

THE METHOD OF LAPLACE AND WATSON'S LEMMA

RICHARD A. ZALIK

ABSTRACT. in this paper we present a different proof of a well known asymptotic estimate for Laplace integrals. The novelty of our approach is that it emphasizes, and rigorously justifies, the appealing heuristic method of Laplace. As a bonus, we also obtain a simple and short proof of Watson's Lemma.

Let a be an element of the extended real number set $[-\infty, \infty]$. If

$$\lim_{x \rightarrow a} f(x)/g(x) = 1$$

we write

$$f(x) \sim g(x), \quad x \rightarrow a$$

and say that f is asymptotic to g , or that g is an asymptotic approximation to f . If there is no risk of ambiguity, we may also write $f \sim g$ for the sake of brevity.

Note that $f \sim g$ if and only if $\lim(f(x) - g(x))/g(x) = 0$. In other words, if and only if the relative error made in approximating f by g tends to zero. In some cases, in particular when the values involved are either very small or very large, it may be more appropriate to estimate the relative rather than the absolute error.

In this article we discuss the problem of obtaining asymptotic estimates for integrals of the form

$$(1) \quad I(x) := \int_J e^{-xp(t)} q(t) dt,$$

where J is a bounded or unbounded interval, $p(x)$ and $q(x)$ are functions satisfying certain properties, and $x \rightarrow \infty$. Most authors, such as Bender and Orszag [1] use an appealing heuristic method attributed to Laplace to obtain an asymptotic estimate for $I(x)$. A rigorous proof may be found in, for example, Erdélyi [2, §2.4] (see also Olver [3, pp.80–82]). We give another rigorous proof of this estimate in Theorem 1. The novelty of our approach consists in breaking down the proof by means of two preliminary lemmas that highlight and rigorously justify the method of Laplace.

Under suitable conditions $I(x)$ has an infinite asymptotic expansion (see for example [3, pp.85–88]). In Theorem 2 we use Lemma 2 to give a simple proof of Watson's lemma, which gives an infinite asymptotic expansion for $I(x)$ when $p(t) = t$. We define asymptotic series in the paragraph preceding the statement of Theorem 2.

We begin with

Lemma 1. *Let J be an interval of the form $[a, \infty)$ or $[a, b]$, $a < b$, and assume that the following conditions are satisfied:*

Key words and phrases. Asymptotic methods; Laplace integrals; Watson's Lemma.
2010 Mathematics Classification: 34E05.

(a) The function $p(t)$ is real-valued and measurable on J and for every point $c > a$ in J ,

$$(2) \quad \inf\{p(t); t \in J \cap [c, \infty)\} > p(a).$$

(b) There is a number $\sigma > 0$ such that $p(t)$ is continuous and strictly increasing on $[a, a + \sigma]$.

(c) The function $q(t)$ is Lebesgue integrable on J .

(d) There are numbers $\alpha > -1$ and $Q \neq 0$ such that

$$q(t) \sim Q(t - a)^\alpha, \quad t \rightarrow a^+.$$

Let $I(x)$ be given by (1) and, for $\delta > 0$ such that $a + \delta \in J$,

$$(3) \quad I(x, \delta) := \int_a^{a+\delta} e^{-xp(t)} q(t) dt.$$

Then $e^{-xp(t)} q(t)$ is Lebesgue integrable on J for every positive x and there is a number $\eta > 0$ such that $p(t)$ is strictly increasing on $[a, a + \eta]$, and $0 < \delta \leq \eta$ implies that

$$(4) \quad I(x) \sim I(x, \delta), \quad x \rightarrow \infty.$$

Remark 1: Equation (2) is needed when $J = [a, \infty)$ to ensure that $p(t)$ remains bounded away from $p(a)$ as t grows. If $J = [a, b]$, (2) is equivalent to $p(t)$ having a unique absolute minimum at a .

Proof. The integrability follows from

$$|e^{-xp(t)} q(t)| = e^{-xp(a)} e^{-x[p(t)-p(a)]} |q(t)| \leq e^{-xp(a)} |q(t)|.$$

From (b) and (d) there is a number $\eta > 0$ such that $p(t)$ is continuous and strictly increasing on $[a, a + \eta]$, and if $t \in (a, a + \eta]$ then

$$|Q^{-1}(t - a)^{-\alpha} q(t) - 1| < 1/2.$$

Thus $Q^{-1}(t - a)^{-\alpha} q(t) > 1/2$, and we conclude that $q(t)$ has constant sign on $(a, a + \eta]$ and that

$$|q(t)| > (1/2)|Q|(t - a)^\alpha \quad \text{on } (a, a + \eta].$$

Let $\delta \in (0, \eta]$ be arbitrary but fixed, and let $J_\delta := J \setminus [a, a + \delta)$ and

$$M_\delta := \inf\{p(t); t \in J_\delta\}.$$

Assume $x > 0$. Since the hypotheses imply that $p(a) < M_\delta$ we have

$$\left| \int_{J_\delta} e^{-xp(t)} q(t) dt \right| \leq \int_{J_\delta} e^{-xp(t)} |q(t)| dt \leq K_1 e^{-M_\delta x}.$$

