ON THE STABILITY OF FRAMES AND RIESZ BASES[1]

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1. Introduction

Orthogonal bases make it possible to represent an element of a Hilbert space as an infinite series. It is the easiest way to represent a complicated vector in terms of simpler ones. This is a problem that appears often in many areas of mathematics, physics, and engineering, like harmonic analysis, differential equations, quantum mechanics, scattering theory, and signal and image processing, to name just a few. Although in theory easy to implement, expansion in orthogonal series is sometimes problematic: It is not always easy to find a suitable orthogonal basis, and there are cases where an expansion in orthogonal series, or even in series generated by more general bases, may not be an adequate representation method.

Frames have many of the desirable properties of bases, while differing in a very important aspect: they may be linearly dependent, and therefore the uniqueness of representation characteristic of bases may be lost. This redundancy has important applications in, for example, signal and image processing, because it leads to robustness: the quality of the signal is less affected by the presence of noise, and the signal may be reconstructed from sampling done at relatively low precision.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| := \langle \cdot, \cdot \rangle^{1/2}$. A sequence $\{f_n, n \in Z^+\} \subset \mathcal{H}$ is called a *frame* if there are constants A and B such that for every $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{n \in Z^+} |\langle f, f_n \rangle|^2 \leq B\|f\|^2.$$

The constants A and B are called *bounds* of the frame. The supremum of all such A and the infimum of all such B are called *best bounds.* If only the right-hand inequality is satisfied for all $f \in \mathcal{H}$, then $\{f_n, n \in Z^+\}$ is called a *Bessel sequence* with bound B. Excellent introductions to the theory of frames and Bessel sequences can be found in [3, 12, 13, 25, 39]. One of the properties that will be used frequently is the following: $\{f_n, n \in Z^+\}$ is a Bessel sequence with bound M if and only if, for every *finite* sequence of scalars $\{c_k\}$,

$$\| \sum_k c_k f_k \|^2 \leq M \sum_k |c_k|^2. \tag{1}$$

(cf. eg [39, p.155, Theorem 3]). As remarked by Chui and Shi in [9, Lemma 4], it is a atraightforward consequence of this statement that $\{f_n, n \in Z^+\}$ is a Bessel sequence with bound M if and only if (1) is satisfied for every sequence $\{c_k\}$ in ℓ^2. A frame is called exact, or a *Riesz basis*, if upon the removal of any single element of the sequence, it ceases to be a frame. However, not every frame is a Riesz basis: As is well known, a sequence $\{f_n, n \in Z^+\} \subset \mathcal{H}$ is a Riesz basis if and only if it is the image of an orthonormal basis under a bounded invertible linear operator $U : \mathcal{H} \to \mathcal{H}$ ([39]). If we say that $\{f_n, n \in Z^+\}$ is a Riesz basis with bounds A and B, we mean that A and B are its frame bounds.

The theory of frames in Hilbert spaces has recently been generalized by S. T. Ali, J. -P. Antoine and J. -P. Gazeau [1] to one where the basis vectors may be labelled using discrete, continuous, or a mixture of the two types of indices.

Frames were introduced by Duffin and Schaeffer [15] to study an irregular sampling problem (for other applications, see, e.g.[14]). In the same paper they introduced the *frame algorithm*, which makes it possible to reconstruct uniquely and stably any element $f \in \mathcal{H}$ from the sequence of coefficients $\{\langle f, f_k\rangle, k \in Z^+\}$, and the frame bounds A and B. The stability that has just been mentioned is a *theoretical* stability: The algorithm works if the elements f_n are exactly known. In some cases, however, it could happen that the f_n are actually approximated by a different set of elements. For example, for the case of wavelet frames in $\mathcal{R}$, i.e. sequences of the form $\{a^{j/2}\phi(a^j x - bk), j, k \in Z\}$, this may happen if, because of problems of numerical computation, ϕ is replaced by some approximation ψ or if, for reasons having to do with sampling, measurement, or machine representation of integers, the k are replaced by approximations $\{\lambda_{j,k}\}$, leading to an irregular sampling problem. These issues can be addressed by a study of the stability of the frame algorithm under small perturbations, which is a motivation for the stability problem to be studied here: Given a sequence $\{g_k\}$ that is in some sense "close" to the frame or Riesz basis $\{f_k\}$, find conditions to ensure that it is also a frame or Riesz basis. For bases, this problem has been studied for many years (cf., e.g., [31, 32, 39], and in Section 3 we have applied some of the techniques that have been used in that context. There is some overlap between the results in this section and those previously and independently obtained by O. Christensen, which will discussed at the appropriate places in the sequel.

