DENSITY AND APPROXIMATION PROPERTIES OF WEAK MARKOV SYSTEMS [1]

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Abstract. We study the density and approximation properties of weak Markov systems defined on a closed interval $[a, b]$.

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1. Introduction

In [10], based on earlier work of Borwein, Bojanov, et al. [1, 2, 3], we studied the density and approximation properties of Markov systems of continuous functions defined on a closed interval $[a, b]$. Here we will extend some of the results of [10] to the weak Markov system setting.

This article is organized as follows: in this section we introduce some of the basic definitions and results from the theory of weak Markov systems. For definitions and results not mentioned here, or for clarification of details, the reader is referred to [10], the books [4, 7, 13, 14, 19], and the survey papers [5, 17]. In Section 2 we generalize to weak Markov systems the differentiation operator introduced in Section 2 of [10]. This generalization is very straightforward, except for the proof of Proposition 3 (a). The results of Section 3 of [10] are extended in Sections 3 and 4: In Section 3 we develop a theory of Chebychev polynomials for weak Markov systems. Although now these polynomials are not necessarily unique, they still have useful and interesting properties. In particular we remark on Theorem 4(d), which shows the existence of disjoint *intervals of equioscillation*. The density results of [10, Section 4], which are given in terms of the distribution of zeros of the Chebychev polynomials, have their counterpart in Section 4 of the present paper. Although the proofs were motivated by arguments used in [10], the task was complicated by the lack of uniqueness of the Chebychev polynomials and by the possibility that the functions in the system may be linearly dependent on a subset of the interval of definition. Finally, in Section 5 we obtain Jackson type theorems. Here the essential idea is to convolve with the Gauss kernel, apply the results of [10] to the Markov systems thus obtained, and then pass to the limit to recover the original system.

Let A be a set of real numbers, let $F(A)$ denote the set of all real–valued functions defined on A, let $G_n := \{g_0, \ldots, g_n\}$ be a sequence of functions, or *system*, and let $S(G_n)$ denote the linear span of $\{g_0, \ldots, g_n\}$. A system of functions $G_n \subset F(A)$ is called a *Chebychev system* or *T–system* if A contains at least $n+1$ points, and all the determinants of the square collocation matrices

$$\bigcup \begin{pmatrix} g_0, \cdots, g_n \\ t_0, \cdots, t_n \end{pmatrix} := \det(g_j(t_i);\ 0 \leq i, j \leq n)$$

with $t_0 < \ldots < t_n$ in A, are positive. If all these determinants are merely nonnegative, and, in addition, the functions in G_n are linearly independent on A, then G_n is called a *weak Chebychev system* or *WT–system*. A system G_n is called a *Markov system* (*weak Markov system*) if $G_k = \{g_0, \ldots, g_k\}$ is a Chebychev system (weak Chebychev system) for each $k = 0, 1, \ldots, n$. If $g_0 = 1$, we say that G_n is *normalized*. If $G := \{g_0, g_1, g_2, \ldots\} \subset F(A)$ and G_n is a (normalized) Markov system (weak Markov system) for all $n \geq 0$, we say that G is a (normalized) infinite Markov system (infinite weak Markov system).

Let $f(t)$ be a real valued function defined on a set A of $n \geq 2$ elements. A sequence $x_0 < \cdots < x_{n-1}$ of elements of A is called a strong alternation of f of length n, if either $(-1)^i f(x_i)$ is positive for all i, or $(-1)^i f(x_i)$ is negative for all i. It is well known that if G_n is a weak Chebychev system on A, then no function in $S(G_n)$ has a strong alternation of length $n + 2$ on A [14, 18, 19]. This property will be used in the proof of Theorem 5 below.

Let $I(A)$ denote the convex hull of A. We call $G_n \subset F(A)$ *r*epresentable if for all $c \in A$ there is a basis U_n of $S(G_n)$, obtained from G_n by a triangular transformation (i. e., $u_0(x) = g_0(x)$ and $u_i - g_i \in S(g_{i-1}), 1 \leq i \leq n$); a strictly increasing function h (an "embedding function") defined on A, with $h(c) = c$; and a set $P_n := \{p_1, \ldots, p_n\}$ of continuous, increasing functions defined on $I(h(A))$, such that for every $t \in A$ and

$1 \le k \le n$,

$$u_k(x) = u_0(x) \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_{k-1}} dp_k(t_k) \cdots dp_1(t_1). \tag{1}$$

In this case we say that (h, c, P_n, U_n) is a *r*epresentation of G_n. An n–dimensional linear space S_n is called *r*epresentable, if it has a representable basis, and (h, c, P_n, U_n) will be called a *r*epresentation for S_n, if it is a representation for some basis of S_n.

The main result of [20] implies that a Markov system on an open interval is representable. However not every Markov system on a closed interval is representable. The representability of weak Markov systems can be characterized in terms of the so–called *Condition E* and *property (M)*:

Let $S(G_n)$ denote the linear span of G_n. We say that G_n satisfies *condition E* if for all $c \in I(A)$ the following two requirements are satisfied:

(a) If G_n is linearly independent on $[c, \infty) \cup A$ then there exists a basis $\{u_0, \dots, u_n\}$ for $S(G_n)$, obtained by a triangular linear transformation, such that for any sequence of integers $0 \le k(0) < \cdots < k(m) \le n$, $\{u_{k(r)}\}_{r=0}^m$ is a weak Markov system on $A \cap [c, \infty)$.

(b) If G_n is linearly independent on $(-\infty, c] \cap A$ then there exists a basis $\{v_0, \dots, v_n\}$ for $G(Z_n)$, obtained by a triangular linear transformation, such that for any sequence of integers $0 \le k(0) < \cdots < k(m) \le n$, $\{(-1)^{r-k(r)} v_{k(r)}\}_{r=0}^m$ is a weak Markov system on $(-\infty, c] \cap A$.

Let $P_n := \{p_1, \dots, p_n\} \subset F(I)$, where I is an interval, let h be a real-valued function defined on A such that $h(A) \subset I$, and let $x_0 < \cdots < x_n$ be points of $h(A)$. We say that P_n satisfies *property* (M) with respect to h at $x_0 < \cdots < x_n$ if there is a sequence $\{t_{i,j} : i = 0, \dots, n; j = 0, \dots, n-i\}$ in $h(A)$ such that

(a) $x_j = t_{0,j} (j = 0, \dots, n)$;

(b) $t_{i,j} < t_{i+1,j} < t_{i,j+1} (i = 0, n-1; j = 0, \dots, n-i)$;

(c) For $i = 1, \dots, n$, and $j = 0, \dots, n-i$ the function $p_i(x)$ is not constant at $t_{i,j}$.