Let $C \in (p(a), M_\delta)$. By continuity, there is a $\delta_1 \in (0, \delta]$ such that $p(a + \delta_1) < C$. Since $p(t)$ is strictly increasing on $[a, a + \delta_1]$, we deduce that $p(t) < C$ for $t \in [a, a + \delta_1]$. Since $q(t)$ has constant sign on $(a, a + \delta]$, we have

$$\begin{aligned} |I(x, \delta)| &= \left| \int_a^{a+\delta} e^{-xp(t)} q(t) dt \right| = \int_a^{a+\delta} e^{-xp(t)} |q(t)| dt \geq \\ &\int_a^{a+\delta_1} e^{-xp(t)} |q(t)| dt \geq (1/2)|Q| \int_a^{a+\delta_1} e^{-xp(t)} (t - a)^\alpha dt \geq K_2 e^{-Cx}. \end{aligned}$$

Thus

$$\left| \frac{\int_{J_\delta} e^{-xp(t)} q(t) dt}{I(x, \delta)} \right| \leq K_3 e^{(C-M_\delta)x} \longrightarrow 0, \quad x \longrightarrow \infty.$$

Since

$$I(x) = I(x, \delta) + \int_{J_\delta} e^{-xp(t)} q(t) dt,$$

the assertion follows. \square

As a consequence of Lemma 1 we obtain the following basic proposition, which will be used to prove both theorems in this paper.

Lemma 2. *Let J be an interval of the form $[a, \infty)$ or $[a, b]$ with $a < b$, and assume that the function $q(t)$ satisfies conditions (c) and (d) of Lemma 1. Then*

$$\int_J e^{-xt} q(t) dt \sim Q \int_J e^{-xt} (t-a)^\alpha dt, \quad x \rightarrow \infty.$$

Proof. If $J = [a, \infty)$, Lemma 1 implies that there exists a number $b > a$ such that

$$\int_J e^{-xt} q(t) dt \sim \int_a^b e^{-xt} q(t) dt, \quad x \rightarrow \infty.$$

Thus we may assume without loss of generality that $J = [a, b]$.

Let

$$\begin{aligned} A(x) &:= \frac{\int_a^b e^{-xt} q(t) dt}{Q \int_a^b e^{-xt} (t-a)^\alpha dt}, & A_1(x, \delta) &:= \frac{\int_a^b e^{-xt} q(t) dt}{\int_a^{a+\delta} e^{-xt} q(t) dt}, \\ A_2(x, \delta) &:= \frac{\int_a^{a+\delta} e^{-xt} q(t) dt}{Q \int_a^{a+\delta} e^{-xt} (t-a)^\alpha dt}, & A_3(x, \delta) &:= \frac{\int_a^{a+\delta} e^{-xt} (t-a)^\alpha dt}{\int_a^b e^{-xt} (t-a)^\alpha dt}. \end{aligned}$$

Applying Lemma 1 we see that there is a $\delta_1 > 0$ such that if $0 < \delta \leq \delta_1$, then

$$(5) \quad \lim_{x \rightarrow \infty} A_1(x, \delta) = \lim_{x \rightarrow \infty} A_3(x, \delta) = 1.$$

Let $\varepsilon > 0$ be given. Then there is a $\delta_2 > 0$ such that if $0 < t-a < \delta_2$, then

$$(6) \quad |Q^{-1}(t-a)^{-\alpha} q(t) - 1| < \varepsilon.$$

Let $\delta := \min(\delta_1, \delta_2)$. Since the Generalized Mean Value Theorem implies there is a $\xi \in (a, a+\delta)$ such that $A_2(x, \delta) = Q^{-1}(\xi-a)^{-\alpha} q(\xi)$, we conclude from (6) that

$$-\varepsilon \leq A_2(x, \delta) - 1 \leq \varepsilon.$$

Since

$$A(x) = A_1(x, \delta) A_2(x, \delta) A_3(x, \delta),$$

we deduce from (5) that

$$1 - \varepsilon \leq \liminf_{x \rightarrow \infty} A(x) \leq \limsup_{x \rightarrow \infty} A(x) \leq 1 + \varepsilon.$$

Since ε is arbitrary, the assertion follows. \square

Theorem 1. *Let J be an interval of the form $[a, \infty)$ or $[a, b]$, $a < b$, and assume that the following conditions are satisfied:*

- (a) *The function $p(t)$ is real-valued and measurable on J and for every point $c > a$ in J , inequality (2) holds.*
- (b) *There is a number $\sigma > 0$ such that $p(t)$ is continuous and strictly increasing on $[a, a+\sigma]$, and $p \in C^1(a, a+\sigma]$.*

(c) There are numbers $P, \mu > 0$ such that

$$(7) \quad p(t) - p(a) \sim P(t - a)^\mu, \quad t \rightarrow a^+,$$

and

$$(8) \quad p'(t) \sim \mu P(t - a)^{\mu-1}, \quad t \rightarrow a^+.$$

(d) The function $q(t)$ is Lebesgue integrable on J .

