0i} for $1 \leq i \leq n$, are regarded as optimal (Fries and Hunter (1980)). In the past decade, significant progress has been achieved in understanding the MA designs. Readers are referred to Wu and Hamada (2000) for an up-to-date and comprehensive account.

Lately, much attention has been given to the construction of "optimal" 2^{n-p} designs for experiments in which factors are of different types or some pre-specified effects require special accommodations. Typical examples of the former case are robust parameter design experiments (RPDEs) involving at least two types of factors: control factors and noise factors, split-plot experiments involving whole-plot factors and subplot factors, and block experiments involving blocking factors and treatment factors. An example of the latter case is an experiment in which some two-factor interactions (2f.i.'s) are presumably important and need to be estimated. The wordlength pattern W_0 is not sufficient to discriminate and select 2^{n-p} designs in these two cases. Wu and Zhu (2003) introduced the wordtype pattern as an extension of W_0 for constructing optimal single arrays for RPDEs. Ke and Tang (2003) proposed the minimum N-aberration criterion for selecting optimal designs in the second case. In both approaches, more detailed aliasing patterns, which are not captured by W_0, were implicitly used. In addition, W_0 is not sufficient to study other properties of 2^{n-p} designs such as the number of clear effects. Wu and Wu (2002) classified the defining words of length 4 into various types so as to characterize designs with maximum numbers of clear 2f.i.'s. Hence, in order to construct optimal designs for complex experiments, an elaborate characterization of the aliasing patterns in a 2^{n-p} design is necessary and critical. In the literature, some possible characterizations such as the letter pattern have already been proposed and discussed.

A related issue is the isomorphism between 2^{n-p} designs. Due to the lack of construction methods, optimal 2^{n-p} designs are traditionally obtained through intensive computer search. Much computing is spent on checking whether two designs are isomorphic to each other or not. Draper and Mitchell (1967, 1968, 1970) proposed using the wordlength pattern and letter pattern comparison to discard isomorphic designs. They conjectured that two designs with equivalent letter patterns are isomorphic. Chen and Lin (1991) constructed a counterexample to disprove the conjecture. Note that both the wordlength pattern and the letter pattern are invariant under isomorphism. Whether there exists a complete set of numerical invariants that can uniquely determine a 2^{n-p} design has long been an outstanding issue.

In this paper, we consider the defining contrast subgroup and all its cosets. Similar to the wordlength pattern, the wordlength pattern of a coset, called the coset pattern, can be defined. All the coset patterns form a coset pattern matrix, which can be viewed as an extension of the letter patterns. The rest of the paper is organized as follows. In Section 2, notation and basic definitions are given, the relationships between wordlength pattern, letter pattern matrix and coset pattern matrix are established and discussed. In Section 3, four pairs of small nonisomorphic 2^{n-p} designs with equivalent letter pattern matrices and nonequivalent coset pattern matrices are presented. In Section 4, small nonisomorphic 2^{n-p} designs with equivalent coset pattern matrices are reported. In Section 5, the minimum M-aberration criterion is proposed and used to rank-order 2^{n-p} designs. Its connections with other existing criteria are discussed. Concluding remarks are given in Section 6. All the designs considered in this paper are of resolution III or higher.

2. Notation and Basic Definitions

We use letters 1, ..., n to denote the factors included in an experiment. A word represents a factorial effect and is denoted by the juxtaposition of the involved letters from the smallest to the largest. The number of letters in a word is called the wordlength. A 2^{n-p} design with n factors and 2^m runs ($m = n-p$), denoted by d, is determined by p independent defining words, which generate the defining contrast subgroup G. $W_0(d)$ is the wordlength pattern of d. For a fixed letter i, let l_{ij} be the number of words in G that involve i and have length j. Then the vector $L_i(d) = (l_{i1}, \ldots, l_{in})$ is called the letter pattern of i, and the n by n matrix $L(d) = (L_1(d)^t, \ldots L_n(d)^t)^t = (l_{ij})$, where $L_i(d)^t$ is the transpose of $L_i(d)$, is called the *letter pattern matrix* (LPM) (Draper and Mitchell (1970)). Two designs d and $\tilde{d}$ are said to have equivalent LPMs if there exists a row permutation P such that $PL(d) = L(\tilde{d})$. Note that to show $L(d)$ and $L(\tilde{d})$ are equivalent, we need verify (i) any row of $L(d)$ is also a row of $L(\tilde{d})$ and vice versa, and (ii) any common row has the same frequency in $L(d)$ and $L(\tilde{d})$. It is clear that $W_0(d) = (\sum_{i=1}^n l_{i1}, \ldots, j^{-1}\sum_{i=1}^n l_{ij}, \ldots, n^{-1}\sum_{i=1}^n l_{in})$. Therefore two designs with equivalent LPMs necessarily have the same wordlength pattern. However, the converse is not true in general. The smallest counterexample includes the following two 2^{8-3} designs: $d'(126, 137, 23458)$ and $d''(126, 347, 1358)$, where 1, 2, 3, 4 and 5 are the independent letters and the words within parentheses are independent defining words. d' and d'' have the same wordlength pattern $W_0 = (0, 0, 2, 1, 2, 2, 0, 0)$. It is easy to verify that $L_1(d') = (0, 0, 2, 0, 0, 2, 0, 0)$, and no letter in d'' has the same pattern as $L_1(d')$. Hence $L(d')$ and $L(d'')$ are not equivalent.

