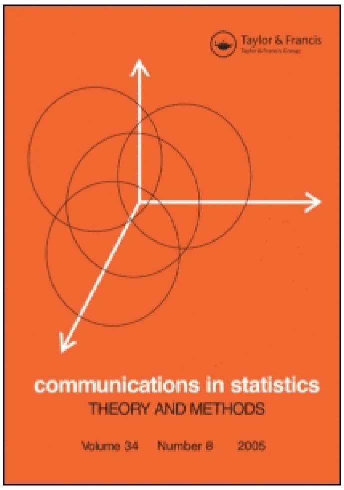

Communications in Statistics - Theory and Methods
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597238

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Online publication date: 10 June 2010

To cite this Article Zeng, Peng , Zhu, Yu and Phelps, Kevin T.(2010) 'Bounds on the Maximum Number of Factors in Designs with Two Distinct Groups of Factors', Communications in Statistics - Theory and Methods, 39: 12, 2302 — 2310
To link to this Article: DOI: 10.1080/03610920903039514
URL: http://dx.doi.org/10.1080/03610920903039514

Communications in Statistics—Theory and Methods, 39: 2302–2310, 2010

ISSN: 0361-0926 print/1532-415X online
DOI: 10.1080/03610920903039514

Bounds on the Maximum Number of Factors in Designs with Two Distinct Groups of Factors

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This article focuses on designs involving two distinct groups of factors. In particular, we assume that between-group interactions are more important than within-group interactions. Under this assumption, a new word-length pattern is proposed to characterize the aliasing severity of a design, and the concepts of resolution and aberration are defined accordingly. Furthermore, we have obtained various bounds on the maximum number of factors that a design with given resolution can accommodate.

Keywords Compromise plan; Fractional factorial design; Minimum aberration; Resolution; Word-length pattern; Word-type pattern.

Mathematics Subject Classification 62K15; 62K05.

1. Introduction

Regular two-level fractional factorial designs (or briefly designs) are widely used in scientific and industrial experiments to investigate factorial effects due to their economic run sizes. For given number of factors and run size, there are usually many distinct designs with different statistical properties. Various criteria have been proposed in the literature to rank and select designs for practical use. Two such examples are the maximum resolution criterion (Box and Hunter, 1961) and the minimum aberration criterion (Fries and Hunter, 1980). They have been further extended to other design scenarios such as split-plot designs (Bingham and Sitter, 1999), blocking designs (Chen and Cheng, 1999), and robust parameter designs (Wu and Zhu, 2003). Readers are referred to Wu and Hamada (2000) for a comprehensive account of two-level fractional factorial designs.

Recently, designs with multiple groups of factors have attracted much attention. In this article, we consider designs involving two groups of factors. The two groups

Received March 20, 2009; Accepted May 8, 2009
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are denoted by G_1 and G_2, and factors in these two groups are referred to as G_1- and G_2-factors, respectively. We assume that interactions involving three or more factors are negligible. Therefore, only main effects and two-factor interactions (2fis) need to be considered in design. Denote $G_i \times G_j$ as the collection of 2fis between one G_i-factor and one G_j-factor ($i, j = 1, 2$). In practice, prior knowledge may exist regarding the relative importance of the 2fis. For example, 2fis in $G_1 \times G_2$ are potentially significant, whereas 2fis in $G_1 \times G_1$ and $G_2 \times G_2$ are likely negligible. Four scenarios have been considered in the literature, which are (1) $G_1 \times G_1$; (2) $G_1 \times G_1$ and $G_2 \times G_2$; (3) $G_1 \times G_1$ and $G_1 \times G_2$; (4) $G_1 \times G_2$. In each scenario above, the listed type of 2fis is considered more important than those unlisted. Scenarios 1–3 were first discussed by Addelman (1962) and were recently studied in Constantine and Xue (1998a,b) and Ke, Tang, and Wu (2005); Scenario 4 was discussed in Sun (1993). All these designs are referred to as compromise plans in the literature.

