A FAMILY OF NONSEPARABLE SCALING FUNCTIONS AND COMPACTLY SUPPORTED TIGHT FRAMELETS

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Abstract. Given integers b and d, with $d > 1$ and $|b| > 1$, we construct even nonseparable compactly supported refinable functions with dilation factor b that generate multiresolution analyses on $L^2(\mathbb{R}^d)$. These refinable functions are nonseparable, in the sense that they cannot be expressed as the product of two functions defined on lower dimensions. We use these scaling functions and a slight generalization of a theorem of Lai and Stöckler to construct smooth compactly supported tight framelets. Both the refinable functions and the framelets they generate can be made as smooth as desired. Estimates for the supports of these refinable functions and framelets, are given.

1. Introduction

Tight wavelet frames have recently become the focus of increased interest because they can be computed and applied just as easily as orthonormal wavelets, but are easier to construct. Moreover, since frames may have redundant terms, they are better suited for data transmission.

Multivariate wavelets obtained by tensor products, not being anisotropic, may introduce distortions in the processing of images. The purpose of this paper is to find nonseparable tight wavelet frames with properties that may make them better suited for image processing and other applications.

Han [11], and independently Ron and Shen [23], found necessary and sufficient conditions for translates and dilates of a set of functions to be a tight wavelet frame. Ron and Shen also formulated what is known as the Unitary Extension Principle (UEP), which, in addition to its other applications, allowed Gröchenig and Ron to construct, for any dilation matrix, compactly supported tight wavelet frames with any desired degree of smoothness (see [10]). Articles studying nonseparable frames or orthogonal wavelets include Ayache [1, 2, 3, 4], Karoui [15], Kovačević and Vetterli [17], Lai [19], Li [21], and Yang and Xue [30]. A closed form representation valid for MRA orthonormal wavelets in $L^2(\mathbb{R}^d)$ was obtained by Zalik. It follows from [31, Theorem 3] and [32, Theorem 3 and Theorem 9]. No such representation is currently known for tight wavelet frames. We point out that there is a typographical error in [31, Theorem 3 and Theorem 4]: where it says *orthogonal matrix* should say *unitary matrix*. In addition to the well–known work of Daubechies, the construction of compactly supported orthonormal wavelets was also studied by Lai [18] and

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Yang and Xue [29]. The construction of orthonormal wavelets can be seen as a matrix extension problem (see for example Han and Zhuang [12, 13] and references thereof), and the present article can be interpreted as the solution of a matrix extension problem for constructing compactly supported tight framelets.

With the definition of the Fourier transform that we shall adopt in the next section, the UEP may be formulated as follows:

Theorem A. *Let $B \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$. Let $\phi \in L^2(\mathbb{R}^d)$ be compactly supported and refinable, i.e.*

$$\widehat{\phi}(B^T\mathbf{t}) = P(\mathbf{t})\widehat{\phi}(\mathbf{t}),$$

where $P(\mathbf{t})$ is a trigonometric polynomial. Assume moreover that $\widehat{\phi}(\mathbf{0}) = 1$, $|\widehat{\phi}(\mathbf{t})| \leq C(1+|\mathbf{t}|)^{-\alpha}$ for some $\alpha > d/2$, and that there are trigonometric polynomials or rational functions Q_ℓ, $\ell = 1, \cdots, N$, that satisfy the condition

$$P(\mathbf{t})\overline{P(\mathbf{t}+\mathbf{j})} + \sum_{\ell=1}^{N} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} = \begin{cases} 1 & if \quad \mathbf{j} = \mathbf{0}, \\ 0 \quad if & \mathbf{j} \in \left((B^T)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d\right) \setminus \mathbb{Z}^d. \end{cases} \tag{1}$$

If

$$\widehat{\psi_\ell}(B^T\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad \ell = 1, \ldots, N,$$

then $\Psi = \{\psi_1, \ldots, \psi_N\}$ is a tight wavelet frame in $L^2(\mathbb{R}^d)$ with frame constant equal to 1.

This version of the UEP is a consequence of [8, Proposition 1.9], except for the value of the frame constant which follows from, e.g. [23, Theorem 6.5].

In [20], Lai and Stöckler provided two methods for constructing compactly supported tight wavelet frames in $L^2(\mathbb{R}^d)$ for the dilation matrix $2I_{d\times d}$. The first method is for refinable functions whose masks P are quadrature mirror filters, whereas the second method is for masks that satisfy the so–called sub-QMF condition:

Theorem B. ([20, Theorem 3.4]) *Let $B = 2I_{d\times d}$, suppose that $P(\mathbf{x})$ is a polynomial that satisfies the condition*

$$\sum_{\boldsymbol{\gamma}\in\{0,1/2\}^d} |P(\mathbf{t}+\boldsymbol{\gamma})|^2 \leq 1,$$

and assume that there are nonnegative polynomials $\widetilde{P}_j(\mathbf{t})$, $j = 1, \cdots, M$, such that

$$\sum_{\boldsymbol{\gamma}\in\{0,1/2\}^d} |P(\mathbf{t}+\boldsymbol{\gamma})|^2 + \sum_{j=1}^{M} |\widetilde{P}_j(2\mathbf{t})|^2 = 1.$$

Then there exist $2^d + M$ trigonometric polynomials Q_ℓ. $\ell = 1, \cdots, 2^d + M$, such that P and the polynomials Q_ℓ satisfy (1).

Lai and Stöckler also gave a number of examples using multivariate box splines. Similar results for vector–valued frames were obtained by Charina, Chui and He [6].

In the present paper we use Theorem A and a generalization of Theorem B to construct another family of tight wavelet frames with compact support, for systems with dilation matrix $bI_{d\times d}$. For all the members of this family, the frame generators are nonseparable and their masks may be obtained by spectral factorization. Also, estimates for their supports and degrees of smoothness are given. We should also point out that all members of this family have the same number of generators for each given dimension d: $|b|^d + 2d$. Thus, in contrast to [24, 25], this number is independent of the degree of smoothness of the frame generators. Since the number of generators for orthonormal wavelets associated with an MRA with dilation matrix $bI_{d\times d}$ is $|b|^d - 1$, we see that $|b|^d + 2d$ is asymptotically close for large values of d.