The stability of frames in Hilbert space has been studied by C. E. Heil and O. Christensen [8] and D. Walnut [37] from the point of view of a more general stability theory for atoms in Banach spaces. This theory was introduced by Feichtinger and Gröchenig ([17, 18, 21]), and one of its features is the generalization of the concept of a frame to Banach spaces: Given a Banach space $\mathcal{X}$, let its norm be denoted by $\|\cdot\|_{\mathcal{X}}$ and its dual by $\mathcal{X}'$. Let $\mathcal{X}_f$ be a Banach space of scalar valued sequences indexed by Z^+, $F := \{f_n, n \in Z^+\} \subset \mathcal{X}', \mathcal{S} : \mathcal{X}_f \to \mathcal{X}$. Following e.g. [21], the ordered pair (F, S) will be called a *Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_f$* provided that:
(a) $\{f_n(x), n \in Z^+\} \in \mathcal{X}_f$ for every $x \in \mathcal{X}$;
(b) There are constants A and B such that, for every $x \in \mathcal{X}$,

$$A\|x\|_{\mathcal{X}} \le \|\{f_n(x), n \in Z^+\}\|_{\mathcal{X}_f} \le B\|x\|_{\mathcal{X}},$$

and
(c) S is bounded and linear, and for every $x \in \mathcal{X}$, $S(\{f_n(x), n \in Z^+\}) = x$.
Just as for Hilbert spaces, A and B are called *bounds* of the frame, and the supremum of all such A and the infimum of all such B are called *best bounds*. S is called the *reconstruction operator*.

In Section 3, stability conditions for frames and Riesz bases in Banach and Hilbert spaces are obtained, which complement the results of Christensen. Since the present paper focuses mainly on Hilbert spaces, direct and elementary proofs of Christensen's results are also given, without using the theory of atoms in Banach spaces. Then, in Section 4, a condition for $\{e^{i\langle k,t\rangle} - e^{i\langle\lambda_k,t\rangle}, k \in Z^d\}$ to be a Bessel sequence in $L^2(I)$, where I is an interval in $\mathcal{R}^\lceil$, is found. This result, together with two of Christensen's theorems, the multivariate version of Chui and Shi's Second Oversampling Theorem [9, Theorem 8], and a variety of other theorems and techniques, are used to study the stability of frames and Riesz bases in three concrete cases: In Section 4 a multivariate version of Kadec's 1/4-theorem is proved, whereas Section

5 and Section 6 deal with wavelet (or affine) frames and Gabor (or Weyl - Heisenberg) frames respectively. All these results include estimates for frame bounds, since they are necessary for implementing the frame algorithm. Although the conjugate gradient acceleration method discussed by Gröchenig [22] does not use these bounds in its implementation, they are still needed to set up stopping criteria and to estimate the speed of convergence. There are very few references on the stability of wavelet and Gabor Riesz bases, and none seem to include any explicit quantitative estimates for regions of stability. The key to our approach is the application of stability results for frames. This allows us to describe regions of validity in terms of frame bounds. Some of the theorems in the last two sections can be used to determine under which conditions a coherent state frame or Riesz basis can be perturbed into another coherent state frame or Riesz basis. (For a discussion of coherent states, from a variety of perspectives, see [14, 16, 25, 29] and references thereof.)

We should point out that there are no Riesz bases of translates, i.e., if $f \in L^2(\mathcal{R}^{\lceil})$, and $\{\lambda_n, n \in Z^+\} \subset \mathcal{R}^{\lceil}$, then $\{f(\cdot - \lambda_n), n \in Z^+\}$ cannot be a Riesz basis of $L^2(\mathcal{R}^{\lceil})$. For $d = 1$ this was shown in [30], and the generalization for arbitrary d seems obvious. For a related result, see [40].

2. Notation.

This section contains notation that has not been defined in the preceding discussion. However, notation that appears in only one section may be defined at the beginning of that section.

