# On Orthonormal Wavelet Bases

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**Abstract**

Given a multiresolution analysis with one generator in $L^2(\mathbb{R}^d)$, we give a characterization in closed form and in the frequency domain, of all orthonormal wavelets associated to this MRA. Examples are given. This theorem corrects a previous result of the author.

## 1 Introduction

In what follows $\mathbb{Z}$ will denote the set of integers, and $\mathbb{R}$ the set of real numbers. We will always assume that $\mathbf{A}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$; that is, $\mathbf{A}\mathbb{Z}^d \subset \mathbb{Z}^d$ and all its eigenvalues have modulus greater than 1; $\mathbf{A}^*$ will denote the transpose of $\mathbf{A}$ and $\mathbf{B} := (A^*)^{-1}$. The underlying space will be $L^2(\mathbb{R}^d)$, where $d \geq 1$ is an integer and $\mathbf{I}$ will stand for the identity matrix. Boldface lowcase letters will denote elements of $\mathbb{R}^d$, which will be represented as column vectors; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; $||\mathbf{x}||^2 := \mathbf{x} \cdot \mathbf{x}$.

Let $\mathbf{A} \in \mathbb{R}^{d \times d}$ and $a := |\det \mathbf{A}|$. For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $D^{\mathbf{A}}$ and the translation operator $T_{\mathbf{k}}$ are defined on $L^2(\mathbb{R}^d)$ by

$$D^{\mathbf{A}} f(\mathbf{t}) := a^{1/2} f(\mathbf{A}\mathbf{t}) \qquad \text{and} \qquad T_{\mathbf{k}} f(\mathbf{t}) := f(\mathbf{t} + \mathbf{k})$$

respectively.

Let $\mathbf{u} = \{u_1, \ldots, u_m\} \subset L^2(\mathbb{R}^d)$; then $T(u_1, \ldots, u_m) = T(\mathbf{u})$, $S(u_1, \ldots, u_m) = S(\mathbf{u})$ and $S(\mathbf{A}; u_1, \ldots, u_m) = S(\mathbf{A}; \mathbf{u})$ are respectively defined by

$$T(\mathbf{u}) := \{T_{\mathbf{k}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}, \qquad S(\mathbf{u}) := \overline{\operatorname{span}}\, T(\mathbf{u}),$$

and

$$S(\mathbf{A}, \mathbf{u}) := \overline{\operatorname{span}}\, \{D^{\mathbf{A}} T_{\mathbf{k}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}.$$

In [5] we formulated a representation theorem for multiresolution analyses having an arbitrary set $u_1, \ldots, u_n$ of scaling functions, i.e., the set of translates of all these functions constitutes an orthonormal basis of $V_0$. However the proof was based on the implicit (and incorrect) assumption that any such function $u_\ell$ is contained in $S(\mathbf{A}, u_\ell)$, and it is therefore not valid. The purpose of this paper is to apply the method of proof

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employed in [5] to prove a representation theorem for MRA's having a single scaling function, and to provide some examples.

A function $f$ will be called $\mathbb{Z}^d$*–periodic* if it is defined on $\mathbb{R}^d$ and $T_{\mathbf{k}} f = f$ for every $\mathbf{k} \in \mathbb{Z}^d$ .

The Fourier transform of a function $f$ will be denoted by $\widehat{f}$ or $\mathfrak{F}(f)$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-i2\pi \mathbf{x}\cdot\mathbf{t}} f(\mathbf{t})\, d\mathbf{t}.$$

The Fourier transform is extended to $L^2(\mathbb{R}^d)$ in the usual way.

Our starting point and motivation is the following well known characterization in Fourier space of affine MRA orthonormal wavelets in $L^2(\mathbb{R})$ (see e.g. Hernández and Weiss [2], Wojtaszczyk [4]) which, with the definition of Fourier transform we have adopted, may be stated as follows.