The remainder of this paper is organized as follows. The next section summarizes the notation and definitions that will be used. Section 3 is devoted to the construction of compactly supported nonseparable smooth scaling functions in a b-MRA in $L^2(\mathbb{R}^d)$, $d > 1$. In Section 4 we construct smooth tight wavelet frames of compactly supported functions with any dilation factor b, from the scaling functions defined in Section 3.

2. Notation and Definitions

The sets of strictly positive integers, integers, and real numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ respectively, and the set of complex numbers by $\mathbb{C}$. Given $c \in \mathbb{R}$ and $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we define $c\mathbf{x} := (cx_1, cx_2, \dots, cx_d)$ and $c(\mathbb{Z}) := \{ck : k \in \mathbb{Z}\}$. Given $m \in \mathbb{N}$, we will also use the following notation: $\mathbf{\Delta}_m := \{0, 1, \dots, m-1\}$, and $\mathbf{\Gamma}_m := \{0, 1/m, \dots, (m-1)/m\}$. By $\mathbf{\Delta}_m^d$ we mean the d–fold cartesian product of $\mathbf{\Delta}_m$ with itself, and $\mathbf{\Gamma}_m^d$ is similarly defined; $\mathbf{t} = (t_1, \dots, t_d)$, $\mathbf{y} = (y_1, \dots, y_d)$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d)$, and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_d)$ will be vectors in $\mathbb{R}^d$, $\mathbf{z} = (z_1, \dots, z_d) = \mathbf{x} + i\mathbf{y} \in \mathbb{C}^d$, and $\mathbf{0}$ will denote the zero vector in $\mathbb{R}^d$. In what follows, b will always be an integer such that $m := |b| > 1$.

A function $\phi \in L^2(\mathbb{R}^d)$, $d > 1$ is said to be separable if there exist $a \in \mathbb{N}$, $1 \le a < d$, and two functions $\varphi \in L^2(\mathbb{R}^a)$ and $\theta \in L^2(\mathbb{R}^{d-a})$ such that ϕ can be expressed as

$$\phi(x_1, \cdots, x_a, x_{a+1}, \cdots, x_d) = \varphi(x_1, \cdots, x_a)\theta(x_{a+1}, \cdots, x_d) \tag{2}$$

If $\phi \in L^2(\mathbb{R}^d)$ is not a separable function, then it is called a nonseparable function.

A sequence $\{\phi_n\}_{n=1}^\infty$ of elements in a separable Hilbert space $\mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|h\|^2 \le \sum_{n=1}^\infty |\langle h, \phi_n\rangle|^2 \le C_2\|h\|^2, \qquad \forall h \in \mathbb{H},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{H}$. The constants C_1 and C_2 are called *frame bounds*. The definition implies that a frame is a complete set of elements in $\mathbb{H}$. A frame $\{\phi_n\}_{n=1}^\infty$ is *tight* if we can choose $C_1 = C_2$.

If there are constants $C_1, C_2 > 0$ such that for any sequence $\{\alpha_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^n)$

$$C_1 \sum_{\mathbf{k}\in\mathbb{Z}^d} |\alpha_{\mathbf{k}}|^2 \le \| \sum_{\mathbf{k}\in\mathbb{Z}^d} \alpha_{\mathbf{k}}\phi_k \|^2 \le C_2 \sum_{\mathbf{k}\in\mathbb{Z}^d} |\alpha_{\mathbf{k}}|^2,$$

then $\{\phi_n\}_{n=1}^\infty$ is called a *Riesz sequence*. It is a *Riesz basis* of the closure of its linear span.

Let $B \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$. A *multiresolution analysis* (MRA) in $L^2(\mathbb{R}^d)$ (generated by B) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:
(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.
(ii) For every $j \in \mathbb{Z}$, $f(\mathbf{x}) \in V_j$ if and only if $f(B\mathbf{x}) \in V_{j+1}$.
(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.
(iv) There exists a function $\phi \in V_0$, called a *scaling function*, such that $\{\ \phi(\cdot-\mathbf{k}); \mathbf{k} \in \mathbb{Z}^d\ \}$ is a Riesz basis for V_0.

Let $B \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$. A set of functions $\Psi = \{\psi_1, \dots, \psi_n\} \subset L^2(\mathbb{R}^d)$ is called a wavelet frame, or framelet, if the system

$$\{|\det B|^{j/2}\psi_\ell(B^j \cdot +\mathbf{k}); j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le n\}$$

is a frame for $L^2(\mathbb{R}^d)$. If this system is a tight frame for $L^2(\mathbb{R}^d)$ then Ψ is called a *tight framelet*.

Let $b \in \mathbb{Z}$ and $|b| > 1$. By an MRA with dilation factor b, or a b–MRA, in $L^2(\mathbb{R}^d)$, we mean a multiresolution analysis with dilation matrix $bI_{d\times d}$, where $I_{d\times d}$ denotes the identity matrix on $\mathbb{R}^d$.

The simplest way to construct an MRA in $L^2(\mathbb{R}^d)$ is by multiplying scaling functions of lower dimensions: assume that φ is a scaling function for a b–MRA in $L^2(\mathbb{R}^a)$, $1 \le a < d$, and that θ is a scaling function for a b–MRA in $L^2(\mathbb{R}^{d-a})$. Then the function ϕ defined as in (2) is the scaling function of a b–MRA in $L^2(\mathbb{R}^d)$.

If ϕ is a scaling function of an MRA in $L^2(\mathbb{R}^d)$, then conditions (i), (ii) ad (iv) imply that ϕ is *refinable*, i.e.,

$$\widehat{\phi}(B^T\mathbf{t}) = P(\mathbf{t})\widehat{\phi}(\mathbf{t}) \quad \text{a.e. on } \mathbb{R}^d, \tag{3}$$

where the *low pass filter* $P(\mathbf{t})$ is a $\mathbb{Z}^d$-periodic essentially bounded function.