We say that a function f is not constant at a point $c \in (a, b)$ if for every $\epsilon > 0$ there are points $x_1, x_2 \in (a, b)$ with $c - \epsilon < x_1 < c < x_2 < c + \epsilon$, such that $f(x_1) \neq f(x_2)$.

Theorem A. [16] *Let $G_n \subset F(A)$. Then the following statements are equivalent:*

(a) *G_n is a normalized weak Markov system that satisfies Condition (E).*

(b) *G_n is representable, and there is a representation (h, c, P_n, U_n) of G_n such that P_n satisfies property (M) with respect to h at some sequence $x_0 < \cdots < x_n$ in $h(A)$.*

(c) *G_n is representable, and for every representation (h, c, P_n, U_n) of Z_n, P_n satisfies Property (M) with respect to h at some sequence $x_0 < \cdots < x_n$ in $h(A)$.*

Note that the original statement of Theorem A contained two typographical errors (A instead of $h(A)$), which we have corrected above.

Since Condition E is usually difficult to verify, we give another condition for representability which is general enough for our purposes. It will shed some light on an additional assumption we will make in Section 4.

Theorem 1. *Let A be a set of real numbers such that $a := \inf A \in A$ and $b := \sup A \in A$, and let G_n be a normalized weak Markov system on A. Then G_n is representable if and only if there are numbers $\alpha < a$ and $\beta > b$, and a weak Markov system F_n on $[\alpha, a] \cup A \cup [b, \beta]$ such that for each $0 \le k \le n$, g_k is the restriction to A of f_k, and the functions in F_n are linearly independent on each of the intervals $[\alpha, a]$ and $[b, \beta]$.*

Proof. If , there are numbers $\alpha < a$ and $\beta > b$, and a weak Markov system F_n on $[\alpha, a] \cup A \cup [b, \beta]$ such that for each $0 \le k \le n$, g_k is the restriction to A of f_k, and the functions in F_n are linearly independent on each of the intervals $[\alpha, a]$ and $[b, \beta]$, the assertion follows directly from [6, Proposition 5.1 and Theorem 5.8]. To prove the converse, let $c \in A$, and let (h, c, P_n, U_n) be a representation for G_n. It suffices to prove the assertion for the system U_n. Let r be a strictly increasing function on $[\alpha, a] \cup A \cup [b, \beta]$ that coincides with h on A, and for $1 \leq k \leq$n let q_k be a continuous increasing function on $[r(\alpha), r(\beta)]$, strictly increasing on each of the intervals $[\alpha, a]$ and $[b, \beta]$, that coincides with p_k on $[r(a), r(b)] = [h(a), h(b)]$. Let $v_0 = 1$,

$$v_1(x) = \int_c^{r(x)} dq_1(t),$$

and

$$v_k(x) = \int_c^{r(x)} \int_c^{t_1} \cdots \int_c^{t_{k-1}} dq_k(t_k) \cdots dq_2(t_2)\, dq_1(t_1), \qquad 2 \leq k \leq n.$$

It is clear that $v_k = u_k$ for each $0 \leq k \leq n$, and from Theorem A or the Lemma of [15] we readily conclude that V_n is a normalized weak Markov system. Since the functions q_k are strictly increasing on each of the intervals $[\alpha, a]$ and $[b, \beta]$, a simple inductive argument involving the number of integrations readily shows that the functions in V_n are linearly independent on each of the intervals $[\alpha, a]$ and $[b, \beta]$. $\square$

An infinite weak Markov system G will be called *finitely representable* if G_n is representable for each $n > 0$. At present, it is not known under what conditions an infinite (weak) Markov system defined on a set A is representable. In other words, the problem of finding conditions under which for every $c \in A$ there is a strictly increasing function h defined on A with $h(c) = c$, an infinite sequence $P := \{p_1, p_2, \ldots\}$ of continuous, increasing functions defined on $I(h(A))$, and an infinite sequence of functions $U := \{u_0, u_1, \ldots\}$, such that (h, c, P_n, U_n) is a representation of G_n for each $n > 0$, is still open.

2. Relative Derivatives For Weak Markov Systems

The following are generalizations of [10, Proposition 1 and Proposition 2] and have exactly the same proof.

Proposition 1. *Let G_n be a representable normalized weak Markov system on a set A, $n > 0$, and let (h, c, P_n, U_n) be a representation for G_n. Then u_1 depends only on g_1 and c. If, moreover, $p_1(c) = 0$, then also $p_1 \circ h$ depends only on g_1 and c.*

Proposition 2. *Let G_n be a representable normalized weak Markov system of continuous functions on a closed interval $[a, b]$ with $n \geq 1$, and let (h, c, P_n, U_n) be a representation for the restriction to (a, b) of the functions in G_n. Then, for $x \in [a, b]$,*

$$u_1(x) = \int_c^{x} dg_1(t)$$

and, if $n \geq 2$,

$$u_k(x) = \int_c^{x} \int_c^{h(t_1)} \cdots \int_c^{t_{k-1}} dp_k(t_k) \cdots dp_2(t_2)\, dg_1(t_1), \qquad 2 \leq k \leq n.$$

Let G_n be a representable normalized weak Markov system of continuous functions on a closed interval $[a, b]$. A representation (h, c, P_n, U_n) of G_n such that h is continuous at a, left–continuous on $(a, b]$, and $p_1(c) = 0$, will be called *standard.* Repeating *verbatim* the discussion in the second paragraph that follows the proof of [10, Proposition 2], we see that every representable weak Markov system of continuous functions on a closed interval $[a, b]$ has a standard representation for every $c \in [a, b]$.

Let I denote an interval and g a continuous real–valued strictly increasing function on I. If f is a real–valued function on I and $x \in I$ then, provided the limit exists, we define:

$$Df(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{g(x+h)-g(x)}$$

The operator D is called the *relative derivative* with respect to g.