**Theorem A**. *Let $\varphi$ be a scaling function for a multiresolution analysis $M$ with associated low pass filter $p$. The following propositions are equivalent:*
*(a) $\psi$ is an MRA orthonormal wavelet associated with $M$.*
*(b) There is a measurable unimodular $\mathbb{Z}$–periodic function $\mu(x)$ such that*

$$\widehat{\psi}(2x) = e^{i2\pi x}\mu(2x)\overline{p(x+1/2)}\widehat{\varphi}(x) \qquad a.e.$$

Recall that a *multiresolution analysis* (MRA) in $L^2(\mathbb{R}^d)$ (generated by $\mathbf{A}$) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.

(ii) For every $j \in \mathbb{Z}, f(\mathbf{t}) \in V_j$ if and only if $f(\mathbf{A}\mathbf{t}) \in V_{j+1}$.

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

(iv) $\bigcap_{j\in\mathbb{Z}} V_j = \emptyset$.

(v) There is a function $u$ (called the *scaling function* of the MRA) such that $T(u)$ is an orthonormal basis of $V_0$.

A finite set of functions $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_m\} \in L^2(\mathbb{R}^d)$ is called an orthonormal wavelet system if the affine sequence

$$\{D_j^{\mathbf{A}} T_{\mathbf{k}} \psi_\ell; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \ell = 1, \cdots, m\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Let $\boldsymbol{\psi} := \{\psi_1, \cdots, \psi_m\}$ be an orthonormal wavelet system in $L^2(\mathbb{R}^d)$ generated by a matrix $\mathbf{A}$; for $j \in \mathbb{Z}$ we define

$$V_j = \sum_{r<j} S(\mathbf{A}^r; \boldsymbol{\psi}).$$

We say that $\boldsymbol{\psi}$ is *associated* with an MRA, if $M := \{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis. If this is the case, we also say that $\boldsymbol{\psi}$ is associated with $M$. Let $W_j$ denote the orthogonal complement of $V_j$ in $V_{j+1}$. Then it is easily seen that $\boldsymbol{\psi}$ is an orthonormal wavelet associated with $M$ if and only if $T(\boldsymbol{\psi})$ is an orthonormal basis of $W_0$.

Let $\mathbf{e} := (1, 0, \cdots, 0)^T \in \mathbb{R}^m$ and let $\mathrm{diag}\,\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{m}}$ denote the $m \times m$ diagonal matrix with $-e^{i\omega}, 1, \cdots, 1$ as its diagonal entries. The following proposition was implicitly established by Jia and Shen in the discussion that follows the proof of [3, Lemma 3.3] (we adopt the convention that $\mathrm{Arg}\, 0 = 0$).

**Theorem B.** *Let* $\mathbf{b} = (b_1, \cdots, b_m)^T \in \mathbb{C}^m$ *be unimodular,* $\omega := \mathit{Arg}\, b_1$ *and* $\mathbf{q} := \mathbf{b} + e^{i\omega}\mathbf{e}$. *Then the matrix*

$$\mathbf{Q} = (q_{r,k})_{r,k=1}^m := \mathrm{diag}\,\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{m}} \left[\overline{\mathbf{I} - 2\mathbf{q}\mathbf{q}^*/\mathbf{q}^*\mathbf{q}}\right]$$

*is unitary. Moreover*

$$q_{r,k} = \begin{cases} b_k & \text{if } r = 1, 1 \le k \le m \\ -\overline{b_r} e^{i\omega} & \text{if } 1 < r \le m, k = 1 \\ \delta_{r,k} - \dfrac{\overline{b_r} b_k}{1 + |b_1|} & \text{if } 1 < r \le m, 1 < k \le m, \end{cases}$$

*where* $\delta_{r,k}$ *is Krönecker's delta.*

The following proposition is a particular case of [5, Theorem 3].

**Lemma C.** Let $u \in L^2(\mathbb{R}^d)$ and assume that $T(u)$ is an orthonormal sequence. Let $\mathbf{A}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, let $\{j_1, \ldots, j_a\}$ be a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$, and let

$$v_k(\mathbf{t}) := a^{1/2} u(\mathbf{A}t + j_k), \quad k = 1, \ldots a. \tag{1}$$

Then $T(v_1, \cdots, v_a)$ is an orthonormal basis of $S(\mathbf{A}; u)$.