Conversely, if ϕ is a refinable function and we define

$$V_j := \overline{\operatorname{span}}\{|\det B|^{j/2}\phi(B^j\mathbf{x} - \mathbf{k}); \mathbf{k} \in \mathbb{Z}\}, \quad \mathbf{j} \in \mathbb{Z}, \tag{4}$$

we see that conditions (i) and (ii) in the definition of MRA are satisfied. Thus, we say that a refinable function $\phi \in L^2(\mathbb{R}^d)$ *generates an MRA* if $\{\phi(\cdot - \mathbf{k}); \mathbf{k} \in \mathbb{Z}^d\}$ is a Riesz sequence and the subspaces V_j satisfy condition (iii).

We normalize the Fourier transform as follows:

$$\widehat{f}(\mathbf{t}) := \int_{\mathbb{R}^d} f(\mathbf{x})e^{-2\pi i\mathbf{x}\cdot\mathbf{t}}d\mathbf{x}.$$

3. Construction of smooth nonseparable scaling functions

In this section we construct smooth nonseparable compactly supported scaling functions of a b-MRA in $L^2(\mathbb{R}^d)$, $d > 1$. We begin with

Proposition 1. *Let $m \in \mathbb{N}$ and*

$$p(t) := \frac{1}{m}\sum_{\delta\in\mathbf{\Delta}_m} e^{-2\pi i\delta t} = \frac{1}{m}\frac{e^{-2\pi imt} - 1}{e^{-2\pi it} - 1}, \tag{5}$$

let $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$, and let the trigonometric polynomial $P(\mathbf{t})$ in $\mathbb{R}^d$ be defined by

$$P(\mathbf{t}) := \frac{1-i}{2}\left(\prod_{j=1}^{d}|p(t_j)|^{2n_1} + i\prod_{j=1}^{d}|p(t_j)|^{2n_2}\right). \tag{6}$$

Then

$$\sum_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}_m^d}|P(\mathbf{t}+\boldsymbol{\gamma})|^2 = \frac{1}{2}\prod_{j=1}^{d}\left(\sum_{\gamma\in\boldsymbol{\Gamma}_m}|p(t_j+\gamma)|^{4n_1}\right)+\frac{1}{2}\prod_{j=1}^{d}\left(\sum_{\gamma\in\boldsymbol{\Gamma}_m}|p(t_j+\gamma)|^{4n_2}\right)\leq 1. \tag{7}$$

Moreover,

$$\sum_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}_m^d}|P(\mathbf{t}+\boldsymbol{\gamma})|^2 < 1,$$

except for a countable set of points.

Proof. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d)$ and $\widetilde{\boldsymbol{\gamma}} = (\gamma_2, \dots, \gamma_d)$. We have

$$\begin{aligned}\sum_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}_m^d}|P(\mathbf{t}+\boldsymbol{\gamma})|^2 &= \sum_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}_m^d}\frac{1}{2}\left(\prod_{j=1}^{d}|p(t_j+\gamma_j)|^{4n_1}+\prod_{j=1}^{d}|p(t_j+\gamma_j)|^{4n_2}\right)\\ &= \sum_{\gamma\in\boldsymbol{\Gamma}_m}|p(t_1+\gamma)|^{4n_1}\left(\sum_{\widetilde{\boldsymbol{\gamma}}\in\boldsymbol{\Gamma}_m^{d-1}}\frac{1}{2}(\prod_{j=2}^{d}|p(t_j+\gamma_j)|^{4n_1})\right)\\ &\quad+\sum_{\gamma\in\boldsymbol{\Gamma}_m}|p(t_1+\gamma)|^{4n_2}\left(\sum_{\widetilde{\boldsymbol{\gamma}}\in\Gamma_m^{d-1}}\frac{1}{2}(\prod_{j=2}^{d}|p(t_j+\gamma_j)|^{4n_2})\right).\end{aligned}$$

Repeating this procedure a finite number of times we obtain the desired identity in (7), and the inequality follows from (5).

Finally note that (5) also implies that $p(t) = 1$ if and only if $t \in \mathbb{Z}$. This implies that the left–hand side of (7) is strictly less than 1 if and only if $\mathbf{t} \notin \mathbb{Z}^d$. □

The following statement may be found in ([28, Appendix A.2]). The proof is straightforward and will be omitted.

Lemma C. *Let C^0 be the class of continuous functions in $L^2(\mathbb{R}^d)$, and let C^r, $r = 1, 2, \dots$ be the class of functions f such that all partial derivatives of f of order not greater than r are continuous and in $L^2(\mathbb{R}^d)$. If*

$$|\widehat{f}(\mathbf{t})| \leq C(1+|\mathbf{t}|)^{-N-\varepsilon}$$

for some integer $N \geq d$ and $\varepsilon > 0$, then f is in C^{N-d}.

We now prove.

Proposition 2. *Let $b \in \mathbb{Z}$, assume that $m := |b| > 1$, and let P be defined by (6) with $n_1 < n_2$. Then the functional equation (3) has a solution ϕ that has the following properties: it is even, continuous, and square–integrable on $\mathbb{R}^d$; if $2n_1 > d+1$ it is in the continuity class C^{2n_1-d-1}; it has support in $[-n_2, n_2]^d$. Moreover, $\widehat{\phi}(\mathbf{0}) = 1$ and $\{\ \phi(\cdot - \mathbf{k}); \mathbf{k} \in \mathbb{Z}^d\ \}$ is a Riesz sequence.*

Proof. For the dyadic case, the existence of a solution of (3) such that $\widehat{\phi}(\mathbf{0}) = 1$ follows from [5, Corollary 5.1]. The proof for arbitrary b is similar and will be omitted. That ϕ is even follows by observing that $P(t)$ is even and

$$\widehat{\phi}(\mathbf{t}) = \prod_{j=1}^{\infty} P(b^{-j}\mathbf{t}).$$

Thus $\widehat{\phi}$ is even, which implies that also ϕ is even.