This article generalizes Scenario 4 to include interactions involving more than two factors. Let $G_1 \times G_2$ denote the collection of interactions involving both G_1- and G_2-factors and $G_1 \times G_1$ (or $G_2 \times G_2$) the collection of interactions involving only G_1-factors (or G_2-factors). We assume that between-group interactions (i.e., those in $G_1 \times G_2$) are more important than within-group interactions (i.e., those in $G_1 \times G_1$ or $G_2 \times G_2$). Under this assumption, the ordinary word-length pattern is no longer appropriate for discriminating and ranking designs, because interactions of the same order may have different importance. Instead, we propose a new word-length pattern based on word-type patterns introduced in Wu and Zhu (2003), which takes into consideration the distinction between the two groups of factors. Furthermore, the concepts of resolution and aberration are redefined according to the new word-length pattern.

Given the resolution of a design, it is usually of practical and theoretical interest to know the maximum number of factors that the design can accommodate. Define $R(r, m)$ as the maximum number of factors that a design of m runs and resolution at least r can accommodate. It is known that $R(3, 2^k) = 2^k - 1$ and $R(4, 2^k) = 2^{k-1}$ for designs not involving multiple groups of factors; see Draper and Lin (1990) for more results. In this article, we further investigate and report various bounds on the maximum number of factors for designs involving two groups of factors. These results provide useful guidelines and information for practical application and further theoretical study of these designs.

The remaining part of this article is organized as follows. Section 2 introduces the new word-length pattern and redefines resolution and aberration. Section 3 discusses some bounds on the maximum number of factors under various scenarios. Section 4 ends this article with some concluding remarks. All the designs discussed in this article have resolution III or higher.

2. New Wordlength Pattern and Resolution

A 2^{n-p} design, which has n two-level factors and 2^k runs ($k = n - p$), is uniquely determined by p independent defining words, which are aliased with the gross mean. The defining contrast subgroup consists of all the defining words. Let W_i be the number of words of length i in the defining contrast subgroup. The word-length pattern is defined to be $W = (W_1, \ldots, W_n)$. A design has resolution i if the first nonzero element in W is W_i. A design D has less aberration than design D'

if $W_i(D) < W_i(D')$ and $W_j(D) = W_j(D')$ for $0 < j < i$. If no other design has less aberration than D, it is called *a minimum aberration* (MA) design.

When a 2^{n-p} design consists of two distinct groups of factors, say, G_1 with $|G_1| = n_1$ and G_2 with $|G_2| = n_2$, where $n = n_1 + n_2$ and $|S|$ denotes the cardinality of set S, the design is referred to as a $2^{(n_1+n_2)-p}$ design. Because of the presence of two groups, the ordinary word-length pattern is not appropriate to characterize the defining contrast subgroup any longer. The set of words of the same length is no longer homogenous and should be further distinguished. For example, a 2fi in $G_1 \times G_2$ should be distinguished from a 2fi in $G_1 \times G_1$, because the former is assumed to be more important than the latter and thus an alias involving the former should be considered more severe. Wu and Zhu (2003) categorized the set of words of length i into $i + 1$ subtypes as $(0, i)$, $(1, i - 1), \dots, (i, 0)$, where (t, s) denotes words involving t G_1-factors and s G_2-factors $(0 \le t, s \le i)$. Denote $W_{i,j}$ as the number of words of subtype (t, s) in the defining contrast subgroup. A word-type pattern is defined to be the collection of $W_{i,j}$, $W = (W_{i,j})_{0\le i\le n_1, 0\le j\le n_2}$, where $W_{i,j} = 0$, $0 \le i + j \le 2$, for designs of resolution III or higher. Clearly, word-type pattern determines word-length pattern by $W_k = \sum_{i+j=k} W_{i,j}$.