Since $\widehat{v}_k(\mathbf{x}) = e^{i2\pi \mathbf{Bx} \cdot j_k} \widehat{u}(\mathbf{Bx})$, a straightforward consequence of Lemma C and [5, Lemma E] is the following

**Corollary 1.** *Let* $u \in L^2(\mathbb{R}^d)$ *and assume that* $T(u)$ *is an orthonormal sequence. Let* $\mathbf{A}$ *be a dilation matrix preserving the lattice* $\mathbb{Z}^d$, $B := (A^*)^{-1}$, *let* $\{j_1, \ldots, j_a\}$ *be a full collection of representatives of* $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$, *and let* $v_k(\mathbf{t})$ *be defined by* (1). *If* $u \in S(A, u)$, *then there are* $\mathbb{Z}^d$*–periodic functions* $q_k \in L^2(\mathbb{T}^d)$ *such that*

$$\sum_{k=1}^{a} |q_k(\mathbf{x})|^2 = 1 \quad a.e., \tag{2}$$

*and*

$$\widehat{u}(\mathbf{x}) = \sum_{k=1}^{a} q_k(\mathbf{x}) \widehat{v}_k(\mathbf{x}) = \sum_{k=1}^{a} q_k(\mathbf{x}) e^{i2\pi \mathbf{Bx} \cdot j_k} \widehat{u}(\mathbf{Bx}) = p(B\mathbf{x}) \widehat{u}(B\mathbf{x}), \tag{3}$$

*where*

$$p(\mathbf{x}) := a^{-1/2} \sum_{k=1}^{a} q_k(A^*\mathbf{x}) e^{i2\pi \mathbf{x}\cdot j_k}.$$

We can now prove

**Theorem 1.** *Let $M$ be a multiresolution analysis generated by $\mathbf{A}$ with scaling function $u$, let $v_k(\mathbf{t})$ be defined by (1), $B := (A^*)^{-1}$, and let the functions $q_k(\mathbf{x})$ be $\mathbb{Z}^d$–periodic, in $L^2(\mathbb{T}^d)$, and satisfy (2) and (3). Let*

$$\alpha(\mathbf{x}) := Arg\, q_1(\mathbf{x}), \tag{4}$$

$$w_{r,k}(\mathbf{x}) := \begin{cases} q_k(\mathbf{x}) & \text{if } r = 1, 1 \le k \le a \\ -\overline{q_r(\mathbf{x})} e^{i\alpha(\mathbf{x})} & \text{if } 1 < r \le a, k = 1 \\ \delta_{r,k} - \dfrac{\overline{q_r(\mathbf{x})} q_k(\mathbf{x})}{1 + |q_1(\mathbf{x})|} & \text{if } 1 < r \le a, 1 < k \le a \end{cases} \tag{5}$$

*and*

$$\widehat{z}_r(\mathbf{x}) := \sum_{k=1}^{a} w_{r,k}(\mathbf{x}) \widehat{v}_k(\mathbf{x}),$$

*and let*

$$\mathbf{Z}(\mathbf{x}) := \left(\widehat{z}_2(\mathbf{x}), \ldots, \widehat{z}_a(\mathbf{x})\right)^T.$$

*Then*

$$\{\psi_1, \ldots, \psi_{(a-1)}\}$$

*is an orthonormal wavelet system associated with $M$ if and only if there exists an $(a-1) \times (a-1)$ unitary matrix function $\mathbf{U}(x)$ such that*

$$(\widehat{\psi}_1(\mathbf{x}), \ldots, \widehat{\psi}_{(a-1)}(\mathbf{x}))^T = \mathbf{U}(\mathbf{x})\mathbf{Z}(\mathbf{x}).$$