Given a representation (h, c, P_n, U_n) of a weak Markov system G_n on $[a, b]$ with $n \geq 1$, we define the operator H_n on $S(G_n)$ (the *weak relative derivative* with respect to (h, c, P_n, U_n)), exactly as in [10]:

$$H_n u_0 := 0, \quad H_n u_1 := 1.$$

If $n \geq 2$,

$$H_n u_2(x) := \int_c^{h(x)} dp_2(t_2).$$

If $n \geq 3$,

$$H_n u_k(x) := \int_c^{h(x)} \int_c^{t_2} \cdots \int_c^{t_{k-1}} dp_k(t_k) \cdots dp_2(t_2), \ 3 \leq k \leq n.$$

And for every $f \in S(G_n)$ by linearity.

We then have:

Proposition 3. *Let $G := \{1, g_1, g_2, \ldots\}$ be a finitely representable normalized infinite weak Markov system on a closed interval $[a, b]$, let $G_n := \{g_0, \ldots, g_n\} \subset G$ with $n \geq 1$, let $c \in [a, b]$, and let H_n be the weak relative derivative with respect to some standard representation (h, c, P_n, U_n) of G_n. Then*

(a) *$\{H_n u_1, H_n u_2, \cdots\}$ is a finitely representable normalized infinite weak Markov system on $[a, b]$.*

(b) *If the functions g_k are all continuous on $[a, b]$, then*

$$u_k(x) = u_0(x) \int_c^x H_n u_k(t)\, dg_1(t), \quad x \in [a, b], \quad 1 \leq k \leq n,$$

(c) *If, moreover, g_1 is strictly increasing, then the operator H_n depends neither on n nor on c, nor on the representation, but only on g_1.*

Proof. Let (h, c, P_n, U_n) be a representation of G_n. From Theorem A we know that P_n satisfies property (M) with respect to h at some sequence $x_0 < \cdots < x_n$ in $h[a, b]$. Let $\overline{P}_{n-1} := \{p_2, \ldots, p_n\}$ and $\overline{U}_{n-1} := \{H_n u_1, \ldots, H_n u_n\}$; then $(h, c, \overline{P}_{n-1}, \overline{U}_{n-1})$ is a representation of $\{H_n g_1, \ldots, H_n g_n\}$, and we readily see that $\overline{P}_{n-1}$ satisfies property (M) with respect to h at some sequence $s_0 < \cdots < s_n$ in $h[a, b]$. Applying again Theorem A, (a) follows.

Part (b) follows directly from Proposition 2. The proof of part (c) is almost identical to that of the corresponding portion of [10, Proposition 3], and will be omitted. □

Just as in [10], applying [10, Lemma 1], Proposition 3 (instead of [10, Proposition 3]), and bearing in mind the argument used to prove the latter part of Proposition 3, we obtain the following generalizations of [10, Theorem 1 and Theorem 2]:

Theorem 2. *Let $G := \{1, g_1, g_2 \ldots\} \subset C([a, b])$ be a finitely representable normalized infinite weak Markov system on $[a, b]$, let $G_n := \{g_0, \ldots, g_n\} \subset G$ with $n \geq 1$, and assume that g_1 is strictly increasing. Then there is a unique linear operator $\widetilde{D}$ defined on $S(G)$ and depending only on g_1, such that if (h, c, P_n, U_n) is a standard representation of G_n with associated operator H_n, then $\widetilde{D} = H_n$ on $S(G_n)$.*

As in [10], $\widetilde{D}$ will be called the *generalized derivative* associated with the system G.

Theorem 3. *Let $G := \{1, g_1, g_2, \ldots\} \subset C([a,b])$ be a finitely representable normalized infinite weak Markov system on $[a,b]$, let $G_n := \{g_0, \ldots, g_n\} \subset G$ with $n \geq 1$, assume that g_1 is strictly increasing, and let (h, c, P_n, U_n) be any standard representation of G_n. Then the generalized derivative $\widetilde{D}$ associated with G has the following properties:*

(a) *The functions $\widetilde{D}g_k$ are continuous at a, left–continuous on $(a,b]$, and if D denotes the relative derivative with respect to g_1, and $f \in S(G)$, then $\widetilde{D}f(x) = Df(x)$ a. e. in $[a,b]$.*

(b)
$$u_k(x) = u_0(x) \int_c^x \widetilde{D}u_k(t)\, dg_1(t), \quad x \in [a,b], \quad k \geq 1.$$

(c) $\{\widetilde{D}g_1, \widetilde{D}g_2, \widetilde{D}g_3 \cdots\}$ *is an normalized infinite weak Markov system on $[a,b]$.*

(d) *For any $n \geq 1$, if (h, c, P_n, U_n) is any standard representation of G_n then*
$$\widetilde{D}u_0 = 0, \quad \widetilde{D}u_1 = 1.$$
If $n \geq 2$,
$$\widetilde{D}u_2(x) = \int_c^{h(x)} dp_2(t_2).$$
If $n \geq 3$,
$$\widetilde{D}u_k(x) = \int_c^{h(x)} \int_c^{t_2} \cdots \int_c^{t_{k-1}} dp_k(t_k) \cdots dp_2(t_2), \ 3 \leq k \leq n.$$

We end this section with the following generalization of [10, Proposition 4]. It has the same proof, except that we need to use Theorem 3(b) instead of [10, Theorem 2(b)].

Proposition 4. *Let G_n be a a representable normalized weak Markov system of continuous functions on $[a,b]$ such that g_1 is strictly increasing, and let $f \in S(G_n)$. Then, for every $x_0, x_1 \in [a,b]$,*
$$f(x_1) = f(x_0) + \int_{x_0}^{x_1} \widetilde{D}f(t)\, dg_1(t).$$

3. Generalized Chebychev Polynomials

For the case of a compact interval, Haar's famous unicity theorem says that an n–dimensional subspace S of $C[a,b]$ has a unique element of best approximation in the norm of the supremum for each $f \in C([a,b])$, if and only if S has a basis that is a Chebychev system on $[a,b]$ [7, 8]. Moreover, if $f \notin S$ and g is the best approximation to f in S, then the function $e := f - g$ has an equioscillation of length $n+1$, i. e., there are points $a \leq x_0 < \cdots < x_n \leq b$ such that
$$e(x_i) = \epsilon(-1)^i \|e\|, \quad i = 0, \ldots, n; \quad \epsilon = 1 \quad \text{o} \quad \epsilon = -1.$$