Let $c(t)$ denote the characteristic function of the set $[0,1]$. Then

$$\widehat{c}(t) = e^{-\pi i t}\frac{\sin \pi t}{\pi t}.$$

Since

$$[0,1) = \bigcup_{k=0}^{m-1}\left[\frac{k}{m}, \frac{k+1}{m}\right),$$

we readily see that

$$\widehat{c}(mt) = p(t)\widehat{c}(t).$$

But $|p(t)|^2$ is an even function; therefore

$$\prod_{j=1}^{\infty} |p(b^{-j}t)|^{2n} = \prod_{j=1}^{\infty} |p(m^{-j}t)|^{2n} = \left|\frac{\sin \pi t}{\pi t}\right|^{2n}. \tag{8}$$

Since

$$|P(\mathbf{t})| \le \prod_{j=1}^{d} |p(t_j)|^{2n_1},$$

we have

$$|\widehat{\phi}(\mathbf{t})| \le \prod_{j=1}^{d}\left(\frac{\sin \pi t_j}{\pi t_j}\right)^{2n_1} \le K(1+|\mathbf{t}|)^{-2n_1}. \tag{9}$$

Thus Lemma C implies that ϕ is in the continuity class C^{2n_1-d-1}.

Let

$$v(\mathbf{t}) := \prod_{j=1}^{d} c(t_j),$$

$$V(\mathbf{t}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{v}(\mathbf{t}+\mathbf{k})|^2,$$

and

$$\Phi(\mathbf{t}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{\phi}(\mathbf{t}+\mathbf{k})|^2.$$

Since $\{\, c(\cdot - \mathbf{k}); \mathbf{k}\in\mathbb{Z}^d \,\}$ is an orthonormal sequence, we know that $V(\mathbf{t}) = 1$ a.e.; but (9) implies that $\Phi(\mathbf{t}) \le V(\mathbf{t})$. Thus, $\Phi(\mathbf{t}) \le 1$ a.e. (This also follows from [26, p. 113]) .

On the other hand

$$|P(\mathbf{t})| \ge \prod_{j=1}^{d} |p(t_j)|^{2n_2},$$

and from (8) we see that

$$|\widehat{\phi}(\mathbf{t})| \geq \prod_{j=1}^{d} \left(\frac{\sin \pi t_j}{\pi t_j} \right)^{2n_2}. \tag{10}$$

Therefore,

$$\Phi(\mathbf{t}) \geq |\widehat{\phi}(\mathbf{t})|^2 \geq C > 0, \quad \text{a.e. in } [-\tfrac{1}{2}, \tfrac{1}{2}]^d.$$

Since $\Phi(\mathbf{t})$ is $\mathbb{Z}^d$–periodic, the preceding inequality holds on $\mathbb{Z}^d$, and we have established that $\{ \phi(\cdot - \mathbf{k}); \mathbf{k} \in \mathbb{Z}^d \}$ is a Riesz sequence. It remains to prove that the support of ϕ is in $[-n_2, n_2]^d$. We will use an argument of Wojtaszczyk [28, p. 79].

The definition of $p(t)$ in Proposition 1 implies that

$$|p(t)|^2 = \frac{1}{m^2} (\sum_{l=0}^{m-1} e^{-2\pi i l t})(\sum_{j=0}^{m-1} e^{2\pi i j t}) = \sum_{k=-(m-1)}^{m-1} a_k e^{-2\pi i k t}, \qquad a_k \in \mathbb{R}.$$

This implies that if $n \in \mathbb{N}$, then

$$|p(t)|^{2n} = \sum_{k=-n(m-1)}^{n(m-1)} c_k e^{-2\pi i k t}, \qquad c_k \in \mathbb{R},$$

which readily yields

$$P(\mathbf{t}) = \sum_{\mathbf{k} \in \Omega} \alpha_{\mathbf{k}} e^{-2\pi i \mathbf{k}\mathbf{t}}, \qquad \alpha_{\mathbf{k}} \in \mathbb{C},$$

where $\Omega = [-n_2(m-1), n_2(m-1)]^d \bigcap \mathbb{Z}^d$.

Given $N \in \mathbb{N}$, let

$$\Pi_N(\mathbf{t}) = \prod_{j=1}^{N} P(b^{-j}\mathbf{t}),$$

and observe that Π_N may be written as

$$\Pi_N(\mathbf{t}) = \sum_{\mathbf{k}^{(1)}, \ldots, \mathbf{k}^{(N)}; \mathbf{k}^{(j)} \in \Omega} \alpha_{\mathbf{k}^{(1)}} \cdots \alpha_{\mathbf{k}^{(N)}} e^{2\pi i \mathbf{t} (\sum_{j=1}^{N} b^{-j} \mathbf{k}^{(j)})}.$$

Since $\mathbf{k}^{(j)} = (k_1^{(j)}, \cdots, k_d^{(j)}) \in \mathbb{Z}^d$ with $|k_s^{(j)}| \leq n_2(m-1)$, $s = 1, \cdots, d$, then $0 \leq \sum_{j=1}^{N} |b^{-j} k_s^{(j)}| \leq 2n_2(m-1) \sum_{j=1}^{\infty} m^{-j} = n_2$. Hence,

$$\sum_{j=1}^{N} b^{-j} \mathbf{k}^{(j)} \in [-n_2, n_2]^d. \tag{11}$$

Assume now that f is a function in the Schwartz class of $\mathbb{R}^d$ such that $\widehat{f}(\mathbf{t}) = 0$ for every $\mathbf{t}$ in $[-n_2, n_2]^d$. Bearing in mind that $|\Pi_N(\mathbf{t})| \leq 1$ for every $\mathbf{t}$ in $\mathbb{R}^d$ and

using the Lebesgue Dominated Convergence Theorem, we obtain:

$$\int_{\mathbb{R}^d} \phi(\mathbf{t})\widehat{f}(\mathbf{t})d\mathbf{t} = \int_{\mathbb{R}^d} \widehat{\phi}(\mathbf{t})f(\mathbf{t})d\mathbf{t} = \lim_{N\longrightarrow\infty} \int_{\mathbb{R}^d} \Pi_N(\mathbf{t})f(\mathbf{t})d\mathbf{t}$$

$$= \lim_{N\longrightarrow\infty} \sum_{\mathbf{k}^{(1)},\ldots,\mathbf{k}^{(N)};\mathbf{k}^{(j)}\in\Omega} \alpha_{\mathbf{k}^{(1)}}\cdots\alpha_{\mathbf{k}^{(N)}} \int_{\mathbb{R}^d} e^{-2\pi i\mathbf{t}(\sum_{j=1}^N b^{-j}\mathbf{k}^{(j)})} f(\mathbf{t})d\mathbf{t}$$

$$= \lim_{N\longrightarrow\infty} \sum_{\mathbf{k}^{(1)},\ldots,\mathbf{k}^{(N)};\mathbf{k}^{(j)}\in\,\Omega} \alpha_{\mathbf{k}^{(1)}}\cdots\alpha_{\mathbf{k}^{(N)}} \widehat{f}(\sum_{j=1}^N m^{-j}\mathbf{k}^{(j)}) = 0,$$

and we conclude that the support of ϕ is in $[-n_2, n_2]^d$.