*Proof.* The existence of functions $q_k(\mathbf{x})$ satisfying (2) and (3) is a consequence of Corollary 1. Setting

$$\widehat{\mathbf{v}}(\mathbf{x}) := (\widehat{v}_1(\mathbf{x}), \cdots, \widehat{v}_a(\mathbf{x}))^T$$

and applying Theorem B, we see that

$$(\widehat{z}_1(\mathbf{x}), \cdots \widehat{z}_a(\mathbf{x}))^T = \mathbf{Q}(\mathbf{x})\widehat{\mathbf{v}}(\mathbf{x}),$$

and that $\mathbf{Q}(\mathbf{x})$ has $(q_1(\mathbf{x}), \cdots, q_a(\mathbf{x}))$ as its first row. Therefore [5, Theorem 8] implies that $\{z_2, \ldots z_a\}$ is an orthonormal wavelet system associated with $M$, which is equivalent to saying that $S(z_2, \ldots z_a)$ is an orthonormal basis generator of $W_0$. Applying now [5, Theorem 5], the assertion follows. □

**Example 1**. Let us verify that Theorem A is a particular case of Theorem 1.

For $d = 1$ and $A = 2$ we have $j_1 = 0$ and $j_2 = 1$, and Corollary 1 implies that

$$p(x) = 2^{-1/2}[q_1(2x) + e^{i2\pi x} q_2(2x)],$$

whence the periodicity of $q_1(x)$ and $q_2(x)$ implies that

$$p(x + 1/2) = 2^{-1/2}[q_1(2x) - e^{i2\pi x} q_2(2x)].$$

On the other hand, since $|q_1(x)|^2+|q_2(x)|^2 = 1$ a.e., (5) implies that $w_{2,1}(x) = -\overline{q_2(x)}e^{i\alpha(x)}$ and

$$w_{2,2}(x) = 1 - \frac{|q_2(x)|^2}{1 + |q_1(x)|} = 1 - \frac{|q_2(x)|^2(1 - |q_1(x)|)}{|q_2(x)|^2} = |q_1(x)|.$$

Since $\mathbf{B} = 1/2$, it follows that $\widehat{v}_1(x) = 2^{-1/2}\widehat{u}(x/2)$ and $\widehat{v}_2(x) = 2^{-1/2}e^{-i\pi x}\widehat{u}(x/2)$, and Theorem 1 implies that

$$\widehat{z}_2(x) = 2^{-1/2}[-e^{i\alpha(x)}\overline{q_2(x)} + e^{i\pi x}|q_1(x)|]\widehat{u}(x/2) = \\ 2^{-1/2}e^{i\pi x}e^{i\alpha(x)}[-\overline{q_2(x)}e^{-i\pi x} + e^{-i\alpha(x)}|q_1(x)|]\widehat{u}(x/2) = \\ 2^{-1/2}e^{-i\pi x}e^{i\alpha(x)}[\overline{q_1(x)} - e^{i\pi x}\overline{q_2(x)}]\widehat{u}(x/2),$$

and therefore

$$\widehat{z}_2(2x) = 2^{-1/2}e^{-i2\pi x}e^{i\alpha(2x)}[\overline{q_1(2x)} - e^{i2\pi x}\overline{q_2(2x)}]\widehat{u}(x) = e^{-i2\pi x}\mu(2x)\overline{p(x + 1/2)}\widehat{u}(x),$$

where $\mu(x) := e^{i\alpha(x)}$ is unimodular and $\mathbb{Z}$-periodic.

**Example 2**. Let

$$\mathbf{A} := \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

and let $\phi(\mathbf{t})$ be the characteristic function of $[0, 1] \times [0, 1]$. Gröchenig and Madych [1] have shown that $\phi$ is a scaling function of an MRA generated by the dilation matrix $\mathbf{A}$ and that the function $\psi$ defined by

$$\psi(\mathbf{t}) := \begin{cases} 1 & \text{if } \mathbf{t} \in [0,1] \times [0,1/2] \\ -1 & \text{if } \mathbf{t} \in [0,1] \times [1/2,1] \\ 0 & \text{otherwise} \end{cases}$$

is a wavelet associated with this MRA. Let us see how this assertion follows from Theorem 1.

