On Multiresolution Analyses Of Multiplicity n

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Abstract

This paper studies multiresolution analyses in $L^2(\mathbb{R}^d)$ that have more than one scaling function and are generated by an arbitrary dilation matrix. It provides a further analysis of a representation theorem obtained by the author for such MRA's.

1 Introduction

The concept of multiresolution analysis of multiplicity n is due to Alpert [1, 2, 3] who introduced his now well known dyadic multiresolution analysis with an arbitrary number of filters in $L^2(\mathbb{R})$. Alpert's results motivated a number of papers, focused on the univariate case, such as Hervé [12, 13], Donovan, Geronimo and Hardin [6, 7], Geronimo and Marcellán [10], Goodman, Lee and Tang [9], Goodman and Lee [8], and Hardin, Kessler and Massopoust [11]. Multiresolution analyses of multiplicity 1 (i.e., with a single scaling function) with arbitrary expansive matrices in $L^2(\mathbb{R})$ were studied by Lemarié [15, 16] and Madych [17], among others, and we should also mention Wojtaszczyk's excellent textbook [25]. Properties of low pass filters and scaling functions in this context were studied by San Antolín [19, 20, 21, 22] and Cifuentes, Kazarian and San Antolín [5]. Saliani [18] extended these results to multiresolution analyses of multiplicity n generated by an expansive matrix. These results were further extended by Soto–Bajo [24] to multiresolution analyses having an arbitrary (not necessarily finite) set of generator functions. In [4], Behera studied multiwavelet packets and frame packets of $L^2(\mathbb{R}^d)$ associated with multiresolution analyses of multiplicity n generated by an expansive matrix. In [26] the author presented a representation theorem for such multiresolution analyses, in [27] he gave some simple examples for the case $n = 1$, and in [23] San Antolín and the author obtained representation theorems for vector valued wavelets. Some of the authors cited above have showed that by using more than one scaling function it is possible to

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construct wavelets that have a set of properties not available for wavelets associated with a multiresolution analysis having a single scaling function; for instance in [6] they constructed wavelets associated with more than two scaling functions having compact support, arbitrary regularity, orthogonality, and symmetry. These results would indicate that the further study of multiresolution analyses of multiplicity n may lead to other interesting results.

In what follows, $\mathbb{Z}$ will denote the set of integers, $\mathbb{Z}_+$ the set of strictly positive integers and $\mathbb{R}$ the set of real numbers; $\mathbb{C}$ will denote the set of complex numbers, and $\mathbf{I}$ will stand for the identity matrix. Boldface lowcase letters will denote elements of $\mathbb{R}^d$; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; the vector norm $||\cdot||$ is defined by $||\mathbf{x}||^2 := \mathbf{x} \cdot \mathbf{x}$. If $\mathbf{A}$ is a matrix $||\mathbf{A}||$ will denote the matrix norm induced by the vector norm $||\cdot||$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ will be denoted by $\langle f, g\rangle$, their bracket product by $[f, g]$, and the norm of f by $||f||$; thus,

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(\mathbf{t})\overline{g(\mathbf{t})}\, d\mathbf{t},$$

$$[f, g](\mathbf{t}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} f(\mathbf{t}+\mathbf{k})\overline{g(\mathbf{t}+\mathbf{k})},$$

and

$$||f|| := \sqrt{\langle f, f\rangle}.$$

The Fourier transform of a function f will be denoted by $\widehat{f}$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-i2\pi\mathbf{x}\cdot\mathbf{t}} f(\mathbf{t})\, d\mathbf{t}.$$

Let $\mathbf{A} \in \mathbb{C}^{d\times d}$ and $|a| := \det(\mathbf{A})$. For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $D_j^{\mathbf{A}}$ and the translation operator $T_{\mathbf{k}}$ are defined on $L^2(\mathbb{R}^d)$ by

$$D_j^{\mathbf{A}} f(\mathbf{t}) := |a|^{j/2} f(\mathbf{A}^j\mathbf{t})$$

and

$$T_{\mathbf{k}} f(\mathbf{t}) := f(\mathbf{t}+\mathbf{k})$$

respectively. Let $\mathbb{T} := [0, 1]$, and let $\mathbb{T}^d$ denote the d–fold cartesian product of $\mathbb{T}$. A function f will be called $\mathbb{Z}^d$*–periodic* if it is defined on $\mathbb{R}^d$ and $T_{\mathbf{k}} f = f$ for every $\mathbf{k} \in \mathbb{Z}^d$.

