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Proof. The definitions imply that ψ is a semiorthogonal wavelet if and only if $\psi \in W_0$. Let

$$r(x) := \left(\sum_{k\in\mathbb{Z}} |\widehat{\psi}(x+2k\pi)|^2 \right)^{1/2}$$

and

$$\widehat{h}(x) := \widehat{\psi}(x)/r(x).$$

Then, as remarked in [3, p. 78], h is an orthonormal wavelet. Since $1/r(x)$ is 2π–periodic and bounded, it follows that $h \in W_0$. Thus h has a representation of the form (1). Setting $\mu(x) := r(x)\nu(x)$, we readily see that (6) and (7) are satisfied.

Conversely, assume there is a 2π–periodic function $\mu(x)$ such that (6) and (7) are satisfied. Let

$$\operatorname{sign}\mu(x) := \begin{cases} \mu(x)/|\mu(x)| & \text{if } \mu(x) \neq 0 \\ 1 & \text{if } \mu(x) = 0 \end{cases} .$$

Setting

$$\widehat{h}_1(2x) := e^{ix}\operatorname{sign}(\mu(x))\overline{p(x+\pi)}\widehat{\varphi}(x),$$

we conclude from Theorem A that h_1 is an orthonormal wavelet. Moreover, (6) implies that

$$\widehat{\psi}(x) = |\mu(x)|\widehat{h}_1(x) \qquad \text{a.e.}$$

Since

$$|\,|\mu(x)| - \sqrt{B}| \le \sqrt{B} - \sqrt{A} < \sqrt{B} \qquad \text{a.e.},$$

we deduce from Proposition 2 that ψ is a Riesz wavelet. Indeed, since $A' = 1$ and $B' = 1$ are Riesz bounds for h_1, setting $C = \sqrt{B} - \sqrt{A}$ and $D = \sqrt{B}$, we see that $0 < C < |D|\sqrt{A'/B'}$ and

$$|\,|\mu(x)| - D| \le C \qquad \text{a.e.}$$

Proposition 2 does not yield sufficiently accurate information about the Riesz bounds of ψ. However, from Proposition 1 we readily see that ψ is a frame wavelet with lower bound A and upper bound B. Since frame bounds and Riesz bounds coincide, we conclude that A and B are a lower and an upper Riesz bound of ψ respectively. □

Clearly Theorem A is a particular case of Theorem 3. From Theorem A and Theorem 3 we obtain

Corollary 2. *A function ψ is a semiorthogonal MRA Riesz wavelet with bounds $0 < A \le B$ if and only if there is a measurable 2π–periodic function $\mu(x)$ that satisfies (7), and an MRA orthonormal wavelet h, such that*

$$\widehat{\psi}(x) = \mu(x)\widehat{h}(x) \qquad \textit{a.e.}$$

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Let $f \in W_r$. The hypotheses imply there is a sequence $\{d_{j,k}; j,k \in \mathbb{Z}\} \in \ell^2(\mathbb{Z}^2)$ such that $f = \sum_{j,k\in\mathbb{Z}} d_{j,k}\psi_{j,k}$ (in the L^2 sense). Define $f_1 := \sum_{k\in\mathbb{Z}} d_{0,k}\psi_{0,k}$ and $f_2 = f - f_1$. Since $f - f_1 \in W_r$ and $f_2 \in W_r^{\perp}$, we conclude that $f_2 = 0$; thus, $\overline{\operatorname{span}\{\psi(\cdot - k); k \in \mathbb{Z}\}} = W_r$. Since $h_{r,0} \in W_r$, this implies that there is a 2π–periodic function $m(x)$ such that

$$\widehat{h_{r,0}}(x) = m(x)\widehat{\psi}(x) \qquad \text{a.e.}$$

Thus Theorem B implies that

$$2^{-r} = 2^{-r}\sum_{k\in\mathbb{Z}} |\widehat{h}(2^{-r}x + 2k\pi)|^2 = |m(x)|^2 \sum_{k\in\mathbb{Z}} |\widehat{\psi}(x + 2^{r+1}k\pi)|^2 \leq |m(x)|^2 B \quad \text{a.e.}$$