There are 2^n factorial effects (including the grand mean) in a 2^{n-p} design. They form an Abelian group, denoted by S. The defining contrast subgroup G consists of the effects aliased with the grand mean, which is denoted by I. 2^{n-p} cosets can be generated by G and they form a partition of S. In order to derive unique representations for the cosets, we define an order, labeled by $\lhd$, among the effects. Suppose $i_1 \cdots i_k$ and $j_1 \cdots j_l$ are two effects, $i_1 \cdots i_k$ is said to be 'smaller' than $j_1 \cdots j_l$, written as $i_1 \cdots i_k \lhd j_1 \cdots j_l$, if $k < l$ or if $k = l$ and $i_1 \cdots i_k$ should be listed ahead of $j_1 \cdots j_l$ lexicographically. For a given coset, the *coset leader* is defined to be the 'smallest' effect in the coset. If a coset has $i_1 \cdots i_k$ as its coset leader, it is represented by $i_1 \cdots i_k G$. It is clear that $\lhd$ can also be applied to the coset leaders, so the cosets can be rank-ordered from the 'smallest' to the 'largest' with the 'smallest' coset receiving rank 0 and the 'largest' receiving rank $2^{n-p} - 1$. The rank of the coset $i_1 \cdots i_k G$ is denoted by $r(i_1 \cdots i_k G)$. In fact, the 'smallest' coset is G itself with I as its coset leader. So $r(IG) = 0$. For convenience, we use $0G$ instead of IG for its representation. If a coset has a main effect as its coset leader, it is called an m.e. coset; Similarly we can define two-factor interaction cosets (or briefly, 2f.i. cosets), three-factor interaction cosets (or briefly, 3f.i. cosets), and so on. The first $n+1$ cosets are $0G$, $1G$, $\ldots$ and nG, followed by the 2f.i. cosets and other higher order cosets. We use $\mathcal{F}_k$ to denote the collection of all the kf.i. cosets and $r(\mathcal{F}_k)$ the collection of ranks of the cosets in $\mathcal{F}_k$.

The wordlength pattern W_0 is the frequency vector of the possible wordlengths in G, or $0G$. Similar frequency vectors can be defined for the other cosets. Suppose $i_1 \cdots i_l G$ is the kth coset, that is, $r(i_1 \cdots i_l G) = k$, with $0 \le k \le 2^{n-p}-1$. Let A_{kj} be the number of words of length j in $i_1 \cdots i_l G$. The vector $W_k = (A_{k1}, \ldots A_{kn})$ is the wordlength pattern, or *coset pattern*, of $i_1 \cdots i_l G$. The $2^{n-p} \times n$ matrix $A(d) = (W_0^t, \ldots, W_{2^{n-p}-1}^t)^t = (A_{kj})$ is called the *coset pattern matrix* (CPM). Note that the first row of $A(d)$ is exactly the wordlength pattern W_0, and the next n rows are the coset patterns of the m.e. cosets, then the 2f.i. coset patterns follow and so on. Suppose $j_1 \cdots j_l$ is an arbitrary effect. We use $W_{[j_1 \cdots j_l]}$ to denote the pattern of the coset to which $j_1 \cdots j_l$ belongs. Clearly, if $j_1 \cdots j_l$ and $i_1 \cdots i_h$ belong to the same coset, then $W_{[j_1 \cdots j_l]} = W_{[i_1 \cdots i_h]}$. Two designs d and $\tilde{d}$ are said to have equivalent coset pattern matrices if there exists a row permutation P such that $PA(d) = A(\tilde{d})$. To show $A(d)$ and $A(\tilde{d})$ are equivalent, the same procedure as used for checking the equivalence of LPMs can be employed.

It can be verified that $\sum_{j=1}^{n} A_{kj} = 2^p - 1$ when $k=0$, is 2^p otherwise, and $\sum_{k=0}^{2^m-1} A_{kj} = \binom{n}{j}$ for $1 \le j \le n$. The next proposition presents the relationship between the wordlength pattern W_0, the letter patterns and the m.e. coset patterns. Note that $A_{01} = A_{02} = 0$ in 2^{n-p} designs with resolution III or higher. For convenience, we define $A_{00} = 1$, and $l_{i(n+1)} = 0$ for $1 \le i \le n$.

Proposition 1. *For a given letter i with $1 \le i \le n$, its letter pattern $L_i = (l_{i1}, \ldots, l_{in})$ and coset pattern $W_i = (A_{i1}, \ldots, A_{in})$ are related to each other through the following: for $1 \le j \le n$,*

$$A_{ij} = A_{0(j-1)} - l_{i(j-1)} + l_{i(j+1)}. \tag{2.1}$$

Proof. The effects of length j in iG that contain i have an one-to-one correspondence to the effects of length $j-1$ in $0G$ that do not contain i. Their total number is $A_{0(j-1)} - l_{i(j-1)}$. The effects of length j in iG that do not contain i have an one-to-one correspondence to the effects of length $j+1$ in $0G$ that contain i. Their total number is $l_{i(j+1)}$. An effect of length j in iG either contains i or does not contain i, hence (2.1) follows.