□

Our estimate for the support of ϕ may also be obtained from (11) using the Paley–Wiener Theorem for several complex variables ([27, Theorem 4.9]).

From, e.g. [16, Theorem J], we obtain the following multivariate version of a result that is well–known for the single variable and dyadic dilation case ([14, p. 46]):

Proposition D. *Let V_j, $j \in \mathbb{Z}$, be a sequence of closed subspaces in $L^2(\mathbb{R}^d)$ satisfying conditions (i), (ii) and (iv) of the definition of b–MRA. Assume moreover that the scaling function ϕ is such that the origin is a point of continuity of $|\widehat{\phi}|$. Then $\overline{\bigcup_{j\in\mathbb{Z}^n} V_j} = L^2(\mathbb{R}^d)$ if and only if $\widehat{\phi}(\mathbf{0}) \neq 0$.*

We thus have:

Proposition 3. *Let ϕ be a function that satisfies the properties described in Proposition 2. Then ϕ generates a b-MRA with low pass filter P.*

Proof. Proposition 2 implies that $\{\phi(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^d\}$ is a Riesz sequence and that ϕ and the trigonometric polynomial P satisfy the refinement equation (3) with $B = bI_{d\times d}$. This implies that $V_j \subset V_{j+1}$, for every $j \in \mathbb{Z}$, where the subspaces V_j are defined by (4) with $B = bI_{d\times d}$. It remains to prove that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$. This follows from Proposition 1 and Proposition D. □

Proposition 4. *Let P be defined by (6) with $m \in \mathbb{N}$ and $n_1 < n_2$. Then there do not exist an integer a, $0 < a < d$, and two functions $f(\mathbf{t}_1)$ and $g(\mathbf{t}_2)$ defined on $L^2(\mathbb{R}^a)$ and $L^2(\mathbb{R}^{d-a})$ respectively and continuous at the origin, such that $f(\mathbf{t}_1)$ is $\mathbb{Z}^a$–periodic, $g(\mathbf{t}_2)$ is $\mathbb{Z}^{d-a}$–periodic, and*

$$P(t_1, \cdots, t_a, t_{a+1}, \cdots, t_d) = f(t_1, \cdots, t_a)g(t_{a+1}, \cdots, t_d) \quad \text{for } (t_1, \cdots, t_d) \in \mathbb{R}^d.$$

Proof. We will argue by contradiction. Assume that there is an integer a, $1 \leq a < d$, and two functions $f(\mathbf{t})$ and $g(\mathbf{t})$ as described in the statement of the proposition. Then, for $\mathbf{t} = (t_1, \cdots, t_d) \in \mathbb{R}^d$,

$$\prod_{s=1}^d |p(t_s)|^{4n_1} + \prod_{s=1}^d |p(t_s)|^{4n_2} = 2f^2(t_1, \cdots, t_a)g^2(t_{a+1}. \cdots, t_d). \tag{12}$$

If $(t_1, \cdots, t_a) = (t_1, \cdots, t_1)$ and $(t_{a+1}, \cdots, t_d) = (t_d, \cdots, t_d)$, where $t_1, t_d \in \mathbb{R}$, we have

$$|p(t_1)|^{4n_1 a}|p(t_d)|^{4n_1(d-a)} + |p(t_1)|^{4n_2 a}|p(t_d)|^{4n_2(d-a)} = 2f^2(t_1, \cdots, t_1)g^2(t_d, \cdots, t_d).$$

In particular, if $t_d = 0$

$$|p(t_1)|^{4n_1a} + |p(t_1)|^{4n_2a} = 2f^2(t_1, \cdots, t_1)g^2(0, \cdots, 0). \tag{13}$$

Observe that $g(0, \cdots, 0) \neq 0$ because $1 = f(0, \cdots, 0)g(0, \cdots, 0)$. Thus, substituting (13) into (12),

$$g^2(0, \cdots, 0)\left(|p(t_1)|^{4n_1a}|p(t_d)|^{4n_1(d-a)} + |p(t_1)|^{4n_2a}|p(t_d)|^{4n_2(d-a)}\right) = 2\left(|p(t_1)|^{4n_1a} + |p(t_1)|^{4n_2a}\right)g^2(t_d, \cdots, t_d).$$

Therefore

$$|p(t_1)|^{4n_1a}\left(g^2(0, \cdots, 0)|p(t_d)|^{4n_1(d-a)} - 2g^2(t_d, \cdots, t_d)\right) = |p(t_1)|^{4n_2a}\left(g^2(0, \cdots, 0)|p(t_d)|^{4n_2(d-a)} - 2g^2(t_d, \cdots, t_d)\right). \tag{14}$$

Since $|p|^{4n_1}$ and $|p|^{4n_2}$ are different continuous functions, then at least one of the sets

$$S_1 := \{\ t \in \mathbb{R} : g^2(0, \cdots, 0)|p(t)|^{4n_1(d-a)} - 2g^2(t_d, \cdots, t_d) \neq 0\ \}$$

or

$$S_2 := \{\ t \in \mathbb{R} : g^2(0, \cdots, 0)|p(t)|^{4n_2(d-a)} - 2g^2(t_d, \cdots, t_d) \neq 0\ \}$$

must be nonempty. Without essential loss of generality we may assume that S_1 is nonempty. Define $S_p := \{\ t \in [-1/2, 1/2] : p(t) \neq 0\ \}$, and observe that S_p may differ from $[-1/2, 1/2]$ in at most finite number of points.