In the sequel, $Z, Z^+, \mathcal{R}$ and $\mathcal{C}$ will respectively denote the integers, the strictly positive integers , the real numbers, and the complex numbers; d and N will always be elements of Z^+. If $a = (a_1, a_2, \ldots, a_d) \in \mathcal{R}^{\lceil}$, then $|a| := (a_1^2 + a_2^2 + \ldots + a_d^2)^{1/2}$, and $|a|_1$ will denote the taxicab norm, v.i.z. $|a|_1 := |a_1| + |a_2| + \ldots + |a_d|$. If $b = (b_1, b_2, \ldots, b_d) \in \mathcal{R}^{\lceil}$, then $a \geq b$ $(a > b)$ or $b \leq a$ $(b < a)$ means that $a_k \geq b_k (a_k > b_k), k = 1, \ldots, d$. If $a \leq b$, then $[a, b] := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ is called an interval with endpoints a and b. The following notation is also used:

$$a^d := \prod_{r=1}^{d} a_r, \quad a^{d/2} := (a^d)^{1/2}, \quad a^{-d/2} := (a^{d/2})^{-1}, \quad 1/a := (a_1^{-1}, a_2^{-1}, \cdots, a_d^{-1}).$$

If $x \in \mathcal{R}$, then $\boldsymbol{x} := (x, x, \cdots, x)$, and $x^a := \prod_{r=1}^{d} x^{a_r}$. Also:

$$\sum_{k \in Z^d} x_k := \lim_{N \to \infty} \sum_{|k| \leq N} x_k.$$

When there is no danger of ambiguity, the integral over $\mathcal{R}^{\lceil}$ will be denoted by "$\int$". The Fourier transform of a function g will be denoted by $\hat{g}$ or by $\mathcal{F}\}$. If $g \in L(\mathcal{R}^{\lceil})$,

$$\hat{g}(x) := \int e^{-2\pi i \langle x, t \rangle} g(t)\, dt, \qquad x \in \mathcal{R}^{\lceil}.$$

The support of the function $g(x)$ will be denoted by $\mathrm{supp}(g)$. The $L^2(\mathcal{R}^{\lceil})$ norm of g will be denoted by $\|g\|$. For $0 < L < \frac{1}{2}$, $B_d(L)$ is defined recursively as follows: $B_1(L) := 1 - cos\, \pi L +$

$sin\,\pi L$, and for $d > 1$,

$$B_d(L) := \{B_{d-1}^{1/2}(L) + B_1^{1/2}(L)[1 + B_{d-1}^{1/2}(L)]\}^2.$$

3. Frames in Banach and Hilbert Spaces.

In this section $\mathcal{X}_\infty$ and $\mathcal{X}_\in$ are Banach spaces and $\mathcal{H}, \mathcal{H}_\infty$, and $\mathcal{H}_\in$ are Hilbert spaces. If $x \in \mathcal{H}$, then $x' \in \mathcal{H}'$ is defined by $x' := \langle \cdot, x \rangle$. For $j = 1, 2$, $C_j : \mathcal{X}_| \to \mathcal{X}_|''$ is defined by $(C_j x)(f) := f(x)$. The operator C_j is called the *canonical embedding* of $\mathcal{X}_|$ into $\mathcal{X}_|''$. If $\mathcal{X}_\infty$ and $\mathcal{X}_\in$ are Hilbert spaces, then $e_j : \mathcal{X}_| \to \mathcal{X}_|'$ is defined by $e_j(x) := x'$. Given a linear operator $U : \mathcal{X}_\infty \to \mathcal{X}_\in$, its (Banach space) adjoint operator will be denoted by U'. If $\mathcal{X}_\infty$ and $\mathcal{X}_\in$ are Hilbert spaces, then the Hilbert space adjoint operator of U will be denoted by U^*; thus $U^* = e_1^{-1} U' e_2$ ([28, 36]). If $x := \{x_k, k \in Z^+\}$ is a scalar sequence, then $P_n(x) := x_n$.