Jones and Karlovitz [11] characterized those finite–dimensional subspaces S of $C[a,b]$ having the property that every function $f \in C[a,b]$ has at least one element of best approximation g in S such that the error function $f - g$ has an equioscillation of length $n+1$. This result was generalized to functions defined in more general sets by Deutsch, Nüremberg and Singer [9], and was further extended by Kamal. His result, which we will use in the sequel, is the following:

Theorem B. [12, Theorem 2.9] *Let Q be a locally compact totally ordered space that contains at least $(n+1)$ points, and let N be an n-dimensional subspace of $C_0(Q)$. Then N is a weak Chebychev subspace if and only if for each $f \in C_0(Q)$ there is $g \in N$ such that $\|f - g\| = d(f, N)$ and $f - g$ equioscillates at (n+1) points of Q.*

We can now prove

Proposition 5. *Let G_n be a normalized representable weak Markov system in $C[a,b]$. Then there is a function $T_n \in C[a,b]$ such that*

(a) $T_n \in S(G_n)$.
(b) *T_n has an equioscillation of length $n+1$.*
(c) $\|T_n\| = 1$ *and* $T_n(b) > 0$.

Proof. Let $T_0 = 1$. If $n > 0$, then by Theorem B there is an element of best approximation q_n to g_n from $S(G_{n-1})$, such that the error function $g_n - q_n$ has an equioscillation of length $n+1$. Setting $T_n := \alpha_n(g_n - t_n)$, where α_n is chosen so that $\|T_n\| = 1$ and $T_n(b) > 0$, the assertion follows. □

A function that satisfies the conclusions of Proposition 5 will be called a *generalized Chebychev polynomial associated with* G_n and denoted by T_n. Note that if G_n is not a Chebychev system the functions T_n may not be unique. If G is an normalized infinite weak Markov system, we may generate a sequence $\{T_0, T_1, T_2, \dots\}$ by selecting one such T_n for each integer n. Such a sequence will be called a *family of generalized Chebychev polynomials associated with* G.

The following theorem should be compared with [10, Corollary 1], which is the corresponding statement for Markov systems.

Theorem 4. *Let G be a normalized infinite weak Markov system in $C[a,b]$, and let $\{T_0, T_1, T_2, \dots\}$ be a family of generalized Chebychev polynomials associated with G. Then, for each $n \geq 0$ we have:*

(a) $S(\{T_0, \dots, T_n\}) = S(G_n)$
(b) *If $y_0 < \dots < y_n$ is an equioscillation for T_n, then T_n is monotonic in each interval $[y_{j-1}, y_j]$; $j = 1, \dots, n$.*
(c) *T_n is constant on $[a, y_0]$ and on $[y_n, b]$.*
(d) *There are points $d_0, d_0^+, \dots, d_n, d_n^+$ such that $a = d_0 \leq d_0^+ < \dots < d_n \leq d_n^+ = b$ and $z_0 < \dots < z_n$ is an equioscillation for T_n, if and only if $z_i \in [d_i, d_i^+]$; $i = 0, \dots, n$.*
(e) *There are points $c_1, c_1^+, \dots, c_n, c_n^+$ such that $d_{i-1}^+ < c_i \leq c_i^+ < d_i$ for $1 \leq i \leq n$, and*
$$T_n^{-1}(\{0\}) = \bigcup_{j=1}^{n} [c_j, c_j^+]$$
(f) *If, moreover, G is finitely representable, then $\widetilde{D}T_n$ has weakly constant sign in each interval $[y_{j-1}, y_j]$; $j = 1, \dots, n$.*

Proof.

(a) Trivial: The functions $T_0, \dots, T_n$ are linearly independent.
(b) Let us assume for example that $T_n(y_{j-1}) = -1 = -T_n(y_j)$ and that T_n is not increasing. Then there are points ξ, η, with $y_{j-1} < \xi < \eta < y_j$, such that $T_n(\xi) > T_n(\eta)$. Setting $\delta := (T_n(\eta) + T_n(\xi))/2$ we see that $y_0, \dots, y_{j-1}, \xi, \eta, y_j, \dots, y_n$ would be a strong alternation of length $n+3$ for $T_n - \delta$, which is a contradiction.
(c) If $d < y_0$ and $T_n(d) = -T_n(y_0)$, then $d, y_0, \dots y_n$ is a strong alternation of length $n+2$ for T_n. Otherwise, if $|T_n(d)| < 1$, setting $\delta = (T_n(d) + T_n(y_0))/2$, we see that $T_n - \delta$ would have a strong alternation of length $n+2$. Thus $T_n(d) = T_n(y_0)$. In similar fashion we see that $T_n(x) = T_n(y_n)$ in $[y_n, b]$.
(d) Let $y_0 < \dots < y_n$ be an equioscillation for T_n in $[a,b]$. For each $j = 0, \dots, n$ let $I_j := \{x \in (y_{j-1}, y_{j+1}) : T_n(x) = T_n(y_j)\}$, where $y_{-1} := a$ and $y_{n+1} := b$. Let $d_j := \inf I_j$, and $d_j^+ := \sup I_j$; in view of (b) and the continuity of T_n we see that $I_j = [d_j, d_j^+]$, whereas (c) implies that $d_0 = a$ and $d_n^+ = b$; it is also clear that $d_{j-1} \leq d_{j-1}^+ < d_j \leq d_j^+$ by construction. Moreover, if $x \in [a,b]$ is such that

$|T_n(x)| = 1$, bearing in mind that $x \in [y_{j-1}, y_j]$ for some j, $0 \leq j \leq n+1$, we conclude that either $T_n(x) = T_n(y_{j-1})$ or $T_n(x) = T_n(y_j)$, whence either $x \in I_{j-1}$ or $x \in I_j$. Therefore

$$\bigcup_{j=0}^{n} I_j = T_n^{-1}(\{-1, 1\}).$$

Thus, if $z_0 < \cdots < z_n$ is an equioscillation we deduce that $\{z_0, \ldots, z_n\} \subset \bigcup_{j=0}^{n} I_j$. Let us assume that for some j, $0 \leq j \leq n$, $z_j \notin I_j$, and let j_0 the first index for which $z_{j_0} \notin I_{j_0}$; then $\{z_{j_0}, \ldots, z_{j_n}\} \subset \bigcup_{j>j_0}^{n} I_j$; this implies that at least two consecutive z_j's must belong to the same interval I_j; but this contradicts the assumption that $z_0 < \cdots < z_n$ is an equioscillation.