Thus, if $t_d \in S_1$ and $t_1 \in S_p$, $t_1 \neq t_d$, (14) yields

$$|p(t_1)|^{4(n_1-n_2)a} = \frac{g^2(0, \cdots, 0)|p(t_d)|^{4n_2(d-a)} - 2g^2(t_d, \cdots, t_d)}{g^2(0, \cdots, 0)|p(t_d)|^{4n_1(d-a)} - 2g^2(t_d, \cdots, t_d)}. \tag{15}$$

Since the variables in (15) are separated, we conclude that there is a constant C such that

$$|p(t_1)|^{4(n_1-n_2)a} = C.$$

In fact, $C = 1$ because

$$\lim_{t\to 0, t\in S_q} |p(t)|^{4(n_1-n_2)a} = |p(0)|^{4(n_1-n_2)a} = 1.$$

We have thus reached a contradiction. □

Combining Propositions 2, 3, and 4 we obtain

Theorem 1. *Let ϕ be a function that satisfies the properties described in Proposition 2. Then $\{\ \phi(\cdot - \mathbf{k}); \mathbf{k} \in \mathbb{Z}^d\ \}$ is a Riesz sequence, and ϕ has the following properties: it is an even, continuous and square–integrable nonseparable scaling function of a b-MRA, with lowpass filter $P(\mathbf{t})$, and its support is in $[-n_2, n_2]^d$. Moreover, if $2n_1 > d + 1$, then ϕ is in the continuity class C^{2n_1-d-1}.*

Proof. The only thing that needs to be established is that ϕ is nonseparable: all the other statements follow directly from Propositions 2 and 3. We proceed by contradiction. Assume there exist $a \in \mathbb{N}$, $1 \leq a < d$, and two functions $\varphi \in L^2(\mathbb{R}^a)$ and $\theta \in L^2(\mathbb{R}^{d-a})$ such that

$$\phi(x_1, \cdots, x_a, x_{a+1}, \cdots, x_d) = \varphi(x_1, \cdots, x_a)\theta(x_{a+1}, \cdots, x_d) \qquad \text{a.e.} \tag{16}$$

Since ϕ is compactly supported, also φ and θ must be compactly supported. Thus $\widehat{\phi}$, $\widehat{\varphi}$ and $\widehat{\theta}$ are continuous, and (16) implies that

$$\widehat{\phi}(t_1, \cdots, t_a, t_{a+1}, \cdots, t_d) = \widehat{\varphi}(t_1, \cdots, t_a)\widehat{\theta}(t_{a+1}, \cdots, t_d).$$

Since $1 = \widehat{\phi}(\mathbf{0}) = \widehat{\varphi}(0, \cdots, 0)\widehat{\theta}(0, \cdots, 0)$, then

$$\widehat{\varphi}(t_1, \cdots, t_a) = [\widehat{\theta}(\mathbf{0})]^{-1}\widehat{\phi}(t_1, \cdots, t_a, 0, \cdots, 0)$$

and

$$\widehat{\theta}(t_{a+1}, \cdots, t_d) = [\widehat{\varphi}(\mathbf{0})]^{-1}\widehat{\phi}(0, \cdots, 0, t_{a+1}, \cdots, t_d).$$

Observe that $P(t_1, \cdots, t_a, 0, \cdots, 0)$ and $P(0, \cdots, 0, t_{a+1}, \cdots, t_d)$ are continuous functions. Thus,

$$\begin{aligned}
&P(t_1, \cdots, t_a, t_{a+1}, \cdots, t_d)\widehat{\phi}(t_1, \cdots, t_a, t_{a+1}, \cdots, t_d)\\
&\quad = \widehat{\phi}(bt_1, \cdots, bt_a, bt_{a+1}, \cdots, bt_d)\\
&\quad = \widehat{\varphi}(bt_1, \cdots, bt_a)\widehat{\theta}(bt_{a+1}, \cdots, bt_d)\\
&\quad = P(t_1, \cdots, t_a, 0, \cdots, 0)P(0, \cdots, 0, t_{a+1}, \cdots, t_d)\widehat{\varphi}(t_1, \cdots, t_a)\widehat{\theta}(t_{a+1}, \cdots, t_d).
\end{aligned}$$

But (10) implies that $\widehat{\phi}(\mathbf{t}) \neq 0$ a.e. By continuity, we therefore conclude that

$$P(t_1, \cdots, t_a, t_{a+1}, \cdots, t_d) = P(t_1, \cdots, t_a, 0, \cdots, 0)P(0, \cdots, 0, t_{a+1}, \cdots, t_d),$$

in contradiction of Proposition 4. □

4. Construction of compactly supported tight framelets

Note that the property that $\{\ \phi(\cdot - \mathbf{k}); \mathbf{k} \in \mathbb{Z}^d\ \}$ is a Riesz sequence is not used in this section.

Using the scaling functions that we obtained in the previous section, we now construct tight framelets $\Psi = \{\psi_1, \ldots, \psi_n\}$ in $L^2(\mathbb{R}^d)$, $d > 1$, with any dilation factor b, such that functions ψ_ℓ, $\ell = 1, \ldots, n$, are smooth and compactly supported. For this purpose we use a generalization of Theorem B for any dilation factor b. The proof is identical and will be omitted.

Theorem 2. *Let $b \in \mathbb{Z}$, assume that $m := |b| > 1$, let $P(\mathbf{t})$ be a trigonometric polynomial defined on $\mathbb{R}^d$ that satisfies the condition*

$$\sum_{\boldsymbol{\gamma} \in \Gamma_m^d} |P(\mathbf{t} + \boldsymbol{\gamma})|^2 \leq 1,$$

and suppose that there exist trigonometric polynomials $\widetilde{P}_1, \ldots, \widetilde{P}_M$ such that

$$\sum_{\boldsymbol{\gamma} \in \boldsymbol{\Gamma}_m^d} |P(\mathbf{t} + \boldsymbol{\gamma})|^2 + \sum_{j=1}^{M} |\widetilde{P}_j(b\mathbf{t})|^2 = 1.$$

Then there exist $m^d + M$ trigonometric polynomials Q_ℓ. $\ell = 1, \cdots, m^d + M$ such that P and the polynomials Q_ℓ satisfy (1).