Two 2^{n-p} designs d and $\tilde{d}$ are said to be *isomorphic* if there exists a relabeling τ of $1, \ldots, n$ of d, that is, $\tau : (1, \ldots, n) \to (\tau(1), \ldots, \tau(n))$, such that $G(\tau(d)) \equiv \{\tau(i_1), \ldots, \tau(i_k) : i_1, \ldots, i_k \in G(d)\} = G(\tilde{d})$, where $\tau(d)$ denotes the relabeled design. It is clear that isomorphic designs necessarily have equivalent CPMs, equivalent LPMs and the same wordlength patterns. Based on complete computer search, we have found that all 8- and 16-run nonisomorphic designs can be distinguished by their wordlength patterns, all 8-, 16-, and 32-run nonisomorphic designs can be distinguished by their CPMs. In Sections 3 and 4, we report all the 32-run designs with equivalent LPMs but nonequivalent CPMs, and some nonisomorphic 64-run designs with equivalent CPMs. These examples demonstrate the difference between the combinatorial and aliasing properties of 2^{n-p} designs.

3. Small Nonisomorphic Designs with Equivalent LPMs

Chen and Lin (1991) reported two nonisomorphic 2^{31-15}_{VII} designs with equivalent LPMs, each of which involves 31 factors and $2^{16} = 65536$ runs. In fact, much smaller counterexamples exist. Using complete computer search, we have identified four pairs of nonisomorphic 32-run designs with equivalent LPMs. The other 32-run designs can be uniquely determined by their LPMs. The first pair includes the following two 2^{12-7} designs: (1) d_{11} (126, 137, 238, 12349, 1235t_0, 45t_1, 12345t_2) and d_{12} (126, 137, 248, 349, 125t_0, 135t_1, 145t_2), where t_0, t_1 and t_2 represent the letters 10, 11 and 12, respectively. All the letters in d_{11} and d_{12} have exactly the same letter pattern $(0, 0, 2, 5, 10, 16, 14, 10, 6, 0, 0, 1)$. Hence $L(d_{11})$ and $L(d_{12})$ are identical. However d_{11} and d_{12} are nonisomorphic. This can be seen by comparing their CPMs, $A(d_{11})$ and $A(d_{12})$. Because d_{11} and d_{12} have identical LPMs, their m.e. coset patterns are also identical (based on (2.1)) and equal to $(1, 2, 5, 16, 26, 28, 26, 16, 5, 2, 1, 0)$. Excluding the m.e. cosets and the defining contrast subgroup, there are 19 remaining cosets for both d_{11} and d_{12}. In d_{11}, all the 19 remaining cosets are 2f.i. cosets, while in d_{12}, the remaining 19 cosets consist of 18 2f.i. cosets and one 3f.i. coset. $A(d_{12})$ contains a 3f.i. coset pattern, but $A(d_{11})$ does not, so they are not equivalent. Hence d_{11} and d_{12} are not isomorphic.

The other three pairs are (2) d_{21} (126, 137, 238, 149, 234t_0, 1235t_1, 145t_2, 2345t_3) and d_{22} (126, 137, 148, 259, 35t_0, 1235t_1, 45t_2, 1245t_3), (3) d_{31} (126, 137, 238, 149, 234t_0, 25t_1, 1235t_2, 145t_3, 2345t_4) and d_{32} (126, 137, 148, 259, 35t_0, 1235t_1, 45t_2, 1245t_3, 1345t_4), and (4) d_{41} (126, 137, 238, 149, 234t_0, 25t_1, 135t_2, 45t_3, 1245t_4, 345t_5) and d_{42} (126, 137, 148, 259, 35t_0, 1235t_1, 45t_2, 1245t_3, 1345t_4, 12345t_5). Due to limited space, we omit similar discussions for the above three pairs. Readers can verify that all of them have equivalent LPMs but their CPMs are not equivalent. These examples show that designs with equivalent LPMs can be further discerned by their CPMs. An immediate question is whether CPMs can uniquely determine fractional factorial designs. The answer is negative, and there exist many designs that have equivalent CPMs but are nonisomorphic. Some small examples are presented in the next section.

4. Nonisomorphic Designs with Equivalent CPMs

Designs with 8-, 16- and 32-run can be uniquely determined by their CPMs, so can 64-run designs with less than 14 factors. Among the 64-run designs involving 14 letters, there are two special pairs. Each pair contains two nonisomorphic designs that have equivalent CPMs. Before presenting the examples, we state a proposition about a necessary relationship between the coset patterns of two isomorphic designs.

Proposition 2. *Suppose two 2^{n-p} designs d and $\tilde{d}$ are isomorphic. Then there exists a relabeling $\tau : (1, \ldots, n) \rightarrow (\tau(1), \ldots, \tau(n))$ such that*

$$W_{[i_1 \cdots i_k]}(d) = W_{[\tau(i_1) \cdots \tau(i_k)]}(\tilde{d}), \tag{4.1}$$

where $i_1 \cdots i_k$ is an arbitrary factorial effect involving $i_1, \ldots, i_k$.