In order to introduce a new word-length pattern for $2^{(n_1+n_2)-p}$ designs, let us first discuss how the ordinary word-length pattern is obtained from the hierarchical ordering principle. The hierarchical ordering principle consists of two basic assumptions: (1) effects of lower order are more important than effects of higher order; (2) effects of the same order are of same importance. The second assumption implies that we should group the elements $W_{i,j}$ in the word-type pattern according to the value of $(i + j)$ and sum them within each group. By ordering them in the increasing order of $(i + j)$ as suggested by the first assumption, we reproduce the ordinary word-length pattern for 2^{n-p} designs,

$$(W_{3,0} + W_{2,1} + W_{1,2} + W_{0,3}, W_{4,0} + W_{3,1} + W_{2,2} + W_{1,3} + W_{0,4}, \dots),$$

where $W_{i,j} = 0$, $0 \le i + j \le 2$ are dropped from the vector for simplicity.

The above interpretation of word-length pattern sheds light on how to construct a similar vector for $2^{(n_1+n_2)-p}$ designs. The key is to group and combine $W_{i,j}$ according to the relative importance of interactions. Notice that the hierarchical ordering principle is not appropriate any more. Specifically, the first assumption remains valid, but the second assumption should be abandoned because between-group interactions are more important than within-group interactions. The words of length three consist of four subtypes: (0, 3), (1, 2), (2, 1), (3, 0). Because of the symmetry of G_1 and G_2, it is not necessary to distinguish subtypes (0, 3) and (3, 0). Similarly, we treat subtypes (1, 2) and (2, 1) the same. A word of subtype (0, 3) or (3, 0) induces the aliasing of a main effect and a within-group 2fi, whereas a word of subtype (1, 2) or (2, 1) induces the aliasing of a main effect and a 2fi in $G_1 \times G_2$ (see Remark 2.1 below). So the latter aliasing is regarded more severe than the former, and subtypes (1, 2) and (2, 1) are more important than subtypes (0, 3) and (3, 0). Consequently, the first two elements in the new word-length pattern, denoted by W^*, are

$$W^* = (W_{1,2} + W_{2,1}, W_{0,3} + W_{3,0}, \dots).$$

The words of length four consist of five subtypes: (0, 4), (1, 3), (2, 2), (3, 1), (4, 0). Following similar arguments, subtype (0, 4) is treated the same as subtype (4, 0), and both of them are less important than the other subtypes. Because subtype (2, 2) induces the aliasing between two 2fis in $G_1 \times G_2$ and subtype (1, 3) or (3, 1) induces the aliasing between a 2fi in $G_1 \times G_2$ and a 2fi in $G_2 \times G_2$ or $G_1 \times G_1$, we regard subtype (2, 2) as more important than subtype (1, 3) or (3, 1). Therefore, including the next three elements in W^* yields

$$W^* = (W_{1,2} + W_{2,1}, W_{3,0} + W_{0,3}, W_{2,2}, W_{1,3} + W_{3,1}, W_{4,0} + W_{0,4}, \dots).$$

Similarly, we are able to write out the remaining elements in W^*. Notice that the elements in W^* are of the form either $W_{i,i}$ or $W_{i,j} + W_{j,i}$. The element $W_{i_1,j_1} + W_{j_1,i_1}$ precedes $W_{i_2,j_2} + W_{j_2,i_2}$ if (i) $i_1 + j_1 < i_2 + j_2$; or (ii) $i_1 + j_1 = i_2 + j_2$ and $|i_1 - j_1| < |i_2 - j_2|$. This rule also applies to elements $W_{i,i}$.

Remark 2.1. A word can induce more than one aliasing relation. For example, suppose that ab_1b_2 is a word of subtype (1, 2), where $a \in G_1$ and $b_1, b_2 \in G_2$. This word induces the aliasing of a and b_1b_2, and also induces the aliasing of b_1 and ab_2. The latter aliasing is regarded more severe than the former, because it involves a 2fi in $G_1 \times G_2$, which is more important than 2fis in $G_2 \times G_2$. When comparing two words of different subtypes, it is only necessary to compare the most severe aliasing relation induced by these words.