(e) Since $T_n(d_{j-1}^{+})T_n(d_j) < 0$ for each $1 \leq j \leq n$, there is a point $x \in (d_{j-1}^{+}, d_j)$ such that $T_n(x) = 0$. Thus $K_j := \{x \in (d_{j-1}^{+}, d_j) : T_n(x) = 0\} \neq \emptyset$. Let $c_j := \inf K_j$ and $c_j^{+} := \sup K_j$. By continuity $T_n(c_j) = 0 = T_n(c_j^{+})$, and (b) implies that $K_j = [c_j, c_j^{+}]$. Moreover, since (e) implies that T_n is constant and nonzero on $[d_j, d_j^{+}]$, it is clear that if $T_n(x) = 0$ for some $x \in [a, b]$, then $x \notin \bigcup_{j=0}^{n} [d_j, {d_j}^{+}]$; therefore $x \in (d_{j-1}^{+}, d_j)$ for some j, i. e. $x \in K_j$.

(f) Let us assume, for instance, that $T_n(y_{j-1}) = -1 = -T_n(y_j)$; therefore (b) implies that T_n is increasing on $[y_{j-1}, y_j]$. If $\widetilde{D}T_n$ is nonnegative in (y_{j-1}, y_j) there must be a point $x_1 \in (y_{j-1}, y_j)$ such that $\widetilde{D}T_n(x_1) < 0$. Since $\widetilde{D}T_n$ is left–continuous, there must be a point $x_0 \in (y_{j-1}, x_1)$ such that $\widetilde{D}T_n < 0$ in $[x_0, x_1]$. Applying Proposition 4 we thus have

$$T_n(x_1) - T_n(x_0) = \int_{x_0}^{x_1} \widetilde{D}T_n(s)ds < 0.$$

Since T_n is increasing on $[y_{j-1}, y_j]$, we have obtained a contradiction.

□

The intervals $[d_j, d_j^{+}]$ will be called *equioscillation intervals* of T_n, the intervals $[c_j, c_j^{+}]$ will be called *zero intervals* of T_n, and the left endpoints c_j of the zero intervals will be called *ℓ–zeros* of T_n.

4. Density of Infinite weak Markov Systems and Zeros of Chebychev Polynomials

In this section we will assume that G is a finitely representable normalized infinite weak Markov system defined on an interval $[a, b]$. Clearly g_1 is increasing on $[a, b]$. If (h, c, P_n, U_n) is a representation of G_n, then $dp_1 = dg_1$; this implies that all the functions in $S(G_n)$ must be constant on the same subintervals of $[a, b]$ where g_1 is constant. To obtain density theorems for $C[a, b]$ we will therefore assume that g_1 is strictly increasing on $[a, b]$. Once such a density theorem is obtained, it is easy to obtain a corresponding density theorem valid in the case where g_1 is not strictly increasing. That theorem would obtain for the subset of functions in $C[a, b]$ that are constant on those subintervals of $[a, b]$ where g_1 is constant.

Let $\{T_n\}_{n\geq 1}$ be a sequence of generalized Chebychev polynomials associated with G. We define

$$M_n := \max\{|c_i - c_{i-1}| : 1 \leq i \leq n+1\},$$

where $c_1, \ldots, c_n$ are the ℓ–zeros of T_n, $c_0 = a$, and $c_{n+1} = b$.

We will also assume that G has the following property: *There are points a_1, b_1, $a \leq a_1 < b_1 \leq b$ such that for every $n \geq 1$, G_n is linearly independent in $[a, a_1] \cup [b_1, b]$.*

Although Theorem 1 implies that such points exist for each n, they depend on n, and they may coalesce with the endpoints; for example, we could have $\lim_{n\to\infty} a_1 = a$. With this additional hypothesis, the restriction of the functions in G to $[a, a_1] \cup [b_1, b]$ is a normalized infinite weak Markov system, and we obtain the following generalization of [10, Lemma 2]:

Lemma 1. *Let G satisfy the hypotheses of the previous paragraphs, let $a < a_1 < b_1 < b$, let f be the function defined in $A := [a, a_1] \cup [b_1, b]$ by*

$$f(x) := \begin{cases} 0 & \text{if } x \in [a, a_1] \\ 1 & \text{if } x \in [b_1, b], \end{cases}$$

and let $S_n \in S(G_n)$ denote a function whose restriction to A is an element of best approximation to f in A such that $f - S_n$ has an equioscillation of length $n+2$ (such a S_n exists by Theorem B). Then

(a) *S_n is increasing on $[a_1, b_1]$.*
(b) *Assume that $\lim_{n\to\infty} M_n = 0$. Then there is a constant K such that*

$$\|f - S_n\|_A \le K M_n/(b_1 - a_1),$$

where $\|\cdot\|_A$ denotes the norm of the supremum in A.

Proof. The proof of (a) is identical to that of [10, Lemma 2], using Theorem B instead of the alternation theorem for Chebychev systems, Proposition 4 instead of [10, Proposition 4], and Theorem 3(c) instead of [10, Theorem 2(c)] to show that $\{\widetilde{D}g_1, \ldots, \widetilde{D}g_n\}$ is a weak Markov system.

To establish (b) we repeat the steps used in the proof of [10, Lemma 2(b)]. The only difference is that in step (iii) we use ℓ–zeros instead of zeros, and Theorem 4 instead of [10, Corollary 1]. □

Theorem 5. *Let $G \subset C[a, b]$ be a finitely representable normalized infinite weak Markov system such that g_1 is strictly increasing on $[a, b]$. Assume there are points a_1, b_1, $a \le a_1 < b_1 \le b$, such that for every $n \ge 1$, G_n is linearly independent in $[a, a_1] \cup [b_1, b]$. Then the following propositions are equivalent:*

(a) *$S(G)$ is dense in $C[a, b]$, in the norm of the supremum.*
(b) *$\lim_{n\to\infty} M_n = 0$.*

Proof. If M_n does not converge to zero as $n \to \infty$, then there is a number $r > 0$ such that for each $k > 0$ we may choose an integer $n_k > 0$ such that $M_{n_k} \ge r$. For each k, let c_{j_k}, c_{j_k+1} be two consecutive ℓ–zeros of T_{n_k} such that $c_{j_k+1} - c_{j_k} = M_{n_k} \ge r$, where $1 \le j_k \le n_k$ depends of n_k. The sequence $\{c_{j_k} : k \ge 0\}$ will have a subsequence $\{\alpha_k : k \ge 1\}$ that converges to a point α_0.