(e) There are numbers $\lambda > 0$ and $Q \neq 0$ such that

$$(9) \quad q(t) \sim Q(t - a)^{\lambda-1}, \quad t \rightarrow a^+.$$

Then $e^{-xp(t)}q(t)$ is Lebesgue integrable on J for every positive x , and if $I(x)$ is given by (1) then

$$(10) \quad I(x) \sim \frac{Q}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{e^{-xp(a)}}{(Px)^{\lambda/\mu}}, \quad x \rightarrow \infty.$$

Remark 2: Conditions (b), (c) and (d) are satisfied if, for instance, there is a positive integer μ and a positive number σ such that $p \in C^\mu[a, a + \sigma]$,

$$p^{(\ell)}(a) = 0, 1 \leq \ell \leq \mu - 1, \quad \text{and} \quad p^{(\mu)}(a) < 0.$$

Proof. From Lemma 1 we deduce that there is a $\delta_0 > 0$ such that $p(t)$ is continuous and strictly increasing on $[a, a + \delta_0]$, $p \in C^1(a, a + \delta_0]$, and (4) is satisfied for any $\delta \in (0, \delta_0]$. Let δ be an arbitrary but fixed number in $(0, \delta_0]$, and let $I(x, \delta)$ be given by (3). Making the change of variable $s = p(t)$ we see that

$$I(x, \delta) = \int_{p(a)}^{p(a+\delta)} e^{-xs} \frac{q[p^{-1}(s)]}{p'[p^{-1}(s)]} ds.$$

Setting $c = p(a)$ and applying (7) we have

$$\lim_{s \rightarrow c^+} \frac{(p^{-1}(s) - a)^\mu}{s - c} = \lim_{t \rightarrow a^+} \frac{(t - a)^\mu}{p(t) - p(a)} = 1/P.$$

Therefore

$$(11) \quad p^{-1}(s) - a \sim \left(\frac{s - c}{P}\right)^{1/\mu}, \quad s \rightarrow c^+.$$

Moreover, (9) implies that

$$q[p^{-1}(s)] \sim Q[p^{-1}(s) - a]^{\lambda-1}, \quad s \rightarrow c^+.$$

Thus, from (11),

$$(12) \quad q[p^{-1}(s)] \sim Q \left(\frac{s - c}{P}\right)^{(\lambda-1)/\mu}, \quad s \rightarrow c^+.$$

On the other hand, (8) and (11) yield

$$(13) \quad p'[p^{-1}(s)] \sim \mu P(p^{-1}(s) - a)^{\mu-1} \sim \mu(P)^{1/\mu}(s - c)^{1-1/\mu}, \quad s \rightarrow c^+.$$

Combining (12) and (13) we conclude that

$$\frac{q[p^{-1}(s)]}{p'[p^{-1}(s)]} \sim (Q/\mu)(P)^{-(\lambda/\mu)}(s - c)^{\lambda/\mu-1}, \quad s \rightarrow c^+,$$

and Lemma 2 implies that

$$\begin{aligned} I(x, \delta) &\sim \int_{p(a)}^{p(a+\delta)} e^{-xs} (Q/\mu)(P)^{-(\lambda/\mu)} (s-c)^{\lambda/\mu-1} ds = \\ &\quad (Q/\mu)(P)^{-(\lambda/\mu)} \int_{p(a)}^{p(a+\delta)} e^{-xs} (s-c)^{\lambda/\mu-1} ds = \\ &\quad (Q/\mu)(P)^{-\lambda/\mu} e^{p(a)x} \int_0^{p(a+\delta)-p(a)} e^{-xt} t^{\lambda/\mu-1} dt. \end{aligned}$$

But Lemma 1 implies that there is a $\delta \in (0, \delta_0]$ such that

$$\int_0^{p(a+\delta)-p(a)} e^{-xt} t^{\lambda/\mu-1} dt \sim \int_0^\infty e^{-xt} t^{\lambda/\mu-1} dt = \Gamma\left(\frac{\lambda}{\mu}\right) x^{-\lambda/\mu}, \quad x \rightarrow \infty,$$

and the assertion follows. \square

In the preceding theorem we have assumed that the minimum of $p(t)$ is unique and occurs at a . In other cases the interval of integration may be subdivided at the maxima and minima of $p(t)$. We may then apply Theorem 1 to intervals where the minimum is at a left end-point, and Corollary 1 below to intervals where the minimum is at a right end-point.