Since $\{(0,0)^T, (1,0)^T\}$ is a a full collection of representatives of $A/A\mathbb{Z}^2$, from Lemma C we deduce that if $v_1(\mathbf{t}) := 2^{-1/2}\phi(A\mathbf{t})$ and $v_2(\mathbf{t}) := 2^{-1/2}\phi(A\mathbf{t}+(1,0)^T)$, then $T(v_1, v_2)$ is an orthonormal basis of $S(A, \phi)$, and a straightforward computation shows that

$$\phi(\mathbf{t}) = 2^{-1/2}\left(v_1(\mathbf{t} - (1,0)^T) + v_2(\mathbf{t} - (1,1)^T\right),$$

which implies that if $\mathbf{x} = (x_1, x_2)^T$, then

$$\widehat{\phi}(\mathbf{x}) = 2^{-1/2}\left(e^{-i2\pi x_1}\widehat{v}_1(\mathbf{x}) + e^{-i2\pi(x_1+x_2)}\widehat{v}_2(\mathbf{x})\right).$$

Thus $q_1(\mathbf{x}) = 2^{-1/2}e^{-i2\pi x_1}$, $q_2(\mathbf{x}) = 2^{-1/2}e^{-i2\pi(x_1+x_2)}$ and $\alpha(\mathbf{x}) = i2\pi x_1$, and proceeding as in Example 1 we see that

$$w_{2,1}(\mathbf{x}) = -\overline{q_2(x)}e^{i\alpha(x)} = 2^{-1/2}e^{-i2\pi x_2} \quad \text{and} \quad w_{2,2}(\mathbf{x}) = |q_1(\mathbf{x})| = 2^{-1/2}.$$

Thus,

$$\widehat{z_2}(\mathbf{x}) = w_{2,1}(\mathbf{x})\widehat{v_1}(\mathbf{x}) + w_{2,2}(\mathbf{x})\widehat{v_2}(\mathbf{x}) = 2^{-1/2}\left(\widehat{v_2}(\mathbf{x}) - e^{-i2\pi x_2}\widehat{v_1}(\mathbf{x})\right),$$

which by Theorem 1 implies that $\sigma(\mathbf{t})$ is a wavelet associated with $A$ if and only if there is a measurable unimodular $\mathbb{Z}^2$–periodic function $\mu(\mathbf{x})$ such that

$$\widehat{\sigma}(\mathbf{x}) = \mu(\mathbf{x})\widehat{z_2}(\mathbf{x}).$$

In particular, $\widehat{\psi}(\mathbf{x}) = e^{-i2\pi x_1}\widehat{z_2}(\mathbf{x})$.

**Example 3**. Gröchenig and Madych have also shown in [1] that the characteristic function $\phi$ of $[0,1]\times[0,1]$ which we considered in the previous example is also a scaling function of an MRA generated by the dilation matrix

$$\mathbf{A} := 2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $a = 4$, from e.g. [5, Theorem H] we know that any orthonormal wavelet associated with this MRA has exactly three generators.. Let us construct an orthonormal wavelet basis using Theorem 1. The vectors $j_1 := (0,0)^T, j_2 := (1,0)^T, j_3 := (0,1)^T$ and $j_4 := (1,1)^T$ are a full collection of representatives of $A/A\mathbb{Z}^2$. Let

$$v_k(\mathbf{t}) := 2\phi(A\mathbf{t} + j_k) = 2\phi(2\mathbf{t} + j_k).$$

Lemma C implies that $T(v_1, v_2, v_3, v_4)$ is an orthonormal basis of $S(A, \phi)$. Moreover, it is easily verified that

$$\phi(\mathbf{t}) = \sum_{k=1}^{4}\phi(2\mathbf{t} - j_k) = (1/2)\sum_{k=1}^{4} v_k(\mathbf{t} - j_k).$$