Suppose the m.e. coset patterns of d and $\tilde{d}$ are $\{W_i(d)\}_{1 \le i \le n}$ and $\{W_j(\tilde{d})\}_{1 \le j \le n}$ respectively. To check whether d and $\tilde{d}$ are isomorphic, we only need to check those relabeling τ's that satisfy $W_i(d) = W_{\tau(i)}(\tilde{d})$ for $1 \le i \le n$. For example, if all the m.e. coset patterns of $\tilde{d}$ are different from each other, we have at most one candidate relabeling to examine.

Table 4.1. m.e. coset patterns of d_{11} and d_{12}.

d_{11}	*m.e.* coset pattern														d_{12}
W_1	1	3	1	14	34	50	58	40	29	19	5	2	0	0	W_1
W_2	1	2	6	9	31	56	52	46	31	14	6	1	1	0	W_2
W_3	1	1	6	12	31	54	52	44	31	17	6	0	1	0	W_6
W_4	1	1	6	13	30	50	56	50	25	13	10	1	0	0	W_4
W_5	1	2	1	19	34	40	58	50	29	14	5	3	0	0	W_5
W_6	1	1	6	14	31	46	52	56	31	9	6	2	1	0	W_3
W_7	1	1	6	13	30	50	56	50	25	13	10	1	0	0	W_{t_0}
W_8	1	1	4	13	38	50	44	50	33	13	8	1	0	0	W_{t_3}
W_9	1	1	5	12	34	54	50	44	29	17	9	0	0	0	W_9
W_{t_0}	1	1	5	13	35	50	46	50	35	13	5	1	1	0	W_7
W_{t_1}	1	0	6	17	31	44	52	54	31	12	6	1	1	0	W_{t_1}
W_{t_2}	1	0	6	16	30	48	56	48	25	16	10	0	0	0	W_{t_2}
W_{t_3}	1	1	5	13	35	50	46	50	35	13	5	1	1	0	W_8
W_{t_4}	1	0	5	17	34	44	50	54	29	12	9	1	0	0	W_{t_4}

The first pair contains the following two 2^{14-8} designs: d_{11} (127, 138, 149, $25t_0$, $236t_1$, $346t_2$, $56t_3$, $2456t_4$) and d_{12} (127, 138, 149, $25t_0$, $236t_1$, $346t_2$, $56t_3$,

$2345t_4$). Both $A(d_{11})$ and $A(d_{12})$ are 64 by 14 matrices, which are equivalent. Due to the large size of $A(d_{11})$ and $A(d_{12})$, only the m.e. coset patterns are given in Table 4.1. Notice that both d_{11} and d_{12} have the same m.e. coset patterns. Excluding $W_4(d_{11}) = W_7(d_{11}) = W_4(d_{12}) = W_{t_0}(d_{12})$ and $W_{t_0}(d_{11}) = W_{t_3}(d_{11}) = W_7(d_{12}) = W_8(d_{12})$, there exists a one-to-one correspondence between $\{W_i(d_{11})\}_{i\neq 4,7,t_0,t_3}$ and $\{W_j(d_{12})\}_{j\neq 4,7,8,t_0}$. According to Proposition 2, to check whether d_{11} and d_{12} are isomorphic, we only need to consider the following four relabelings:

$$\tau_1 : (1,\ldots,t_4) \to (\tau_1(1),\ldots,\tau_1(t_4)) = (1,2,6,4,5,3,t_0,t_3,9,7,t_1,t_2,8,t_4),$$
$$\tau_2 : (1,\ldots,t_4) \to (\tau_2(1),\ldots,\tau_2(t_4)) = (1,2,6,t_0,5,3,4,t_3,9,7,t_1,t_2,8,t_4),$$
$$\tau_3 : (1,\ldots,t_4) \to (\tau_3(1),\ldots,\tau_3(t_4)) = (1,2,6,4,5,3,t_0,t_3,9,8,t_1,t_2,7,t_4),$$
$$\tau_4 : (1,\ldots,t_4) \to (\tau_4(1),\ldots,\tau_4(t_4)) = (1,2,6,t_0,5,3,4,t_3,9,8,t_1,t_2,7,t_4).$$

Assume that τ_1 is an isomorphic relabeling. Then $W_{[12]}(d_{11}) = W_{[\tau_1(1)\tau_1(2)]}(d_{12}) = W_{[12]}(d_{12})$. Note that 127 is a defining word for both d_{11} and d_{12}. Thus $12 = 7$ holds for d_{11} and d_{12}. Because 12 and 7 belong to the same coset in d_{11} as well as in d_{12}, we have $W_{[12]}(d_{11}) = W_{[7]}(d_{11})$ and $W_{[12]}(d_{12}) = W_{[7]}(d_{12})$. This implies that $W_7(d_{11}) = W_{[7]}(d_{11}) = W_{[7]}(d_{12}) = W_7(d_{12})$, which is false according to Table 4.1. This contradiction leads to the conclusion that τ_1 is not an isomorphic relabeling. Similarly, we can show that τ_2, τ_3 and τ_4 are not isomorphic mappings either. Hence, d_{11} and d_{12} are nonisomorphic.