In summation: If M_n does not converge to zero as $n \to \infty$, then there is a number $r > 0$ and sequences $\{r(k) : k \ge 1\}$ and $\{c_{j_{r(k)}} : k \ge 1\}$, such that $c_{j_{r(k)}+1} - c_{j_{r(k)}} = M_{r(k)} \ge r$, and $\lim c_{j_{r(k)}} = \alpha_0$.

Let

$$\alpha := \alpha_0 + \frac{2r}{10},\ \beta := \alpha_0 + \frac{8r}{10},$$

and let k_0 be such that if $k \ge k_0$, then $|c_{j_{r(k)}} - \alpha_0| < \frac{r}{10}$. Assume $k \ge k_0$; then $c_{j_{r(k)}} \in [\alpha_0 - r/10, \alpha_0 + r/10]$. Thus $\alpha_0 - r/10 \le c_{j_{r(k)}} \le \alpha_0 + r/10 < \alpha < \beta$, and therefore $0 < \beta - c_{j_{r(k)}} \le 9r/10$. Since $c_{j_{r(k)}+1} - c_{j_{r(k)}} \ge r$, we conclude that $\beta < c_{j_{r(k)}+1}$; thus $[\alpha, \beta] \subset (c_{j_{r(k)}+1}, c_{j_{r(k)}})$. Since $c_{j_{r(k)}}$ and $c_{j_{r(k)}+1}$ are consecutive ℓ–zeros of $T_{r(k)}$, this implies that $[\alpha, \beta]$ cannot contain an ℓ–zero of $T_{r(k)}$. From Theorem 3(e) we therefore conclude that either $[\alpha, \beta]$ contains one left endpoint of an equioscillation interval of $T_{r(k)}$, or $[\alpha, \beta]$ contains no left endpoint of an equioscillation interval of $T_{r(k)}$. We will consider the first alternative: the proof of the second alternative is similar and will be omitted.

Assume that $d_m^{r(k)} \in [\alpha, \beta]$, and let D denote the set of the remaining left endpoints of equioscillation intervals of $T_{r(k)}$. Thus $D \subset [a, \alpha] \cup [\beta, b]$ and D has $r(k)$ elements. Choosing now $\alpha < x_1 < x_2 < x_3 < x_4 < \beta$, let $f(x) \in C([a, b])$ be defined by

$$f(x) := \begin{cases} 0 & x \in [a, \alpha] \cup [\beta, b] \\ 2 & x = x_1,\ x = x_3 \\ -2 & x = x_2,\ x = x_4 \end{cases},$$

and by linear interpolation elsewhere in $[a, b]$.

Assume that for some n there is a function $q \in S(G_n)$ such that $\|f - q\| < 1/2$. Let $k \geq k_0$, $r(k) \geq n$, and let g be an element of best approximation from $S(G_{r(k)})$ to f. Since $S(G_n) \subset S(G_{r(k)})$, we see that $\|f - g\| \leq \|f - q\| < 1/2$. The definition of f implies that

$$|g(d)| < 1/2, \quad d \in D.$$

We therefore conclude that

$$\mathrm{sign}[T_{r(k)} - g](d) = \mathrm{sign} T_n(d), \quad d \in D.$$

Moreover, $\{x_1, x_2, x_3, x_4\}$ is a strong alternation for $T_{r(k)} - g$ in (α, β). Selecting three of these four points appropriately and joining them to the set D, we see that $T_{r(k)} - g$ has a strong alternation of length $r(k) + 3$. Since $T_{r(k)} - g \in G_{r(k)}$, and $G_{r(k)}$ is a weak Markov system, and therefore cannot have a strong alternation of length larger than $r(k) + 1$, we have obtained a contradiction. This shows that $(a) \Rightarrow (b)$.

To prove the converse let us assume, as in [10, Theorem 3], that $S(G)$ is not dense. Then there is a nonzero Borel measure, μ such that for every function $g \in S(G)$

$$\int_a^b g(t) d\mu(t) = 0.$$

Let $\varepsilon > 0$, b_2, be such that $a_1 < b_2 < b_1$ and a_2 such that $a_1 < a_2 < b_2$ and $\mu([a_2, b_2]) < \mu([a, b])/6$. The proof now is the same as in [10, Theorem 3, $(b) \Rightarrow (a)$], with a_2, b_2 instead of a_1, b_1, and Lemma 1 replacing [10, Lemma 2] because, since $[a, a_1] \cup [b_1, b] \subset [a, a_2] \cup [b_2, b]$, Lemma 1 is applicable. □

Corollary 1. *If* $\lim_{n\to\infty} M_n = 0$ *for* one *associated family of Tchebychev polynomials, then* $\lim_{n\to\infty} M_n = 0$ *for* every *associated family of Tchebychev polynomials.*

5. Jackson Type Theorems for Finite Systems

In this section we will assume that $G_n \subset C[a, b]$ is a normalized weak Markov system in $[a, b]$. Let $E_n(f)$ denote the distance from f to $S(G_n)$ in the norm of the supremum, and let $\underline{a} := a - 1$ and $\underline{b} := b + 1$.

For each $g \in G_n$ let us define $\overline{g}$ on $[\underline{a}, \underline{b}]$ as follows:

$$\overline{g}(x) := \begin{cases} g(a) & \text{if} \quad \underline{a} \leq x \leq a \\ g(x) & \text{if} \quad a < x < b \\ g(b) & \text{if} \quad b \leq x \leq \underline{b} \end{cases}.$$

It is clear that $\overline{G_n} := \{\overline{g_0}, \ldots, \overline{g_n}\}$ is a normalized weak Markov system on $[\underline{a}, \underline{b}]$.

Let $L(s) := \frac{1}{\sqrt{2\pi}} e^{-s^2/2}$ and $L_k(s) := kL(ks)$; $k \geq 1$. For every $f \in C[\underline{a}, \underline{b}]$ we set $f^{(k)} := f * L_k$, i. e.

$$f^{(k)}(x) = \int_{\underline{a}}^{\underline{b}} f(s) L_k(x - s) ds = \int_{-\infty}^{\infty} f(s) L_k(x - s) ds,$$

where in the second integral we understand f to equal 0 outside the interval $[\underline{a}, \underline{b}]$.