On the other hand, since

$$\sum_{k\in\mathbb{Z}} |\widehat{h}(x + 2^{1-r}k\pi)|^2 = \sum_{\ell=0}^{2^r-1}\sum_{n\in\mathbb{Z}} |\widehat{h}(x + 2^{1-r}\ell\pi + 2n\pi)|^2 = 2^r \qquad \text{a.e.},$$

we see that

$$1 = 2^{-r}\sum_{k\in\mathbb{Z}} |\widehat{h}(2^{-r}x + 2^{1-r}k\pi)|^2 = |m(x)|^2 \sum_{k\in\mathbb{Z}} |\widehat{\psi}(x + 2k\pi)|^2 \geq |m(x)|^2 A \qquad \text{a.e.}$$

We therefore conclude that

$$2^{-r}B^{-1} \leq |m(x)|^2 \leq A^{-1} \qquad \text{a.e.},$$

and Proposition 1 implies that $h_{r,0}$ is a frame wavelet. This implies that

$$\{2^{j/2}h(2^j t - 2^r k); j,k \in \mathbb{Z}\} \tag{4}$$

is a frame wavelet. Since

$$\{2^{j/2}h(2^j t - k); j,k \in \mathbb{Z}\} \tag{5}$$

is a Riesz basis by hypothesis and (4) is a subsequence of (5), this is incompatible with the definition of Riesz basis, and we have obtained a contradiction. □

We are now ready to prove

Theorem 3. *Let φ be a scaling function for an MRA $M = \{V_j; j \in \mathbb{Z}\}$, and let p be the associated low pass filter. The following propositions are equivalent:*
(a) ψ is a semiorthogonal Riesz wavelet associated with M with bounds $0 < A \leq B$.
(b) There is a measurable 2π–periodic function $\mu(x)$ such that

$$\widehat{\psi}(2x) = e^{ix}\mu(2x)\overline{p(x+\pi)}\widehat{\varphi}(x) \qquad \textit{a.e.} \tag{6}$$

and

$$A \leq |\mu(x)|^2 \leq B \qquad \textit{a.e.} \tag{7}$$

Proof. We proceed by contradiction. Assume ψ is a Riesz wavelet in V_j with bounds $0 < A \le B$, and let φ be a scaling function for M with associated low pass filter p. Since $\psi \in V_0$, there is a 2π–periodic function $s(x)$ such that

$$\widehat{\psi}(x) = s(x)\widehat{\varphi}(x) \qquad \text{a.e.}$$

This implies that

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi}(x+2k\pi)|^2 = |s(x)|^2 \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(x+2k\pi)|^2 \qquad \text{a.e.,}$$

whence from Theorem B we conclude that

$$A \le |s(x)|^2 \le B \qquad \text{a.e.}$$

Moreover, the well known identity

$$|p(x)|^2 + |p(x+\pi)|^2 = 1$$

implies that $|p(x)| \le 1$. Thus, if

$$m(x) := \frac{s(2x)p(x)}{s(x)},$$

we infer that $|m(x)|^2 \le B/A$. Since

$$\widehat{\psi}(2x) = s(2x)p(x)\widehat{\varphi}(x) = m(x)\widehat{\psi}(x)$$

and $m \in L^2(0, 2\pi)$ and is 2π–periodic, we conclude that $\psi(t/2)$ is in the closure of the linear span of the sequence $\{\psi(t-k); k \in \mathbb{Z}\}$. This is incompatible with the assumption that $\{\psi_{j,k}; j,k \in \mathbb{Z}\}$ is a Riesz basis, and we have reached a contradiction. □