The following algorithm for obtaining the polynomials Q_ℓ is implicitly in the proof of Theorem 2:

Let

$$\mathcal{P}(\mathbf{t}) := \left(P(\mathbf{t}+\boldsymbol{\gamma}); \boldsymbol{\gamma} \in \boldsymbol{\Gamma}_m^d\right)^T,$$

and let

$$\mathcal{M}(\mathbf{t}) := m^{-d/2}\left(e^{i2\pi\boldsymbol{\delta}\cdot(\mathbf{t}+\boldsymbol{\gamma})}; \boldsymbol{\gamma} \in \boldsymbol{\Gamma}_m^d, \boldsymbol{\delta} \in \boldsymbol{\Delta}_m^d\right) \tag{17}$$

be the polyphase matrix, where $\boldsymbol{\gamma}$ denotes the row index and $\boldsymbol{\delta}$ denotes the column index. Let the $m^d \times 1$ matrix function $G(\mathbf{t})$ be defined by

$$G(\mathbf{t}) := \mathcal{M}^T(\mathbf{t})\mathcal{P}(\mathbf{t}) = \left(L_{\boldsymbol{\delta}}(b\mathbf{t}); \boldsymbol{\delta} \in \boldsymbol{\Delta}_m^d\right)^T, \tag{18}$$

let $N := m^d + M$ and let the $N \times 1$ matrix function $\mathcal{G}(\mathbf{t})$ be defined by

$$\mathcal{G}(\mathbf{t}) := \left(L_{\boldsymbol{\delta}}(b\mathbf{t}); \boldsymbol{\delta} \in \boldsymbol{\Delta}_m^d, \widetilde{P}_j(b\mathbf{t}); 1 \le j \le M\right)^T,$$

and

$$\widetilde{\mathcal{Q}}(\mathbf{t}) := I_{N\times N} - \mathcal{G}(\mathbf{t})\mathcal{G}^T(\mathbf{t}).$$

Let $H(\mathbf{t})$ denote the first $m^d \times N$ block matrix of $\widetilde{\mathcal{Q}}(\mathbf{t})$, and

$$\mathcal{Q}(\mathbf{t}) := \mathcal{M}(\mathbf{t})H(\mathbf{t}).$$

Then the polynomials $Q_1(\mathbf{t}), \dots, Q_N(\mathbf{t})$ are the first row of $\mathcal{Q}(\mathbf{t})$.

We now need an auxiliary proposition.

Let h and v be trigonometric polynomials on $\mathbb{R}$ such that

$$|h(bt)|^2 = 1 - \sum_{\gamma\in\boldsymbol{\Gamma}_m} |p(t+\gamma)|^{4n_1} \quad \text{and} \quad |v(bt)|^2 = 1 - \sum_{\gamma\in\boldsymbol{\Gamma}_m} |p(t+\gamma)|^{4n_2}.$$

To see that these polynomials exist note that, for example,

$$1 - \sum_{\gamma\in\boldsymbol{\Gamma}_m} |p(t+\gamma)|^{4n_1} \ge 0,$$

and the assertion follows applying a lemma of Riesz (cf., e.g., [7, Lemma 6.1.3]), [22, Lemma 10, p. 102]). A similar argument shows that there exist two trigonometric polynomials u and w on $\mathbb{R}$, such that

$$|u(bt)|^2 = \sum_{\gamma\in\boldsymbol{\Gamma}_m} |p(t+\gamma)|^{4n_1} \quad \text{and} \quad |w(bt)|^2 = \sum_{\gamma\in\boldsymbol{\Gamma}_m} |p(t+\gamma)|^{4n_2}.$$

The coefficients of the polynomials h, v, u and w may be obtained by *spectral factorization* ([9]).

We have

Proposition 5. *Let $b \in \mathbb{Z}$, assume that $m := |b| > 1$, let p, n_1, n_2 and P be as in Proposition 1, let the trigonometric polynomials h, v, u and w be as described in*

the preceding paragraph, let the trigonometric polynomials $\widetilde{P}_j$ *be defined by*

$$\begin{aligned}
\widetilde{P}_j(\mathbf{t}) &:= \frac{\sqrt{2}}{2}h(t_j)\prod_{s=j+1}^{d}u(t_s),\ j=1,\cdots,d-1,\\
\widetilde{P}_d(\mathbf{t}) &:= \frac{\sqrt{2}}{2}h(t_d)\\
\widetilde{P}_{d+j}(\mathbf{t}) &:= \frac{\sqrt{2}}{2}v(t_j)\prod_{s=j+1}^{d}w(t_s),\ j=1,\cdots,d-1,\\
\widetilde{P}_{2d}(\mathbf{t}) &:= \frac{\sqrt{2}}{2}v(t_d),
\end{aligned}$$

and let the trigonometric polynomials $L_{\boldsymbol{\delta}}(b\mathbf{t})$, $\boldsymbol{\delta}\in\boldsymbol{\Delta}_m^d$ *be defined by* (18). *Then*

$$\sum_{\boldsymbol{\delta}\in\boldsymbol{\Delta}_m^d}|L_{\boldsymbol{\delta}}(\mathbf{t})|^2+\sum_{j=1}^{2d}|\widetilde{P}_j(\mathbf{t})|^2=1.$$