We redefine resolution and aberration in terms of W^* as follows. A $2^{(n_1+n_2)-p}$ design has resolution $r.s$ if it has resolution r in the usual sense and $s = \min\{|i - j|; W_{i,j} \neq 0 \text{ and } i + j = r\}$. The integer part of the newly defined resolution is consistent with the resolution of usual 2^{n-p} designs if we ignore the difference between two groups of factors, and the fractional part is used to distinguish different situations when the integer parts are equal. Notice that the fractional part cannot be any value. For example, a design can only have resolution III.1, III.3, IV.0, IV.2, IV.4, etc. The definition of aberration resembles the ordinary one, and the only difference is to replace W by W^*.

3. Maximum Number of Factors

This section discusses the maximum number of factors in detail. For $2^{(n_1+n_2)-p}$ designs, we need to consider the presence of two groups of factors. Denote $R^*(r.s, n_1, m)$ as the maximum number of G_2-factors for a $2^{(n_1+n_2)-p}$ design that has resolution $r.s$, $n_1 G_1$-factors and m runs.

Projective geometry has been used as a convenient tool in the study of fractional factorial designs. Lemma 1 in Chen and Hedayat (1996) asserted that a 2^{n-p} design is uniquely determined by a set of n distinct points in $PG(n-p-1, 2)$, which denotes the projective geometry of dimension $n-p-1$ over $GF(2)$. Or, equivalently, a 2^{n-p} design induces a two-way partition $\{G_1, G_2\}$ of $PG(n-p-1, 2)$, where $|G_1| = n$ and $|G_2| = 2^{n-p} - 1 - n$. Similarly, a $2^{(n_1+n_2)-p}$ design induces a three-way partition $\{G_1, G_2, G_3\}$, where $|G_1| = n_1$, $|G_2| = n_2$, and $|G_3| = n_3 = 2^{n_1+n_2-p} - 1 - n_1 - n_2$. $R^*(r.s, n_1, m)$ is the maximum of $|G_2|$.

For ease of presentation, we need to introduce some additional notations. For any $a, b \in PG(k-1, 2)$, their interaction is denoted by $a + b$. Three points a, b, and c form a line if any point is the interaction of the other two. For any subset $S \subset PG(k-1, 2)$, define $a + S = \{a + s : s \in S\}$. For any two subsets $S_1, S_2 \subset PG(k-1, 2)$, define $S_1 + S_2 = \{a + b : a \in S_1, b \in S_2\}$.

$R^*(3.1, n_1, 2^k)$ and $R^*(4.0, n_1, 2^k)$ can be easily obtained from $R(3, 2^k)$ and $R(4, 2^k)$, as stated in Theorem 3.1, because a $2^{(n_1+n_2)-p}$ design is also a 2^{n-p} design.

Theorem 3.1. *$R^*(3.1, n_1, 2^k) = 2^k - 1 - n_1$ and $R^*(4.0, n_1, 2^k) = 2^{k-1} - n_1$.*

Proof. A $2^{(n_1+n_2)-p}$ design of resolution III.1 is also a resolution III design in the usual sense, which implies $R^*(3.1, n_1, 2^k) \leq R(3, 2^k) - n_1 = 2^k - 1 - n_1$. We can construct a $2^{(n_1+n_2)-p}$ design with 2^k runs and $n_2 = 2^k - 1 - n_1$ by randomly assigning n_1 points in $PG(k-1, 2)$ to G_1 and the remaining to G_2. Therefore, $R^*(3.1, n_1, 2^k) = 2^k - 1 - n_1$.