Corollary 1. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be an MRA. If $j \ne 0$ there is no function u such that $\{u(t-k); k \in \mathbb{Z}\}$ is a Riesz basis of V_j.*

Proof. If $\{u(t-k); k \in \mathbb{Z}\}$ is a Riesz basis of V_j then there is a function v such that $\{v(t-k); k \in \mathbb{Z}\}$ is an orthonormal basis of V_j [6, pp. 48–51]. Setting $U_r := V_{r+j}$ we see that $\{U_r; r \in \mathbb{Z}\}$ is an MRA. If W_0' denotes the orthogonal complement of U_0 in U_1, then there is an orthonormal (and therefore Riesz) wavelet h' in $W_0' \subset U_1 = V_{j+1}$. If $j < 0$ this implies that V_0 contains an orthonormal wavelet, which is a contradiction. On the other hand if $j > 0$, let h be an orthonormal wavelet in W_0. Since $W_0 \subset V_1 = U_{1-j} \subset U_0$, we see that U_0 contains an orthonormal wavelet, and we have a contradiction in this case as well. □

Theorem 1 is supplemented by the following

Theorem 2. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be an MRA, and assume that $\psi \in W_r$ for some $r \ne 0$. Then ψ cannot be a Riesz wavelet.*

Proof. Since

$$V_0 = \bigoplus_{j<0} W_j,$$

the assertion for $r < 0$ follows from Theorem 1. Assume now that $r > 0$. Let $h \in W_0$ be an orthonormal wavelet, and let ψ be a Riesz wavelet in W_r with bounds A and B. Then $\psi_{-r,0} \in W_0$. From, e. g., [6, p. 57, Lemma 2.11] we know that W_0 is closed under integral translations (this can also be easily obtained from the definition of W_0). Since $r > 0$, we deduce that $\psi_{-r,k} \in W_0$ for every $k \in \mathbb{Z}$. Thus $\psi_{j,k} \in W_{j+r}$ for all $j,k \in \mathbb{Z}$.

Proposition 1. *Let $v \in L^2(\mathbb{R})$ and let $\mu(x)$ be a measurable 2π–periodic function. Let ψ be a function such that*

$$\widehat{\psi}(x) = \mu(x)\widehat{v}(x) \qquad a.e. \tag{2}$$

Then:

(a) *If there are constants $B, D > 0$ such that $|\mu(x)|^2 \leq B$. a. e. and v is a Bessel wavelet with bound D, then ψ is a Bessel wavelet with bound BD.*
(b) *If there are constants $0 < A \leq B, 0 < C \leq D$ such that $A \leq |\mu(x)|^2 \leq B$ a.e., and v is a frame wavelet with bounds C and D, then ψ is a frame wavelet with bounds AC and BD.*

Proof. We will prove (a) only. The proof of (b) will then reduce to finding the lower bound, which is done in a similar way.

Let $f \in L^2(\mathbb{R})$. By periodization, as in e.g. [4, p. 270, (2.2)] we have:

$$\sum_{j,k\in\mathbb{Z}} |\langle f, v_{j,k}\rangle|^2 = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} 2^j \int_0^{2\pi} \left| \sum_{r\in\mathbb{Z}} \widehat{f}(2^j(x+2r\pi))\overline{\widehat{v}(x+2r\pi)}\right|^2 dx. \tag{3}$$

Applying (3), we also have:

$$\begin{aligned}\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 &= \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} 2^j \int_0^{2\pi} \left| \sum_{r\in\mathbb{Z}} \widehat{f}(2^j(x+2r\pi))\overline{\widehat{\psi}(x+2r\pi)}\right|^2 dx \\ &= \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} 2^j \int_0^{2\pi} \left| \overline{\mu(x)} \sum_{r\in\mathbb{Z}} \widehat{f}(2^j(x+2r\pi))\overline{\widehat{v}(x+2r\pi)}\right|^2 dx \\ &\leq B\frac{1}{2\pi}\sum_{j\in\mathbb{Z}} 2^j \int_0^{2\pi} \left| \sum_{r\in\mathbb{Z}} \widehat{f}(2^j(x+2r\pi))\overline{\widehat{v}(x+2r\pi)}\right|^2 dx.\end{aligned}$$

Since

$$\sum_{j,k\in\mathbb{Z}} |\langle f, v_{j,k}\rangle|^2 \leq D\|f\|^2,$$

the assertion follows from (3). □

We now turn to Riesz wavelets.