Proof. By Proposition 1 we know that

$$\sum_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}_m^d}|P(\mathbf{t}+\boldsymbol{\gamma})|^2<1,$$

except for a countable set of points. We also have

$$\begin{aligned}
&\sum_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}_m^d}|P(\mathbf{t}+\boldsymbol{\gamma})|^2+\sum_{j=1}^{2d}|\widetilde{P}_j(b\mathbf{t})|^2\\
=\ &\frac{1}{2}\prod_{s=1}^{d}\left(\sum_{\gamma\in\boldsymbol{\Gamma}_m}|p(t_s+\gamma)|^{4n_1}\right)+\frac{1}{2}\prod_{s=1}^{d}\left(\sum_{\gamma\in\boldsymbol{\Gamma}_m}|p(t_s+\gamma)|^{4n_2}\right)\\
&+\frac{1}{2}\sum_{j=1}^{d-1}|h(bt_j)|^2\prod_{s=j}^{d}|u(bt_s)|^2+\frac{1}{2}|h(bt_d)|^2\\
&+\frac{1}{2}\sum_{j=1}^{d-1}|v(bt_j)|^2\prod_{s=j}^{d}|w(bt_s)|^2+\frac{1}{2}|v(bt_d)|^2
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{2}(|u(bt_1)|^2+|h(bt_1)|^2)\prod_{s=2}^{d}|u(bt_s)|^2+\frac{1}{2}(|w(bt_1)|^2+|v(bt_1)|^2)\prod_{s=2}^{d}|w(bt_s)|^2\\
& +\frac{1}{2}\sum_{j=2}^{d-1}|h(bt_j)|^2\prod_{s=j+1}^{d}|u(bt_s)|^2+\frac{1}{2}|h(bt_d)|^2\\
& +\frac{1}{2}\sum_{j=2}^{d-1}|v(bt_j)|^2\prod_{s=j+1}^{d}|w(bt_s)|^2+\frac{1}{2}|v(bt_d)|^2\\
= \quad & \frac{1}{2}\prod_{s=2}^{d}|u(bt_s)|^2+\frac{1}{2}\prod_{s=2}^{d}|w(bt_s)|^2+\frac{1}{2}\sum_{j=2}^{d-1}|h(bt_j)|^2\prod_{s=j+1}^{d}|u(bt_s)|^2+\frac{1}{2}|h(bt_d)|^2\\
& +\frac{1}{2}\sum_{j=2}^{d-1}|v(bt_j)|^2\prod_{s=j+1}^{d}|w(bt_s)|^2+\frac{1}{2}|v(bt_d)|^2.
\end{aligned}
$$

Repeating this procedure a finite number of times, we finally obtain

$$
\begin{aligned}
& \sum_{(\rho_1,\cdots,\rho_d)\in\mathbf{\Gamma}_m^d}|P(t_1+\rho_1,\dots,t_d+\rho_d)|^2+\sum_{j=1}^{2d}|\widetilde{P}(bt_1,\dots,bt_d)|^2\\
= \quad & \frac{1}{2}|u(bt_d)|^2+\frac{1}{2}|w(bt_d)|^2+\frac{1}{2}|h(bt_d)|^2+\frac{1}{2}|v(bt_d)|^2=1,
\end{aligned}
$$

i.e.,

$$
\sum_{\boldsymbol{\gamma}\in\mathbf{\Gamma}_m^d}|P(\mathbf{t}+\boldsymbol{\gamma})|^2+\sum_{j=1}^{2d}|\widetilde{P}_j(b\mathbf{t})|^2=1.
$$

Finally, note that, since $\mathcal{M}(\mathbf{t})$ defined by (17) is unitary,

$$
\sum_{\boldsymbol{\delta}\in\mathbf{\Delta}_m^d}|L_{\boldsymbol{\delta}}(b\mathbf{t})|^2=\mathcal{P}^T(\mathbf{t})\mathcal{M}(\mathbf{t})\mathcal{M}^T(\mathbf{t})\mathcal{P}(\mathbf{t})=\sum_{\boldsymbol{\gamma}\in\mathbf{\Gamma}_m^d}|P(\mathbf{t}+\boldsymbol{\gamma})|^2.
$$

□

The following theorem describes the construction of a tight smooth framelet of compact support for any dilation factor b.

Theorem 3. *Let $b\in\mathbb{Z}$ be such that $m:=|b|>1$, let $n_1,n_2\in\mathbb{N}$ with $n_1<n_2$ and $2n_1>d+1$, let $N:=m^d+2d$, and let the trigonometric polynomials $Q_1(\mathbf{t}),\dots,Q_N(\mathbf{t})$ be obtained by the algorithm described in Theorem 2 with $P(\mathbf{t})$ defined by (6) and the trigonometric polynomials $\widetilde{P}_j(\mathbf{t})$, $j=1,\dots,2d$ defined as in Proposition 5. Let ϕ be a function that satisfies the properties described in Proposition 2,*

$$
\widehat{\psi_\ell}(B^T\mathbf{t}):=Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}),\quad \ell=1,\dots,N,
$$

and let $\Psi=\{\psi_1,\dots,\psi_N\}$ be the set of inverse Fourier transforms of the functions $\widehat{\psi_\ell}$ defined in the preceding displayed identity. Then Ψ is a tight framelet in $L^2(\mathbb{R}^d)$ with dilation factor b and frame constant equal to 1, and the functions $\psi_\ell(\mathbf{t})$ are square–integrable on $\mathbb{R}^d$, have support in $[-mn_2,mn_2]^d$, and are in the continuity class C^{2n_1-d-1}.

Proof. That $\Psi = \{\psi_1, \dots, \psi_N\}$ is a tight framelet follows from Proposition 5, Theorem 2 and Theorem A.

Since the functions Q_ℓ are trigonometric polynomials and therefore bounded on $\mathbb{R}^d$, the smoothness of the functions ψ_ℓ follows from (9) and Lemma C.

If

$$I_n := [-(m-1)n_2, (m-1)n_2]^d \cap \mathbb{Z}^d,$$

then (6) and (1) imply that the polynomials Q_ℓ have representations of the form

$$Q_\ell(\mathbf{t}) = \sum_{\mathbf{k} \in I_n} \alpha_{\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{t}}.$$

This implies that

$$\widehat{\psi_\ell}(\mathbf{t}) = \sum_{\mathbf{k} \in I_n} \alpha_{\mathbf{k}} \phi(\mathbf{t} - \mathbf{k}),$$

and since the support of ϕ is in $[-n_2, n_2]^d$, we conclude that the supports of the functions $\psi_\ell(\mathbf{t})$ are in $[-mn_2, mn_2]^d$. □

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