The second pair consists of the following two 2^{14-8} designs: d_{21} (127, 138, 249, $34t_0$, $125t_1$, $136t_2$, $456t_3$, $23456t_4$) and d_{22} (127, 138, 239, $1234t_0$, $45t_1$, $12346t_2$, $156t_3$, $2456t_4$). In a similar way, readers can verify that d_{21} and d_{22} have equivalent CPMs but are not isomorphic. Based on complete computer search and comparison, there are 4579 nonisomorphic 2^{14-8} designs. Among these designs, only the two pairs of designs discussed above cannot be discriminated by their CPMs. There are 11635 nonisomorphic 2^{15-9} designs, among which there exist 22 pairs of designs with equivalent CPMs. They are listed in Table 4.2 with given defining words. Counterexamples for 64-run designs with more than 15 factors are currently under construction and will be reported elsewhere.

In the rest of this section, we introduce a stronger version of equivalence between CPMs. Recall that the equivalence between CPMs defined in Section 2 is not restricted by $\lhd$, the coset ordering scheme defined there. For clarity, we rename the equivalence as the weak equivalence. To check if two designs have weakly equivalent CPMs, one only needs to check if both CPM's share the same coset patterns and the same coset pattern frequencies, so it is computationally convenient. If $\lhd$ is taken into consideration, we only need focus on the permutations of m.e. coset patterns, because once the permutation of m.e. coset patterns is fixed, it automatically results in the permutation of higher order coset patterns. This leads to the definition of strong equivalence between CPMs.

Table 4.2. 2^{15-9} nonisomorphic designs with equivalent CPMs.

Pair number	Design 1
1	127, 138, 249, $35t_0$, $1245t_1$, $46t_2$, $156t_3$, $2356t_4$, $13456t_5$
2	127, 138, 249, $34t_0$, $125t_1$, $135t_2$, $146t_3$, $12356t_4$, $23456t_5$
3	127, 138, 249, $35t_0$, $45t_1$, $12345t_2$, $16t_3$, $1346t_4$, $2456t_5$
4	127, 138, 249, $34t_0$, $15t_1$, $1236t_2$, $246t_3$, $356t_4$, $1456t_5$
5	127, 138, 149, $234t_0$, $25t_1$, $345t_2$, $36t_3$, $56t_4$, $123456t_5$
6	127, 138, 249, $34t_0$, $15t_1$, $235t_2$, $126t_3$, $456t_4$, $13456t_5$
7	127, 138, 249, $34t_0$, $15t_1$, $245t_2$, $126t_3$, $356t_4$, $13456t_5$
8	127, 138, 249, $34t_0$, $15t_1$, $245t_2$, $1236t_3$, $1246t_4$, $123456t_5$
9	127, 138, 249, $34t_0$, $15t_1$, $245t_2$, $1236t_3$, $1246t_4$, $1356t_5$
10	127, 138, 239, $14t_0$, $15t_1$, $2345t_2$, $26t_3$, $346t_4$, $456t_5$
11	127, 138, 249, $34t_0$, $125t_1$, $135t_2$, $246t_3$, $356t_4$, $23456t_5$
12	127, 138, 249, $34t_0$, $15t_1$, $245t_2$, $346t_3$, $12356t_4$, $12456t_5$
13	127, 138, 149, $234t_0$, $25t_1$, $345t_2$, $36t_3$, $456t_4$, $12456t_5$
14	127, 138, 249, $34t_0$, $15t_1$, $1235t_2$, $246t_3$, $56t_4$, $3456t_5$
15	127, 138, 249, $34t_0$, $15t_1$, $245t_2$, $136t_3$, $12356t_4$, $2456t_5$
16	127, 138, 239, $14t_0$, $234t_1$, $145t_2$, $2346t_3$, $256t_4$, $13456t_5$
17	127, 138, 249, $34t_0$, $125t_1$, $135t_2$, $46t_3$, $2356t_4$, $23456t_5$
18	127, 138, 239, $14t_0$, $25t_1$, $45t_2$, $1236t_3$, $46t_4$, $3456t_5$
19	127, 138, 249, $34t_0$, $15t_1$, $45t_2$, $26t_3$, $136t_4$, $12356t_5$
20	127, 138, 239, $14t_0$, $25t_1$, $45t_2$, $136t_3$, $1256t_4$, $1456t_5$
21	127, 138, 239, $14t_0$, $25t_1$, $45t_2$, $1236t_3$, $46t_4$, $1256t_5$
22	127, 138, 239, $14t_0$, $24t_1$, $15t_2$, $235t_3$, $345t_4$, $13456t_5$
Pair number	**Design 2**
1	127, 138, 249, $35t_0$, $1245t_1$, $46t_2$, $2346t_3$, $156t_4$, $2356t_5$
2	127, 138, 239, $1234t_0$, $1235t_1$, $45t_2$, $146t_3$, $256t_4$, $12456t_5$
3	127, 138, 249, $35t_0$, $45t_1$, $12345t_2$, $16t_3$, $2356t_4$, $2456t_5$
4	127, 138, 249, $34t_0$, $125t_1$, $1345t_2$, $126t_3$, $1356t_4$, $3456t_5$
5	127, 138, 249, $34t_0$, $15t_1$, $2345t_2$, $2346t_3$, $56t_4$, $12356t_5$
6	127, 138, 239, $14t_0$, $1235t_1$, $1245t_2$, $246t_3$, $56t_4$, $13456t_5$
7	127, 138, 239, $14t_0$, $1235t_1$, $45t_2$, $246t_3$, $12356t_4$, $1456t_5$
8	127, 138, 249, $34t_0$, $125t_1$, $135t_2$, $26t_3$, $356t_4$, $1456t_5$
9	127, 138, 249, $34t_0$, $125t_1$, $135t_2$, $26t_3$, $356t_4$, $13456t_5$
10	127, 138, 239, $14t_0$, $15t_1$, $26t_2$, $346t_3$, $2356t_4$, $456t_5$
11	127, 138, 239, $14t_0$, $1235t_1$, $45t_2$, $246t_3$, $12356t_4$, $13456t_5$
12	127, 138, 239, $14t_0$, $25t_1$, $345t_2$, $1246t_3$, $12356t_4$, $3456t_5$
13	127, 138, 249, $34t_0$, $15t_1$, $2345t_2$, $246t_3$, $356t_4$, $12356t_5$
14	127, 138, 239, $14t_0$, $1235t_1$, $245t_2$, $46t_3$, $56t_4$, $123456t_5$
15	127, 138, 239, $14t_0$, $1235t_1$, $45t_2$, $146t_3$, $12356t_4$, $2456t_5$
16	127, 138, 249, $34t_0$, $1234t_1$, $15t_2$, $46t_3$, $2356t_4$, $123456t_5$
17	127, 138, 239, $14t_0$, $15t_1$, $2345t_2$, $246t_3$, $356t_4$, $23456t_5$
18	127, 138, 239, $14t_0$, $25t_1$, $45t_2$, $136t_3$, $246t_4$, $3456t_5$
19	127, 138, 249, $34t_0$, $15t_1$, $45t_2$, $26t_3$, $136t_4$, $2356t_5$
20	127, 138, 239, $14t_0$, $25t_1$, $45t_2$, $46t_3$, $1256t_4$, $356t_5$
21	127, 138, 239, $14t_0$, $25t_1$, $45t_2$, $136t_3$, $246t_4$, $1256t_5$
22	127, 138, 239, $14t_0$, $234t_1$, $15t_2$, $45t_3$, $245t_4$, $12356t_5$