Proposition 2. *Let $\mu(x)$ be a measurable 2π–periodic function, and let $v \in L^2(\mathbb{R})$ be a Riesz wavelet with bounds $0 < A \leq B$. If ψ is defined by (2) and there are constants C and D, $0 < C < |D|\sqrt{A/B}$ such that*

$$|\mu(x) - D| \leq C \qquad a.e.,$$

then ψ is a Riesz wavelet.

Proof. Let $u := \psi - D\,v$. From (2), $\widehat{u}(x) = (\mu(x) - D)\widehat{v}(x)$. Since v is a Riesz wavelet with upper bound B, applying Proposition 1 we conclude that u is a Bessel wavelet with bound $C^2B < |D|^2A$. Since $D\,v$ is a Riesz wavelet with lower bound $|D|^2A$, the assertion follows from [5, Theorem 5]. □

Theorem 1. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be an MRA, and assume that $\psi \in V_j$ for some $j \leq 0$. Then ψ cannot be a Riesz wavelet.*

ψ we mean the frame, Bessel, or Riesz bounds, as the case may be, of the affine sequence generated by ψ. A Riesz wavelet ψ is said to be *semiorthogonal* if for every $j, k, \ell, m \in \mathbb{Z}$, whenever $j \neq \ell$,

$$\langle \psi_{j,k}, \psi_{\ell,m} \rangle = 0.$$

A *multiresolution analysis* (MRA) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R})$ such that:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.

(ii) For every $j \in \mathbb{Z}, f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$.

(iii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$.

(iv) There is a function φ such that $\{\varphi(t-k); k \in \mathbb{Z}\}$ is an orthonormal basis of V_0.

It follows from the preceding definition that there is a 2π–periodic function $p \in L^2[0, 2\pi]$ such that

$$\widehat{\varphi}(2x) = p(x)\widehat{\varphi}(x) \qquad \text{a.e.}$$

The function φ is called a *scaling function* for the MRA, and p is called the *low pass filter* associated with φ [6, p. 53].

By W_r we will denote the orthogonal complement of V_r in V_{r+1}. Thus, $V_{r+1} = V_r \oplus W_r$.

The following results will be used in the sequel:

Theorem A [6, p. 57]. *If φ is a scaling function for an MRA $\{V_j; j \in \mathbb{Z}\}$ and p is the associated low pass filter, then h is an orthonormal wavelet in W_0 if and only if there is a measurable unimodular and 2π–periodic function $\nu(x)$, such that*

$$\widehat{h}(2x) = e^{ix}\, \nu(2x)\, \overline{p(x+\pi)}\, \widehat{\varphi}(x) \qquad \text{a.e.} \tag{1}$$

Theorem B [6, pp. 49–50]. *Let $\varphi \in L^2(\mathbb{R})$ and constants $0 < A \leq B$ be given. Then the sequence $\{\varphi(x-k); k \in \mathbb{Z}\}$ is a Riesz basis of the closure of its linear span and has bounds A and B, if and only if*

$$A \leq \sum_{k \in \mathbb{Z}} |\widehat{\varphi}(x + 2k\pi)|^2 \leq B \qquad \text{a.e.}$$

In particular, $\{\varphi(x-k); k \in \mathbb{Z}\}$ is an orthonormal basis of the closure of its linear span if and only if

$$\sum_{k \in \mathbb{Z}} |\widehat{\varphi}(x + 2k\pi)|^2 = 1 \qquad \text{a.e.}$$