Let $\mathbf{u} = \{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)$; then

$$T(\mathbf{u}) = T(u_1, \cdots, u_m) := \{T_{\mathbf{k}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}$$

and

$$S(\mathbf{u}) = S(u_1, \cdots, u_m) := \overline{\operatorname{span}}\, T(\mathbf{u}),$$

where the closure is in $L^2(\mathbb{R}^d)$. $S(\mathbf{u})$ is called a *finitely generated shift–invariant space* or FSI and the functions u_ℓ are called the *generators* of $S(\mathbf{u})$. In this case we will also use the symbols $T(u_1, \cdots, u_n)$ and $S(u_1, \cdots, u_n)$ to denote $S(\mathbf{u})$ and $T(\mathbf{u})$ respectively.

We also define

$$T(\mathbf{A}^j;\mathbf{u}) = T(\mathbf{A}^j;u_1,\cdots,u_m) := \{D_j^{\mathbf{A}}T_{\mathbf{k}}u_\ell; \ell=1,\cdots m, \mathbf{k}\in\mathbb{Z}^d\},$$

and

$$S(\mathbf{A}^j;\mathbf{u}) = S_j(\mathbf{A}^j;u_1,\cdots,u_m) := \overline{\operatorname{span}}\, T(\mathbf{A}^j;\mathbf{u}).$$

Given a sequence of functions $\mathbf{u} := \{u_1,\cdots,u_m\}$ in $L^2(\mathbb{R}^d)$, by $G[u_1,\cdots,u_m](\mathbf{x})$, $G_{\mathbf{u}}(\mathbf{x})$ or $G(\mathbf{x})$ we will denote its *Gramian matrix*, viz.

$$G(\mathbf{x}) := \Big([\widehat{u_\ell},\widehat{u_j}](\mathbf{x})\Big)_{\ell,j=1}^{m}.$$

Let $\Lambda\subset\mathbb{Z}$ and $\mathbf{u}=\{u_k; k\in\Lambda\}\subset S\subset L^2(\mathbb{R}^d)$. If S is a shift–invariant space then $\mathbf{u}$ is called a *basis generator* of S, if for every $f\in S$ there are $\mathbb{Z}^d$–periodic functions p_k, uniquely determined by f (up to a set of measure 0), such that

$$\widehat{f} = \sum_{k\in\Lambda} p_k\widehat{u_k}.$$

In what follows we will assume that $\mathbf{A}$ is a fixed matrix preserving the lattice $\mathbb{Z}^d$, i.e. $\mathbf{A}\mathbb{Z}^d\subset\mathbb{Z}^d$. We will also assume that $\mathbf{A}$ is expansive, that is, there exist constants $C>0$ and $\delta>1$ such that for every $j\in\mathbb{Z}_+$ and $\mathbf{x}\in\mathbb{R}^d$

$$||\mathbf{A}^j\mathbf{x}|| \geq C\delta^j||\mathbf{x}||.$$

Lemma 1. $\mathbf{A}$ *is expansive if and only if all its eigenvalues have modulus larger that* 1.

Proof. Suppose first that all the eigenvalues of $\mathbf{A}$ have modulus larger that 1. If $\mathbf{A}$ is a Jordan block, then $\mathbf{A}=\lambda\mathbf{I}+\mathbf{N}$, where $\mathbf{N}$ has 1's on the superdiagonal and 0's elsewhere, and from e.g. [14, Lemma 3.1.4] we deduce that

$$\mathbf{A}^j\mathbf{x} = \sum_{k=0}^{d}\binom{j}{k}\lambda^k N^{j-k}\mathbf{x}$$

whence the assertion readily follows, and therefore it also follows when $\mathbf{A}$ is in Jordan canonical form. In general, if $\mathbf{Q}$ is the Jordan form of $\mathbf{A}$, then $\mathbf{Q}=\mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ and we have

$$C\delta^j||\mathbf{y}|| \leq ||\mathbf{Q}^j\mathbf{y}|| = ||\mathbf{B}^{-1}\mathbf{A}^j\mathbf{B}\mathbf{y}|| \leq ||\mathbf{B}^{-1}||\,||\mathbf{A}^j\mathbf{B}\mathbf{y}||.$$

Setting $\mathbf{x}=\mathbf{B}\mathbf{y}$ the assertion readily follows.

Conversely, if $\mathbf{A}$ has an eigenvalue λ with modulus less or equal to 1 and $\mathbf{v}$ is an eigenvector for λ with $||\mathbf{v}||=1$, then $\mathbf{A}\mathbf{v}=\lambda\mathbf{I}$; hence $||\mathbf{A}^j\mathbf{v}||=|\lambda|^j\leq|\lambda|$, and $||\mathbf{A}^j\mathbf{v}||$ remains bounded. So $\mathbf{A}$ is not expansive. □

The previous proof was suggested by Wayne Lawton. An elementary proof may be found in San Antolín's thesis [19, Lema A.12].