Definition 4.1. Suppose the CPMs of two 2^{n-p} designs are $A(d)$ and $A(d')$ and the rows of A are ordered from the smallest to the largest by $\lhd$. $A(d)$ and $A(d')$ are said to be *strongly equivalent* if there exists a relabeling τ such that $A(\tau(d)) = A(d')$.

Note that having strongly equivalent CPMs is a necessary condition for two designs to be isomorphic. Computer-aided comparisons show that all the counterexamples given in this section have weakly equivalent CPMs but not strongly equivalent CPMs. Hence, they are not isomorphic. Strong equivalence is more powerful in discriminating designs than the weak equivalence. However there is a trade-off in that the effects in each cosets which have the same order as the coset leaders need be recorded. Incorporating the ordered coset patterns, on average, we can significantly speed up the isomorphism checking algorithm proposed in Chen, Sun and Wu (1993). For practical convenience, we only use m.e. cosets and 2f.i. cosets. Assume d_1 and d_2 are two designs with $A(d_1)$ and $A(d_2)$, and we record the 2f.i.'s in the 2f.i. cosets of d_1. Then we only need to consider the relabeling τ such that the ordered m.e. coset patterns and 2f.i. coset patterns of $\tau(d_1)$ match those of d_2. Most times, the number of these relabelings is extremely small and we can quickly determine whether d_1 and d_2 are isomorphic or not. In our search of all nonisomorphic designs, we observed that the needed amount of time has been dramatically reduced by partially checking the strong equivalence of CPMs. We further conjecture that designs with strongly equivalent CPMs are indeed isomorphic.

5. Minimum M-Aberration Criterion Based on CPMs

Fractional factorial designs are often rank-ordered by their wordlength patterns, and most existing optimality criteria such as the maximum resolution and minimum aberration criteria are also based on wordlength pattern. Because the CPM of a design contains much more information about effect aliasing than its wordlength pattern, it can be used to discriminate or rank-order designs more properly.

First, we briefly review an interpretation of the minimum aberration criterion used in Wu and Zhu (2003). If an effect of order i is aliased with another effect of order j, the aliasing between these two effects is said to be of type (i, j). Without loss of generality, we always assume that $i \leq j$. Different aliasing types imply different aliasing severities. If type (i, j) is considered to be more severe than type (i', j'), it is written as $(i, j) > (i', j')$, where $>$ means 'more severe than'. A possible scheme to rank-order all the aliasing types from the most severe to the least severe is

$$(i_1, j_1) > (i_2, j_2), \text{ if } i_1 + j_1 < i_2 + j_2; \text{ or if } i_1 + j_1 = i_2 + j_2 \text{ and } j_1 - i_1 < j_2 - i_2.$$

Let $N_{(i,j)}$ be the number of pairs of aliased effects of type (i,j) and $N = (N_{(1,2)}, N_{(2,2)}, N_{(1,3)}, \ldots)$, where the components of N are arranged in the order listed above. N represents the overall aliasing severity of a design, so it can be used to rank-order designs. In fact, there exists a one-to-one correspondence between N and the wordlength pattern W_0, and sequentially minimizing N is equivalent to sequentially minimizing W_0. Hence, N and W_0 are equivalent to each other.