Let ψ be a frame wavelet in $L^2(\mathbb{R})$; for $j \in \mathbb{Z}$, let P_j denote the closure of the linear span of $\{\psi_{j,k}; k \in \mathbb{Z}\}$, and let $V_j := \sum_{r<j} P_r$. Note that $\psi \in V_1$. We say that ψ is *associated* with an MRA, or that ψ is an MRA wavelet, if $M := \{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis; we also say that ψ is associated with M.

We begin the statement and proof of our results with the following proposition, of some independent interest:

ON MRA RIESZ WAVELETS

R. A. ZALIK

Abstract. We investigate the properties of univariate MRA Riesz wavelets. In particular we obtain a generalization to semiorthogonal MRA wavelets, of a well-known representation theorem for orthonormal MRA wavelets.

In what follows $\mathbb{Z}$ will denote the integers and $\mathbb{R}$ the real numbers; t and x will always denote real variables. The Fourier transform of a function f will be denoted by $\widehat{f}$. If $f \in L(\mathbb{R})$,

$$\widehat{f}(x) := \int_{\mathbb{R}} e^{-txi} f(t)\, dt.$$

Let $\mathbb{H}$ be a (separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| := \langle \cdot, \cdot \rangle^{1/2}$. A sequence $\{f_k, k \in \mathbb{Z}\} \subset \mathbb{H}$ is called a *frame* if there are constants $0 < A \leq B$ such that for every $f \in \mathbb{H}$

$$A\|f\|^2 \leq \sum_{k \in \mathbb{Z}} |\langle f, f_k \rangle|^2 \leq B\|f\|^2.$$

The constants A and B are called (lower and upper) *bounds* of the frame. If only the right-hand inequality in the preceding displayed formula is satisfied for all $f \in \mathbb{H}$, then $\{f_k, k \in \mathbb{Z}\}$ is called a *Bessel sequence* with bound B. A sequence $\{f_k, k \in \mathbb{Z}\} \subset \mathbb{H}$ is called a *Riesz basis* if its linear span is dense in $\mathbb{H}$ and there are constants A and B, $A > 0$, such that for every sequence $\{c_k, k \in \mathbb{Z}\} \subset \ell^2$,

$$A \sum_{k \in \mathbb{Z}} |c_k|^2 \leq \left\| \sum_{k \in \mathbb{Z}} c_k f_k \right\|^2 \leq B \sum_{k \in \mathbb{Z}} |c_k|^2.$$

The constants A and B are called (lower and upper) *bounds* of the Riesz basis. Every Riesz basis is a frame, every orthonormal basis is a Riesz basis with bounds $A = B = 1$, and Riesz bounds and frame bounds coincide. A sequence $\{f_k, k \in \mathbb{Z}\} \subset \mathbb{H}$ is a Riesz basis if and only if it is a frame having the additional property that upon the removal of any element from the sequence, it ceases to be a frame [1, 2, 7].

In this paper the underlying Hilbert space will be $L^2(\mathbb{R})$ with the usual inner product and norm, and we will study binary affine sequences generated by a single function, i. e. sequences of the form $\{\psi_{j,k}; j, k \in \mathbb{Z}\}$, where $\psi \in L^2(\mathbb{R})$ and $\psi_{j,k}(t) := 2^{j/2}\psi(2^j t - k)$. A function $\psi \in L^2(\mathbb{R})$ will be called a frame wavelet, a Bessel wavelet, a Riesz wavelet, or an orthonormal wavelet if the affine sequence $\{\psi_{j,k}; j, k \in \mathbb{Z}\}$ it generates is respectively a frame, a Bessel sequence, a Riesz basis, or an orthonormal basis of $L^2(\mathbb{R})$. When we refer to the bounds of a wavelet

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