A *multiresolution analysis* (MRA) of multiplicity n in $L^2(\mathbb{R}^d)$ (generated by $\mathbf{A}$) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.

(ii) For every $j \in \mathbb{Z}$, $f(\mathbf{t}) \in V_j$ if and only if $f(\mathbf{A}\mathbf{t}) \in V_{j+1}$.

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

(iv) There are functions $\mathbf{u} := \{u_1, \cdots, u_n\}$ such that $T(\mathbf{u})$ is an orthonormal basis of V_0.

From [18, Lemma 17] we know that if $\{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis, then

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}. \tag{1}$$

This generalizes a result of Cifuentes, Kazarian and San Antolín, which was established for multiresolution analyses of multiplicity 1 (cf. [5, Lemma 4]).

It follows from the definition of multiresolution analysis that there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j} \in L^2(\mathbb{T}^d)$ such that the functions u_ℓ satisfy the *scaling identity*

$$\widehat{u}_\ell(\mathbf{A}^*\mathbf{x}) = \sum_{j=1}^{n} p_{\ell,j}(\mathbf{x})\widehat{u}_j(\mathbf{x}), \quad j,\ell = 1, \cdots, n \quad a.e.,$$

where $\mathbf{A}^*$ is the transpose of $\mathbf{A}$. The functions u_ℓ are called *scaling functions* for the multiresolution analysis, and the functions $p_{\ell,j}$ are called the *low pass filters* associated with $\mathbf{u}$.

A finite set of functions $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_m\} \in L^2(\mathbb{R}^d)$ will be called an orthonormal wavelet system if the affine sequence

$$\{D_j^{\mathbf{A}} T_{\mathbf{k}} \psi_\ell; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \ell = 1, \cdots, m\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Let $\boldsymbol{\psi} := \{\psi_1, \cdots, \psi_m\}$ be an orthogonal wavelet system in $L^2(\mathbb{R}^d)$ generated by a matrix $\mathbf{A}$, let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis and let W_j denote the orthogonal complement of V_j in V_{j+1}. We say that $\boldsymbol{\psi}$ is *associated* with an MRA, if $T(\boldsymbol{\psi})$ is an orthonormal basis of W_0.

2 Representation of orthonormal wavelets.

For $k > 1$ let $\mathrm{diag}\,\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_\mathrm{k}$ denote the $k \times k$ diagonal matrix with $-e^{i\omega}, 1, \cdots, 1$ as its diagonal entries. With the convention that $\mathrm{Arg}\, 0 = 0$ we have

Theorem 1. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n with scaling functions $\mathbf{u} := \{u_1, \cdots, u_n\}$, generated by a matrix $\mathbf{A}$ that preserves the lattice*

$\mathbb{Z}^d$. *For* $1 \le \ell \le n$, *let* $\{v_{\ell,1}, \cdots, v_{\ell,|a|}\}$ *be an orthonormal basis generator of* $S(\mathbf{A}, u_\ell)$, *let* $\mathbf{e} := (1, 0, \cdots, 0) \in \mathbb{R}^k$, *and*

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{|a|} b_{\ell,j}(\mathbf{x}) \widehat{v}_{\ell,j}(\mathbf{x}), \tag{2}$$

$$\mathbf{b}_\ell(\mathbf{x}) := (b_{\ell,1}(\mathbf{x}), \cdots, b_{\ell,|a|}(\mathbf{x}))^T, \quad \delta_\ell(\mathbf{x}) := e^{iArg\, b_{\ell,1}(\mathbf{x})}, \quad \mathbf{q}_\ell(\mathbf{x}) := \mathbf{b}_\ell(\mathbf{x}) + \delta_\ell(\mathbf{x})\mathbf{e},$$

$$\widehat{\mathbf{v}}(\mathbf{x}) := (\widehat{v}_{1,1}(\mathbf{x}), \cdots, \widehat{v}_{1,|a|}(\mathbf{x}), \cdots, \widehat{v}_{n,1}(\mathbf{x}), \cdots, \widehat{v}_{n,|a|}(\mathbf{x}))^T,$$

and

$$\mathbf{Q}_\ell(\mathbf{x}) := \operatorname{diag}\{-\delta_\ell(\mathbf{x}), 1, \cdots, 1\}_{|a|} \left[\overline{\mathbf{I} - 2\mathbf{q}_\ell(\mathbf{x})\mathbf{q}_\ell(\mathbf{x})^* / \mathbf{q}_\ell(\mathbf{x})^* \mathbf{q}_\ell(\mathbf{x})} \right].$$