Next, we refine $N_{(i,j)}$ based on CPMs and propose more sensitive criteria to discriminate designs. Suppose e_1 and e_2 are two effects of order i and j respectively, and they are aliased with each other. Then e_1 and e_2 must belong to the same coset, which we assume to be $i_1 \cdots i_k G$. So, $i_1 \ldots i_k$ is the 'smallest' effect aliased with both e_1 and e_2, and we claim that the aliasing between e_1 and e_2 is of type $(i,j)_k$. Note that $k \leq i$. In general, an aliasing type (i,j) can be further classified into i subtypes, which are $(i,j)_1$, $(i,j)_2, \ldots, (i,j)_i$. For example, type (1,2) has one subtype $(1,2)_1$, while type (2,2) has two subtypes: $(2,2)_1$ and $(2,2)_2$. The 2f.i.'s that form aliased pairs of the subtype $(2,2)_1$ must belong to some m.e. cosets, while the 2f.i.'s that form aliased pairs of the subtype $(2,2)_2$ must belong to some 2f.i. cosets. We need to further rank-order the subtypes according to their aliasing severities. Indisputedly, $(1,2)_1$ is the most severe. The next two subtypes are $(2,2)_1$ and $(2,2)_2$. The subtype $(2,2)_1$ concerns the aliasing between 2f.i.'s which are aliased with certain main effects, and the impact of these interactions on the main effects are taken into consideration by the subtype $(1,2)_1$. The subtype $(2,2)_2$ concerns the aliasing between 2f.i.'s which are not aliased with any main effects. Hence, we consider $(2,2)_2$ to be more severe than $(2,2)_1$ because the former indicates a design's capacity to accommodate 2f.i.'s conditioned on $(1,2)_1$. Following similar arguments, we have

$$(1,2)_1, (2,2)_2, (2,2)_1, (1,3)_1, (2,3)_2, (2,3)_1, (1,4)_1, (3,3)_3, (3,3)_2, (3,3)_1$$

in descending severity. Generally, for any two given subtypes $(i_1, j_1)_{k_1}$ and $(i_2, j_2)_{k_2}$, they can be rank-ordered by the following scheme:

$$(i_1, j_1)_{k_1} > (i_1, j_2)_{k_2} \text{ if } (i_1, j_1) > (i_2, j_2); \text{ or if } i_1 = i_2, j_1 = j_2 \text{ and } k_1 > k_2. \tag{5.1}$$

For a given subtype $(i,j)_k$, we define $M_{(i,j)_k}$ to be the number of pairs of aliased effects which are of the subtype $(i,j)_k$. $M_{(i,j)_k}$ can be calculated from the coset pattern matrix A as follows,

$$M_{(i,j)_k} = \begin{cases} \sum_{h \in r(\mathcal{F}_k)} \frac{1}{2} A_{hi}(A_{hi} - 1), & \text{if } i = j; \\ \sum_{h \in r(\mathcal{F}_k)} A_{hi} A_{hj}, & \text{if } i \neq j. \end{cases} \tag{5.2}$$

When $i = 1$, $M_{(1,j)_1}$ can also be determined by W_0, because $M_{(1,j)_1} = (j+1)A_{0,j+1} + (n-j+1)A_{0,j-1}$. Since $(i,j)_k$ are the subtypes of (i,j) with $1 \le k \le i$, it is easy to see $N_{(i,j)} = \sum_{k=1}^{i} M_{(i,j)_k}$. Using the ordering scheme for the aliasing subtypes above, we define

$$M = (M_{(1,2)_1}, M_{(2,2)_2}, M_{(2,2)_1}, M_{(1,3)_1}, M_{(2,3)_2}, M_{(2,3)_1}, M_{(1,4)_1}, M_{(3,3)_3}, M_{(3,3)_2}, M_{(3,3)_1}, \ldots) \tag{5.3}$$

and call M the *aliasing type pattern* of a design. Based on M, we define M-aberration and the minimum M-aberration criterion as follows.

Definition 5.1. Let $M(d_1)$ and $M(d_2)$ be the aliasing type patterns of two designs d_1 and d_2. Assume $M_{(i^0,j^0)_{k^0}}$ is the first component where $M(d_1)$ and $M(d_2)$ differ from each other. If $M_{(i^0,j^0)_{k^0}}(d_1) < M_{(i^0,j^0)_{k^0}}(d_2)$, d_1 is said to have less M-aberration than d_2. If there does not exist a design that has less M-aberration than d_1, then d_1 is said to have minimum M-aberration.