Since

$$\mathfrak{F}\{v_k(\cdot - j_k)\}(\mathbf{x}) = e^{-i2\pi\mathbf{x}\cdot j_k}\widehat{v_k}(\mathbf{x})$$

we see that

$$\widehat{\phi}(\mathbf{x}) = (1/2)\sum_{k=1}^{4} e^{-i2\pi\mathbf{x}\cdot j_k}\widehat{v_k}(\mathbf{x}),$$

and therefore

$$q_k(\mathbf{x}) = (1/2)e^{-i2\pi\mathbf{x}\cdot j_k}, k = 1, \ldots 4.$$

Since $\alpha(\mathbf{x}) = 0$, (5) implies that

$$w_{r,k}(\mathbf{x}) := \begin{cases} \frac{1}{2}e^{-i2\pi\mathbf{x}\cdot j_k} & \text{if } r = 1, 1 \le k \le 4 \\ \\ -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_r} & \text{if } 1 < r \le 4, k = 1 \\ \\ -\frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_k - j_r)} & \text{if } 1 < r \le 4, 1 < k \le 4, k \ne r. \\ \\ \frac{5}{6} & \text{if } 1 < r \le 4, 1 < k \le 4, k = r. \end{cases}$$

Thus,

$$\widehat{z}_2(\mathbf{x}) = -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_2}\widehat{v}_1(\mathbf{x}) + \frac{5}{6}\widehat{v}_2(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_3-j_2)}\widehat{v}_3(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_4-j_2)}\widehat{v}_4(\mathbf{x})$$

$$\widehat{z}_3(\mathbf{x}) = -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_3}\widehat{v}_1(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_2-j_3)}\widehat{v}_2(\mathbf{x}) + \frac{5}{6}\widehat{v}_3(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_4-j_3)}\widehat{v}_4(\mathbf{x})$$

and

$$\widehat{z}_4(\mathbf{x}) = -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_4}\widehat{v}_1(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_2-j_4)}\widehat{v}_2(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_3-j_4)}\widehat{v}_3(\mathbf{x}) + \frac{5}{6}\widehat{v}_4(\mathbf{x}).$$

i.e.,

$$z_2(t) = -\frac{1}{2}v_1(\mathbf{t} + j_2) + \frac{5}{6}v_2(\mathbf{t}) - \frac{1}{6}v_3(\mathbf{t} + (j_3 - j_2)) - \frac{1}{6}v_4(\mathbf{t} + (j_4 - j_2)),$$

$$z_3(\mathbf{t}) = -\frac{1}{2}v_1(\mathbf{t} + j_3) - \frac{1}{6}v_2(\mathbf{t} + (j_2 - j_3)) + \frac{5}{6}v_3(\mathbf{t}) - \frac{1}{6}v_4(\mathbf{t} + (j_4 - j_3)),$$

and

$$z_4(\mathbf{t}) = -\frac{1}{2}v_1(\mathbf{t} + j_4) - \frac{1}{6}v_2(\mathbf{t} + (j_2 - j_4)) - \frac{1}{6}v_3(\mathbf{t} + (j_3 - j_4)) + \frac{5}{6}v_4(\mathbf{t}).$$

Applying Theorem 1 we conclude that $\{z_2, z_3, z_4\}$ is an orthonormal wavelet system associated with the dilation matrix $\mathbf{A}$, and that $\{\psi_1, \psi_2, \psi_3\}$ is an orthonormal wavelet system associated with $\mathbf{A}$ if and only if there exists a $3\times 3$ unitary matrix function $\mathbf{U}(x)$ such that

$$(\widehat{\psi}_1(\mathbf{x}), \widehat{\psi}_2(\mathbf{x}), \widehat{\psi}_3(\mathbf{x}))^T = \mathbf{U}(\mathbf{x})(\widehat{z}_2(\mathbf{x}), \widehat{z}_3(\mathbf{x}), \widehat{z}_4(\mathbf{x}))^T.$$

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