Let $a := \det \mathbf{A}$, $m := |a|n$, *and let*

$$\mathbf{Q}(\mathbf{x}) = \Big(q_{\ell,k}(\mathbf{x}) \Big)_{\ell,k=1}^{m}$$

be the $m \times m$ *block diagonal matrix*

$$\mathbf{Q}_1(\mathbf{x}) \oplus \mathbf{Q}_2(\mathbf{x}) \oplus \cdots \oplus \mathbf{Q}_m(\mathbf{x}) = \begin{pmatrix} \mathbf{Q}_1(\mathbf{x}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_n(\mathbf{x}) \end{pmatrix}.$$

If

$$(\widehat{y}_1(\mathbf{x}), \cdots, \widehat{y}_m(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x}) \widehat{\mathbf{v}}(\mathbf{x}),$$

then

$$y_{(\ell-1)|a|+1} = u_\ell; \quad 1 \le \ell \le n, \tag{3}$$

and

$$\{y_{(\ell-1)|a|+k}; 1 \le \ell \le n,\ 2 \le k \le |a|\}$$

is an orthonormal wavelet system associated with M.

The preceding theorem was proved in [26, Theorem 9] but there was arguably a gap in the proof, which is bridged by the following

Lemma 2. *Let* $m = n(|a| - 1)$, *let* $\{V_j; j \in \mathbb{Z}\}$ *be a multiresolution analysis and assume that* $\{u_\ell; \ell = 1, \cdots, n\}$ *is an orthonormal basis generator of* V_0. *Then*

$$V_1 = S(\mathbf{A}, u_1) \oplus S(\mathbf{A}, u_2) \oplus \cdots \oplus S(\mathbf{A}, u_n).$$

Proof. From [26, Theorem 3] we know that there exist functions $v_{\ell,k}$, $\ell = 1, \cdots n$, such that $\{v_{\ell,1}, \cdots v_{\ell,|a|}\}$ is an orthonormal basis generator of $S(\mathbf{A}, u_\ell)$. Therefore $\{v_{\ell,k}; 1 \le \ell \le n, 1 \le k \le |a|\}$ is an orthogonal basis generator of $S(\mathbf{A}, u_1) \oplus S(\mathbf{A}, u_2) \oplus \cdots \oplus S(\mathbf{A}, u_n)$, which is a subspace of V_1. But [26, Theorem 3] also tells us that every Riesz generator of V_1 has $|a|n$ functions, and the assertion readily follows from [26, Theorem 1]. □

We can actually say more: the following theorem elucidates the structure of a multiresolution analysis of multiplicity n:

Theorem 2. *Let $\{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis and assume that $\{u_\ell; \ell = 1, \cdots, n\}$ is an orthonormal basis generator of V_0. Then*
(a) If $j > 0$,

$$V_j = S(\mathbf{A}^j, u_1) \oplus S(\mathbf{A}^j, u_2) \oplus \cdots \oplus S(\mathbf{A}^j, u_n).$$

(b)

$$L^2(\mathbb{R}^d) = \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_1)} + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_2)} + \cdots\cdots + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_n)}$$

Proof. From [26, Theorem 4] we know that every orthonormal basis generator of $V_j, j > 0$, must have $n|a|^j$ functions, and an argument similar to the one employed in the proof of Lemma 2 yields (a).

To prove (b), let $f \in L^2(\mathbb{R}^d)$ and let $\varepsilon > 0$ be given; then there is a $j \in \mathbb{Z}_+$ and a $g \in V_j$ such that $||f - g|| < \varepsilon$. Since *a fortiori* g belongs to the closed set

$$\overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_1)} + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_2)} + \cdots\cdots + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_n)}$$

and ε is arbitrary, the assertion follows. □

Theorem 3. *Let $m = n(|a| - 1)$, let the functions y_k, $k = 1 \cdots |a|n$ be constructed as in Theorem 1, and let $q_j = y_{j+\ell}$ for $\ell = 1, \cdots n$, $j = \ell|a|, \cdots, (\ell + 1)(|a| - 1)$. Then $\{w_1, \cdots w_r\}$ is an orthonormal wavelet system if and only if $r = m$ and there exists an orthogonal matrix $Q(x)$ such that*

$$(w_1, \cdots w_m)^T = Q(x)(q_1, \cdots q_m)^T.$$

Proof. From Theorem 1 we know that $\{q_1, \cdots q_m\}$ is an orthonormal basis system or, equivalently, that it is an orthonormal basis generator of $S(\mathbf{u})$. The assertion now readily follows from [26, Corollary 3 and Theorem 5]. □

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