In order to distinguish the aberration based on W_0 and the M-aberration, we call the former the W_0-aberration. If two designs d_1 and d_2 have the same aliasing type pattern, that is, $M(d_1) = M(d_2)$, then $N(d_1) = N(d_2)$, which further implies that $W_0(d_1) = W_0(d_2)$. But the converse does not generally hold. Thus, M can be used to distinguish nonisomorphic designs that have the same W_0. For example, for the two 2^{8-3} designs d' and d'' discussed in Section 2, $W_0(d') = W_0(d'') = (0,0,2,1,2,2,0)$, $M(d') = (6,2,1,4,46,\ldots)$ and $M(d'') = (6,3,0,4,48,\ldots)$. d' and d'' cannot be discriminated by their W_0-aberration. Based on their aliasing type patterns, d' has less M-aberration and it should be rank-ordered ahead of d'', which is consistent with their rankings given in Chen, Sun and Wu (1993) based on their overall properties.

In rank-ordering designs with different wordlength patterns, using W_0-aberration and using M-aberration do not always lead to the same conclusions. There exist many cases where the ranks based on M-aberration are different from those based on W_0-aberration. After careful examination, we conclude that M-aberration gives more proper rankings than W_0-aberration in terms of the overall properties of a design. Recall that a m.e or a 2f.i. is said to be clear if it is not aliased with any other m.e.'s or 2f.i.'s. We have also observed that M-aberration favors designs with more clear effects than W_0-aberration. However, the minimum M-aberration criterion is not equivalent to selecting designs with the maximum number of clear effects, and it does not always rank-order a design with more clear effects ahead of another design with less clear effects. Due to limited space, only one example is discussed in detail in the following.

Example 5.1. Consider the following two 2^{14-8} designs: d_1 (1237, 1248, 1259, 2345t_0, 136t_1, 146t_2, 156t_3, 3456t_4) and d_2 (1237, 1248, 1259, 1345t_0, 2345t_1,

$136t_2$, $146t_3$, $12346t_4$), with $W_0(d_1) = (0,0,0,22,40,36,56,\ldots)$ and $W_0(d_2) = (0,0,0,22,40,41,48,\ldots)$. It is known that d_1 is the minimum W_0-aberration design and d_2 is the second best according to its W_0-aberration. The aliasing type patterns of d_1 and d_2 are $M(d_1) = (0,66,0,88,400,0,200,276,504,\ldots)$ and $M(d_2) = (0,66,0,88,400,0,200,264,554,\ldots)$. It is easy to see that d_2 has less M-aberration than d_1. In fact, d_2 is the minimum M-aberration design. A detailed comparison of d_1 and d_2 supports the conclusion based on M. For example, all the main effects in both d_1 and d_2 are clear, but d_1 has only eight clear 2f.i.'s while d_2 has 16 clear 2f.i.'s.

The aliasing type pattern M includes all possible aliasing types. When only certain effects or aliasing types are of interest, M can be much simplified. For example, if we only consider the aliasing types involving main effects, then M becomes $M_1 = (M_{(1,2)_1}, M_{(1,3)_1}, \ldots, M_{(1,n)_1})$, where n is the total number of factors. In fact, $M_{(1,j)_1} = \mathrm{tr}(C_j^t C_j)$ as defined in Tang and Deng (1999). If we only consider the aliasing types involving at least a main effect or a two-factor interaction, then M reduces to $M_2 = (M_{(i,j)_k})_{1\le i\le 2; 1\le k\le 2}$. Another way to simplify M is to assume that higher order effects are negligible.

We also want to mention the connection between M and the estimation capacity introduced in Cheng, Steinberg and Sun (1999). Let E_k $(1 \le k \le \binom{n}{2})$ be the number of estimable models consisting of all the main effects and k two-factor interactions. It is clear that $E_k = \sum_{\substack{i_1<\cdots<i_k \\ i_1,\ldots,i_k \in r(\mathcal{F}_2)}} \prod_{j=1}^k A_{i_j,2}$. By simple mathematical manipulation, we have $E_1 = n(n-1)/2 - M_{(1,2)_1}$ and $E_2 = E_1(E_1 + 1)/2 - M_{(2,2)_2}$. Since $M_{(1,2)_1}$ and $M_{(2,2)_2}$ are the first two terms in M, minimum M-aberration designs also have maximum estimation capacity in terms of E_1 and E_2. We have rank-ordered the 16-, 32- and 64-run (with resolution IV or higher) designs based on their aliasing type patterns. The complete tables can be requested from the authors.

6. Concluding Remarks

Compared to the wordlength pattern, the coset pattern matrix contains much more information regarding the aliasing properties and structure of a design. Hence, it can be used to derive more sensitive optimality criteria for selecting optimal designs. The minimum M-aberration criterion proposed in Section 5 is one of many possibilities. Furthermore, the coset pattern matrix provides detailed information for lower order effects of a design, hence it can be used to guide the selection of optimal designs for experiments where more than one type of factors is involved or some lower order effects require special consideration, such as RPDEs, split-plot designs and block designs. We will pursue further research along these lines in the future.

The isomorphism between designs can also be defined through their design matrices. Lately, Clark and Dean (2001) studied the Hamming distances between the rows of fractional factorial designs and all their projection designs and derived a criterion and an algorithm to verify if two designs are equivalent. Xu and Deng (2003) defined the p-dimensional K-value distribution and proposed a criterion called the moment aberration projection to rank and classify designs. The focuses of their studies are on nonregular designs in which explicit aliasing relations are not usually available, while our focus is on regular fractional factorial designs.

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