Under these conditions, $\lim_{k\to\infty} f^{(k)} = f$, uniformly on every closed subinterval of $(\underline{a}, \underline{b})$, and, if $d = \underline{a}$, or $d = \underline{b}$, then $f^{(k)}(d)$ converges to $\frac{1}{2}f(d)$. Moreover, $\overline{G}_n^{(k)} := \{\overline{g_0}^{(k)}, \ldots, \overline{g_n}^{(k)}\}$ is an ECT system on $[\underline{a}, \underline{b}]$ [13, pag. 15]. In particular, the functions in $\overline{G}_n^{(k)}$ are an ECT system on $[a, b]$, and $\overline{g}^{(k)}$ converges uniformly to $\overline{g}$ on $[a, b]$, for every $g \in S(G_n)$.

Let $g^{(k)}$ denote the restriction to the interval $[a, b]$ of $\overline{g}^{(k)} \in S(\overline{G_n}^{(k)})$, and let $G_n^{(k)} := \{g_0^{(k)}, \ldots, g_n^{(k)}\}$. Each $g \in S(G_n)$ is in one–to–one correspondence with $\overline{g} \in S(\overline{G_n})$, which (for each k) is in one–to–one correspondence with $\overline{g}^{(k)} \in S(\overline{G_n}^{(k)})$, which in turn are in one–to–one correspondence with its restrictions $g^{(k)}$:

$$g \longleftrightarrow \overline{g} \longleftrightarrow \overline{g}^{(k)} \longleftrightarrow g^{(k)}.$$

However, it is clear that $g^{(k)} \neq g * L_k$.

We now need a slight generalization of the main result of [11]. The proof is similar.

Lemma 2. *Let $f \in C[a, b]$, and let $\{f_k\} \subset C[a, b]$ be a sequence that converges uniformly to f in $[a, b]$. For each $k \geq 1$ let m_k be the element of best approximation to f_k from $S(G_n^{(k)})$. Then there is a subsequence $\{k_j\}$ such that $\{m_{k_j}\}$ converges uniformly in $[a, b]$ to an element of best approximation m to f from $S(G_n)$. Moreover, $f - m$ has an equioscillation.*

Proof. Since $\|f_k - m_k\| \leq \|f_k - 0\|$, and therefore $\|m_k\| \leq 2\|f_k\|$, we see that $\{m_k\}$ is uniformly bounded. Since $m_k = \sum_{i=0}^n \alpha_i^k g_i^{(k)}$, there is a subsequence $\{k_{1,j}\}$ and numbers $\alpha_0, \ldots, \alpha_n$, such that $\alpha_i^k \to \alpha_i$, $i = 0, \ldots, n$ if $k = k_{1,j} \to \infty$.

Let $m = \sum_{i=0}^n \alpha_i g_i$, and $g \in S(G_n)$. Since $m \in S(G_n)$, we see that $f_k - g^{(k)} \to f - g$ and $f_k - m_k \to f - m$ uniformly in $[a, b]$. Since $\|f_k - m_k\| \leq \|f_k - g^{(k)}\|$ we conclude that

$$\|f - m\| \leq \|f - g\|,$$

i. e., m is an element of best approximation to f.

Moreover, if $a \leq x_0^k < \ldots < x_{n+1}^k \leq b$ is an equioscillation for $f_k - m_k$, then there is a subsequence $\{k_{2,j}\}$ of $\{k_{1,j}\}$ and $a \leq x_0, \ldots, x_{n+1} \leq b$ such that

$$[f_k - m_k](x_i^k) = \varepsilon(-1)^i \|f_k - m_k\|, \quad \varepsilon = 1 \text{ or } \varepsilon = -1, \text{ (constant for } k = k_{2,j}),$$

and $x_i^k \to x_i$, $i = 0, \ldots, n$ if $k = k_{2,j} \to \infty$. Thus

$$[f - m](x_i) = \varepsilon(-1)^i \|f - m\|, \quad \varepsilon = 1 \text{ or } \varepsilon = -1.$$

This implies that $[f - m](x_i) = -[f - m](x_{i+1}) \neq 0$, $1 \leq i \leq n$, and therefore that the points x_i are all different. □

For each $k \geq 1$ and $a \leq x_0 < \ldots < x_{n+1} \leq b$, let

$$\mathbf{D}^{(k)} := \{g_i^{(k)}(x_{j+1}) - g_i^{(k)}(x_j) : 1 \leq i \leq n;\ 0 \leq j \leq n\},$$

let $\mathbf{D}_j^{(k)}$ be obtained from $\mathbf{D}$ by deleting the j^{th} column, and $d_j^{(k)} := \det \mathbf{D}_j^{(k)}$, for $0 \leq j \leq n$. In [10, Lemma 5] we showed that $d_j^{(k)} \geq 0$ and $\sum_{j=0}^n d_j^{(k)} > 0$. Setting

$$a_j^{(k)} := \frac{d_j^{(k)}}{2\sum_{j=0}^n d_j^{(k)}}$$

and

$$\delta^{(k)} := \sup_{a \leq x_0 < \ldots < x_{n+1} \leq b} \left\{ \sum_{j=0}^n a_j^{(k)} [g_1^{(k)}(x_{j+1}) - g_1^{(k)}(x_j)] \right\},$$

then, if $\omega(f)$ denotes the modulus of continuity of f, we have:

Theorem 6. *Let $f \in C[a,b]$ and $\delta := \overline{\lim}_{k\to\infty}\delta^{(k)}$. Then $\delta < \infty$ and*

$$E_n(f) \leq \frac{3}{2}\omega(f \circ g_1^{-1}; \delta).$$

Proof. Since $|a_j^{(k)}| \leq \frac{1}{2}$ and $\|g_1^{(k)}\| \leq \|g_1\|$ for every k, we see that the sequence $\{\delta^{(k)}\}$ is bounded; thus $\delta = \overline{\lim}_{k\to\infty}\delta^{(k)} < \infty$. Let m_k be the element of best approximation to f from $S(G_n^{(k)})$. Since $\{f^{(k)}\}$ converges to f, uniformly in $[a,b]$, applying Lemma 2 we obtain a subsequence $\{k_j\}$ and an element of best approximation m, to f from $S(G_n)$, such that $\{m_{k_j}\}$ converges uniformly to m in $[a,b]$. Now, let us choose a subsequence $\{k_{1,j}\}$ of $\{k_j\}$ such that $\delta^{(k_{1,j})} \to \delta$ when $j \to \infty$. For convenience, let us denote it again by $\{k\}$.