Following the similar argument as above, $R^*(4.0, n_1, 2^k) \leq R(4, 2^k) - n_1 = 2^{k-1} - n_1$. A saturated resolution IV design has the structure $\{a\} \cup \{a + PG(k-2, 2)\}$, where $a \notin PG(k-2, 2)$. We can construct a $2^{(n_1+n_2)-p}$ design with 2^k runs and $n_2 = 2^{k-1} - n_1$ by randomly assigning n_1 points in the saturated design to G_1 and the remaining to G_2. Therefore, $R^*(4.0, n_1, 2^k) = 2^{k-1} - n_1$. □

Next, we consider $2^{(n_1+n_2)-p}$ designs with resolution III.3. Because $W_{1,2} = W_{2,1} = 0$, a main effect is not aliased with a 2fi in $G_1 \times G_2$. However, it may be aliased with a within-group 2fi because at least one of $W_{3,0}$ and $W_{0,3}$ is nonzero. The following lemma establishes an upper bound for n_2.

Lemma 3.1. *For a $2^{(n_1+n_2)-p}$ design of resolution III.3 or higher, $n_2 \leq 2^{k-1} - n_1 + 3W_{3,0}/n_1$.*

Proof. Let $N_{i,j,k}$ be the number of lines such that i points come from G_1, j points from G_2 and k points from G_3. For a resolution III.3 design, $W_{1,2} = N_{1,2,0} = 0$ and $W_{2,1} = N_{2,1,0} = 0$, which means there is no line crossing G_1 and G_2. For any point in G_1 and any point in G_2, their interaction has to be in G_3, which implies $n_1 n_2 = N_{1,1,1}$. For any point in G_1 and any point in G_3, their interaction may be in G_1, G_2, or G_3, which implies

$$n_1 n_3 = N_{1,1,1} + 2N_{2,0,1} + 2N_{1,0,2}.$$

For any two points in G_1, their interaction may be in G_1 or G_3, which implies

$$\frac{1}{2} n_1(n_1 - 1) = 3W_{3,0} + N_{2,0,1}.$$

Combining the above equations yields

$$\begin{aligned} 2N_{1,0,2} &= n_1 n_3 - n_1 n_2 - n_1(n_1 - 1) + 6W_{3,0} \\ &= n_1(2^k - 2(n_1 + n_2)) + 6W_{3,0}. \end{aligned}$$

The result immediately follows from $N_{1,0,2} \geq 0$. □

It is easy to know from Lemma 3.1 that when $W_{3,0} = 0$, the maximum number of G_2-factors in a resolution III.3 design is $2^{k-1} - n_1$. One construction for this case is the same as the resolution IV.0 design in the proof of Theorem 3.1. When $W_{3,0} > 0$, is it possible to find a resolution III.3 design with larger number of G_2-factors?

The answer is negative. To obtain this result, we need first introduce the following lemma.

Lemma 3.2. *For any subset $S \subset PG(k-1, 2)$, if there is no line crossing S and $PG(m-1, 2)$, $(0 < m < k)$, then $\{a + S\} \cap \{b + S\} = \varnothing$ for any two distinct points $a, b \in PG(m-1, 2)$.*

Proof. If $\{a + S\} \cap \{b + S\} \neq \varnothing$, then there must exist two distinct points $c_1, c_2 \in S$ such that $a + c_1 = b + c_2$, which yields $a + b = c_1 + c_2$. Because $a + b \in PG(m-1, 2)$, there is a line crossing $PG(m-1, 2)$ and S, which contradicts the assumption. Therefore, this lemma holds. □

Theorem 3.2. $R^*(3.3, n_1, 2^k) = 2^{k-1} - n_1$.

Proof. When $W_{3,0} = 0$, $n_2 \leq 2^{k-1} - n_1$ due to Lemma 3.1, and the equality is attainable. When $W_{0,3} = 0$, we can also obtain $n_2 \leq 2^{k-1} - n_1$ following the similar argument as in Lemma 3.1. Then it is enough to further show that $n_2 \leq 2^{k-1} - n_1$ when $W_{3,0} \neq 0$ and $W_{0,3} \neq 0$.