Making the change of variable $t \rightarrow -t$ we obtain

Corollary 1. *Let J be an interval of the form $(-\infty, b]$ or $[a, b]$, $a < b$, and assume that the following conditions are satisfied:*

(a) *The function $p(t)$ is real-valued and measurable on J and for every point $c < b$ in J ,*

$$\inf\{p(t); t \in J \cap (-\infty, c]\} > p(b).$$

(b) *There is a number $\sigma > 0$ such that $p(t)$ is continuous and strictly decreasing on $[b - \sigma, b]$, and $p \in C^1[b - \sigma, b]$.*

(c) *There are numbers $P, \mu > 0$ such that*

$$p(t) - p(b) \sim P(b-t)^\mu, \quad t \rightarrow b^-.$$

and

$$p'(t) \sim -\mu P(b-t)^{\mu-1}, \quad t \rightarrow b^-.$$

(d) *The function $q(t)$ is Lebesgue integrable on J .*

(e) *There are numbers $\lambda > 0$ and $Q \neq 0$ such that*

$$q(t) \sim Q(b-t)^{\lambda-1}, \quad t \rightarrow b^-.$$

Then $e^{-xp(t)}q(t)$ is Lebesgue integrable on J for every positive x , and if $I(x)$ is given by (1) then

$$I(x) \sim \frac{Q}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{e^{-xp(b)}}{(Px)^{\lambda/\mu}}, \quad x \rightarrow \infty.$$

Before proceeding further, let us recall the definition of asymptotic expansion in the particular format that we require here. Let $(f_k)_{k=0}^\infty$ be a sequence of functions, let $(a(k))_{k=0}^\infty$ be a sequence of scalars such that the set of integers k for which $a(k) \neq 0$ is infinite, and let $m(k) := \inf\{r \geq k : a(r) \neq 0\}$. We say that

$$f(x) \sim \sum_{k=0}^{\infty} a(k) f_k, \quad x \rightarrow a,$$

if for every $n > 0$

$$f(x) - \sum_{k=0}^{n-1} a(k)f_k \sim a(m(n))f_{m(n)}, \quad x \rightarrow a,$$

From Lemma 2 we obtain a simple proof of Watson's Lemma ([2, §2.4], [3, p.71]):

Theorem 2. (*Watson's Lemma*) Let J be an interval of the form $[0, \infty)$ or $[0, b]$, $b > 0$, and assume that $q(t)$ is Lebesgue integrable on J and that there are $\alpha > -1$ and $\beta > 0$ such that

$$q(t) \sim \sum_{k=0}^{\infty} a_k t^{\alpha+\beta k}, \quad t \rightarrow 0^+.$$

If

$$I(x) := \int_J e^{-xt} q(t) dt,$$

then

$$I(x) \sim \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha + \beta k + 1)}{x^{\alpha+\beta k+1}}, \quad x \rightarrow \infty.$$

Proof. Defining if necessary $q(t)$ to equal 0 on $[b, \infty)$ we may assume, without essential loss of generality, that $J = [0, \infty)$.

Let $n \geq 0$ be an arbitrary integer, and let N denote the smallest integer $k \geq n$ such that $a_{k+1} \neq 0$. By definition,

$$q(t) - \sum_{k=0}^n a_k t^{\alpha+\beta k} \sim a_{N+1} t^{\alpha+\beta(N+1)}, \quad t \rightarrow 0^+.$$

Applying Lemma 2 we conclude that

$$\int_0^{\infty} \left[q(t) - \sum_{k=0}^n a_k t^{\alpha+\beta k} \right] e^{-xt} dt \sim \int_0^{\infty} t^{\alpha+\beta(N+1)} e^{-xt} dt, \quad x \rightarrow \infty,$$

i.e.,

$$I(x) - \sum_{k=0}^n a_k \frac{\Gamma(\alpha + \beta k + 1)}{x^{\alpha+\beta k+1}} \sim \frac{\Gamma(\alpha + \beta(N+1) + 1)}{x^{\alpha+\beta(N+1)+1}}, \quad x \rightarrow \infty,$$

and the assertion follows. \square

The interested reader will have no difficulty in applying Lemma 1 and Lemma 2 to obtain a version of Watson's lemma for an arbitrary $p(t)$, as in [3, p.86].

REFERENCES

- [1] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, New York, 1978. Reprint, Springer-Verlag, New-York 1999.
- [2] A. Erdélyi, *Asymptotic Expansions*, Dover, New York, 1956.
- [3] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

DEPARTMENT OF MATHEMATICS AND STATISTICS, AUBURN UNIVERSITY, AL 36849-5310
E-mail address: zalik@auburn.edu