Let $f_1 := f \circ g_1^{-1}$, $\varepsilon > 0$, and let $r_0 > 0$ be such that if $r < r_0$, then $\omega(f_1; r) < \varepsilon/2$. Choosing k_0 so that if $k \geq k_0$ then $|\delta^{(k)} - \delta| < r_0/2$, we have:

$$\omega(f_1; \delta^{(k)}) \leq \omega(f_1; \delta) + \omega(f_1; r_0/2) < \omega(f_1; \delta) + \varepsilon/2$$

and

$$\omega(f_1; \delta) \leq \omega(f_1; \delta^{(k)}) + \omega(f_1; r_0/2) < \omega(f_1; \delta^{(k)})) + \varepsilon/2.$$

Thus $\omega(f \circ g^{-1}; \delta^{(k)}) \to \omega(f \circ g^{-1}; \delta)$ as $k \to \infty$.

Applying [10, Theorem 6] to $G_n^{(k)}$ we have:

$$E_n^{(k)}(f) := \|f - m_k\| \leq \frac{3}{2}\omega(f \circ g_1^{-1}; \delta^{(k)}).$$

Making $k \to \infty$, the assertion follows □

For each $t \in [a,b]$ and $k \geq 1$, let $\sigma_{k,t}$ be the function defined in $[a,b]$ as follows:

$$\sigma_{k,t}(s) = \begin{cases} 0 & \text{for } a \leq s \leq t \\ g_1^{(k)}(s) - g_1^{(k)}(t) & \text{for } t < s \leq b \end{cases}.$$

If t is arbitrary but fixed, it is clear that as $k \to \infty$ the function $\sigma_{k,t}$ converges uniformly on $[a,b]$ to

$$\sigma_t(s) = \begin{cases} 0 & \text{for } a \leq s \leq t \\ g_1(s) - g_1(t) & \text{for } t < s \leq b \end{cases}.$$

Let $\theta_{k,t}$ the element of best approximation to $\sigma_{k,t}$ from $S(G_n^{(k)})$, and let $E_n^{(k)}(\sigma_{k,t}) := \|\sigma_{k,t} - \theta_{k,t}\|$.

Lemma 3. *Let $n \geq 0$ be arbitrary but fixed. The sequence $\{E_n^{(k)}(\sigma_{k,t}) : k \geq 1\}$, of functions of t, is uniformly bounded and uniformly continuous.*

Proof. The uniform boundednes follows from

$$|E_n^{(k)}(\sigma_{k,t})| = \|\sigma_{k,t} - \theta_{k,t}\| \leq \|\sigma_{k,t} - 0\| \leq 2\left\|g_1^{(k)}\right\| \leq 2\|g_1\|$$

If $t_1, t_2 \in [a,b]$, then

$$\|\sigma_{k,t_1} - \theta_{k,t_1}\| \leq \|\sigma_{k,t_1} - \theta_{k,t_2}\| \leq \|\sigma_{k,t_1} - \sigma_{k,t_2}\| + \|\sigma_{k,t_2} - \theta_{k,t_2}\|.$$

Therefore

$$E_n^{(k)}(\sigma_{k,t_1}) - E_n^{(k)}(\sigma_{k,t_2}) \leq \|\sigma_{k,t_1} - \sigma_{k,t_2}\|.$$

A similar argument yields

$$E_n^{(k)}(\sigma_{k,t_2}) - E_n^{(k)}(\sigma_{k,t_1}) \leq \|\sigma_{k,t_2} - \sigma_{k,t_1}\|.$$

Thus

$$|E_n^{(k)}(\sigma_{k,t_2}) - E_n^{(k)}(\sigma_{k,t_1})| \leq \|\sigma_{k,t_2} - \sigma_{k,t_1}\|.$$

But $\|\sigma_{k,t_2} - \sigma_{k,t_1}\| \le |g_1^{(k)}(t_2) - g_1^{(k)}(t_1)|$ (cf. [10, Lemma 6]). However,

$$|g_1^{(k)}(t_2) - g_1^{(k)}(t_1)| \le \int_{-\infty}^{\infty} |g_1(t_2 - s) - g_1(t_1 - s)| L_k(s)\, ds,$$

which implies that the sequence $\{g_1^{(k)} : k \ge 1\}$ is uniformly continuous, which in turn implies that $\{E_n^{(k)}(\sigma_{k,t}) : k \ge 1\}$ is uniformly continuous. □

Theorem 7. *Let $G_n \subset C[a,b]$ be a normalized weak Markov system in $[a,b]$. Then*

(a) *$E_n(\sigma_t)$ is a continuous function of t.*

(b) *If*

$$\Delta_n := \max_{a \le t \le b} E_n(\sigma_t),$$

then

$$\delta \le \sqrt{[g_1(b) - g_1(a)]\Delta_n}.$$

Proof. Applying Lemma 3 and Arzelá's theorem we see that there is a sequence $\{k_{1,j}\}$ such that $E_n^{(k_{1,j})}(\sigma_{k_{1,j},\cdot})$ converges uniformly on $[a,b]$ to a continuous function E.

For each fixed $t \in [a,b]$, Lemma 2. implies that there is a subsequence $\{k_{2,j}\}$ of $\{k_{1,j}\}$ such that $E_n^{(k_{2,j})}(\sigma_{k_{2,j},t})$ converges to $E_n(\sigma_t)$. This shows that $E(t) = E_n(\sigma_t)$, and (a) follows.

Setting $k = k_{2,j}$ and $\Delta_n^{(k)} := \max_{a \le t \le b} E_n^{(k)}(\sigma_{k,t})$, we see that $\Delta_n^{(k)} \to_{k\to\infty} \Delta_n$. Applying [10, Theorem 7] to $G_n^{(k)}$, we have:

$$\delta^{(k)} \le \sqrt{[g_1^{(k)}(b) - g_1^{(k)}(a)]\Delta_n^{(k)}}.$$

Finally, if $k = k_{3,j}$ is a subsequence of $k_{2,j}$ such that $\lim_{k\to\infty} \delta^{(k)} = \delta$, we have:

$$\delta = \lim_{k\to\infty} \delta^{(k)} \le \lim_{k\to\infty} \sqrt{[g_1^{(k)}(b) - g_1^{(k)}(a)]\Delta_n^{(k)}} = \sqrt{[g_1(b) - g_1(a)]\Delta_n}.$$

□

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