When $W_{3,0} \neq 0$, assume that $a, b, a + b \in G_1$ without loss of generality. Sets G_1, G_2, $\{a + G_2\}$, $\{b + G_2\}$, and $\{a + b + G_2\}$ are mutually disjoint due to Lemma 3.2 and $W_{1,2} = W_{2,1} = 0$. Therefore,

$$
\begin{aligned}
&|G_1| + |G_2| + |a + G_2| + |b + G_2| + |a + b + G_2| \\
&\quad = n_1 + n_2 + n_2 + n_2 + n_2 = n_1 + 4n_2 \leq 2^k - 1.
\end{aligned}
$$

When $W_{0,3} \neq 0$, following the similar argument, $4n_1 + n_2 \leq 2^k - 1$. Therefore, $5n_1 + 5n_2 \leq 2(2^k - 1)$, which implies $n_2 \leq 2(2^k - 1)/5 - n_1 < 2^{k-1} - n_1$.

Therefore, $R^*(3.3, n_1, 2^k) = 2^{k-1} - n_1$. □

When the resolution of a $2^{(n_1+n_2)-p}$ design is less than or equal to IV.0, we are able to identify $R^*(r.s, n_1, m)$ exactly for any n_1. For designs with higher resolution, it is difficult to obtain similar results due to their complicated structures. In the following we only report some upper bounds for $R^*(r.s, n_1, m)$ when $r.s > 4.0$.

A $2^{(n_1+n_2)-p}$ design of resolution IV.2 has $W_{1,2} = W_{2,1} = W_{3,0} = W_{0,3} = W_{2,2} = 0$. Hence, any main effect is not aliased with any 2fi and is clearly estimable. Additionally, between-group 2fis are not aliased with each other, but they may be aliased with within-group 2fis. When the latter 2fis are assumed negligible due to prior knowledge, the former are estimable.

Theorem 3.3. $R^*(4.2, n_1, 2^k) \leq [2^k/(n_1 + 1)] - 1$, *where $[x]$ denotes the largest integer smaller than or equal to x.*

Proof. Because $W_{1,2} = W_{2,1} = 0$, for any two points $a \in G_1$ and $b \in G_2$, their interaction $a + b \in PG(k-1, 2) \backslash \{G_1 \cup G_2\}$. For two different pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$ such that $a_1, a_2 \in G_1$ and $b_1, b_2 \in G_2$, we have $a_1 + b_1 \neq a_2 + b_2$ because $W_{2,2} = 0$. Therefore,

$$
n_1 + n_2 + n_1 n_2 \leq 2^k - 1,
$$

which implies $n_2 \leq 2^k/(n_1 + 1) - 1$. □

The upper bound in Theorem 3.3 is attainable for some values of n_1. For example, when $n_1 = 2^m - 1$ $(0 < m < k)$, a resolution IV.2 design with $n_2 = 2^{k-m} - 1$ has the structure $G_1 = PG(m-1, 2)$ and $G_2 = PG(k-1-m, 2)$.

In a $2^{(n_1+n_2)-p}$ design of resolution IV.4, all between-group 2fi's are clearly estimable, which is also discussed by Ke, Tang, and Wu (2005). The proof of Theorem 3.4 is similar to that of Theorem 4 in Ke, Tang, and Wu (2005).

Theorem 3.4. $R^*(4.4, n_1, 2^k) \leq [(2^k + 1 - 2n_1)/(n_1 + 2)]$.

Proof. Because $W_{1,2} = W_{2,1} = W_{3,0} = W_{0,3} = W_{2,2} = 0$, sets G_1, G_2, and $G_1 + G_2$ are mutually disjoint. For any point $a \in G_1$ and $b \in G_2$, $a + (G_1 \backslash \{a\})$ is disjoint with $b + (G_2 \backslash \{b\})$, and both of them are mutually disjoint with the previous three sets because $W_{1,3} = W_{3,1} = 0$. Therefore,

$$n_1 + n_2 + (n_1 - 1) + n_1 n_2 + (n_2 - 1) \leq 2^k - 1,$$

which yields the results. □

Because $R^*(4.4, n_1, 2^k) \geq 1$ for a $2^{(n_1+n_2)-p}$ design, the number of factors in G_1 necessarily satisfies $n_1 \leq (2^k - 1)/3 = 2^{k-2} + (2^{k-2} - 1)/3$. The upper bound in Theorem 3.4 is attainable for some value of n_1. For example, when $n_1 = 2^{k-2} - 1$, $[(2^k + 1 - 2n_1)/(n_1 + 2)] = 2$, in this case, we can construct the design as $G_1 = a + PG(k-3, 2)$ and $G_2 = \{a, b\}$, where $a, b \notin PG(k-3, 2)$; also see Ke, Tang, and Wu (2005).

In a $2^{(n_1+n_2)-p}$ design of resolution V.1, all main effects and 2fis are clearly estimable.

Theorem 3.5. $R^*(5.1, n_1, 2^k) \leq [\sqrt{2^{k+1} - 7/4} - 1/2] - n_1$.

Proof. Because sets G_1, G_2, $G_1 + G_1$, $G_1 + G_2$, and $G_2 + G_2$ are mutually disjoint, we have

$$n_1 + n_2 + \frac{1}{2} n_1 (n_1 - 1) + n_1 n_2 + \frac{1}{2} n_2 (n_2 - 1) \leq 2^k - 1,$$

or equivalently,

$$n_2^2 + (2n_1 + 1) n_2 + n_1^2 + n_1 - 2(2^k - 1) \leq 0.$$

Notice that $\Delta = (2n_1 + 1)^2 - 4\{n_1^2 + n_1 - 2(2^k - 1)\} = 8(2^k - 1) + 1 \geq 0$, we have

$$\frac{-(2n_1 + 1) - \sqrt{\Delta}}{2} \leq n_2 \leq \frac{-(2n_1 + 1) + \sqrt{\Delta}}{2}.$$

Hence, the theorem is proved. □

From Theorem 3.5, we are also able to obtain an upper bound on the maximum number of factors for a resolution V design, $R(5, 2^k) \leq [\sqrt{2^{k+1} - 7/4} - 1/2]$. When $k = 4$, $R(5, 2^4) = 5$ attains the upper bound. However, when $5 < k \leq 12$, the upper bound is strictly larger than $R(5, 2^k)$ reported by Draper and Lin (1990).

4. Conclusions and Discussions

The value of $R^*(r.s, n_1, m)$ can provide useful guidelines for practical experiment planning. In this article, we have obtained the exact value of $R^*(r.s, n_1, m)$ when $r.s \leq 4.0$. When $r.s > 4.0$, to obtain the exact value of $R^*(r.s, n_1, m)$ usually requires computer search. The algorithm discussed in Draper and Lin (1990) can be easily modified and adopted for this purpose.

Word-type patterns can capture more detailed features of a design than the ordinary word-length pattern when there are multiple groups of factors involved in an experiment. They can be further used to define a vector similar to the ordinary word-length pattern in order to incorporate prior knowledge on the relative importance of interactions. Based on this vector, resolution and aberration can be redefined accordingly. This approach was also employed by Bingham and Sitter (2003) and Wu and Zhu (2003) when studying robust parameter designs. The article follows the same approach and focuses on a generalized version of Scenario 4 of the compromise plans. As a matter of fact, the reported results can be extended to the other scenarios.

Nonregular designs are not considered in this article because of the intractability of their complex aliasing structures. Recently J-characteristics, playing a similar role as word-length in regular designs, have been proposed for the study of non-regular designs; see Tang (2001) and references therein. We believe that it is possible to generalize J-characteristics to non-regular designs with two distinct groups of factors, and we are not aware of any existing results in the literature. We will pursue this extension of our work in the future.

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