Three different spaces are involved in the definition of a Banach frame: $\mathcal{X}, \mathcal{X}'$, and $\mathcal{X}_\int$. The first theorem explores what effect a linear homeomorphism has on the frame structure, when applied to each one of these spaces:

Theorem 1. *Let $(\{f_n, n \in Z^+\}, S)$ be a Banach frame with respect to $\mathcal{X}_\int$ in $\mathcal{X}_\infty$ with best bounds A_1 and B_1.*

(a) If $U : \mathcal{X}_\infty \to \mathcal{X}_\in$ is a linear homeomorphism, then $(\{(U^{-1})' f_n, n \in Z^+\}, US)$ is a Banach frame with respect to $\mathcal{X}_\int$ in $\mathcal{X}_\in$, and its best bounds A_2, B_2 satisfy the inequalities

$$A_1 \|U\|^{-1} \le A_2 \le A_1 \|U^{-1}\|, \quad and \quad B_1 \|U\|^{-1} \le B_2 \le B_1 \|U^{-1}\|.$$

(b) Assume that $\mathcal{X}_\infty$ and $\mathcal{X}_\in$ are reflexive spaces, and let $V : \mathcal{X}_\infty' \to \mathcal{X}_\in'$ be a linear homeomorphism. If $T := C_2^{-1}(V')^{-1} C_1$, then $(\{(T^{-1})' f_n, n \in Z^+\}, TS)$ is a Banach frame with respect to $\mathcal{X}_\int$ in $\mathcal{X}_\in$, and its best bounds A_2, B_2 satisfy the inequalities

$$A_1 \|T\|^{-1} \le A_2 \le A_1 \|T^{-1}\|, \quad and \quad B_1 \|T\|^{-1} \le B_2 \le B_1 \|T^{-1}\|.$$

(c) Let $\mathcal{X}_\sqcup$ be a Banach space of scalar valued sequences indexed by Z^+, and assume that $W : \mathcal{X}_\int \to \mathcal{X}_\sqcup$ is an isometric isomorphism. If $g_n(x) := P_n W\{f_n(x), n \in Z^+\}$, then $(\{g_n, n \in Z^+\}, SW^{-1})$ is a Banach frame with respect to $\mathcal{X}_\sqcup$ in $\mathcal{X}_\infty$, with best bounds A_1 and B_1.

Proof. (a) Let x denote an arbitrary element of $\mathcal{X}_\infty$, and let y denote an arbitrary element of $\mathcal{X}_\in$. By hypothesis,

$$A_1 \|x\|_{\mathcal{X}_\infty} \le \|\{f_n(x), n \in Z^+\}\|_{\mathcal{X}_\int} \le B_1 \|x\|_{\mathcal{X}_\infty}.$$

But

$$\|y\|_{\mathcal{X}_\in} = \|UU^{-1}y\|_{\mathcal{X}_\in} \le \|U\| \; \|U^{-1}y\|_{\mathcal{X}_\infty}.$$

Thus,

$$A_1 \|U\|^{-1} \; \|y\|_{\mathcal{X}_\in} \le A_1 \; \|U^{-1}y\|_{\mathcal{X}_\infty} \le \|\{f_n(U^{-1}y), n \in Z^+\}\|_{\mathcal{X}_\int}$$

$$(= \|(U^{-1})' f_n(y), n \in Z^+\|_{\mathcal{X}_\int}) \le B_1 \; \|U^{-1}y\|_{\mathcal{X}_\infty} \; \le B_1 \; \|U^{-1}\| \; \|y\|_{\mathcal{X}_\in}.$$

Moreover, $(U^{-1})' f_n(y) = f_n(U^{-1}y) \in \mathcal{X}_\int$, $(U^{-1})'S$ is linear and bounded, and

$$U\,S\{(U^{-1})' f_n(y), n \in Z^+\} = U\,S\{f_n(U^{-1}y), n \in Z^+\} = U\,U^{-1}y = y.$$

We have therefore established that $(\{(U^{-1})' f_n, n \in Z^+\}, U\,S)$ is a frame with respect to $\mathcal{X}_\int$, and that

$$A_1\|U\|^{-1} \leq A_2,\ B_2 \leq B_1\|U^{-1}\|.$$

On the other hand,

$$A_2\|y\|_{\mathcal{X}_\in} \leq \|\{(U^{-1})' f_n(y), n \in Z^+\}\|_{\mathcal{X}_\int} \leq B_2\|y\|_{\mathcal{X}_\in}.$$

But

$$\|x\|_{\mathcal{X}_\infty} \leq \|U^{-1}\|\ \|Ux\|_{\mathcal{X}_\in},$$

whence

$$A_2\|U^{-1}\|^{-1}\ \|x\|_{\mathcal{X}_\infty} \leq A_2\|Ux\|_{\mathcal{X}_\in} \leq \|\{(U^{-1})' f_n(Ux), n \in Z^+\}\|_{\mathcal{X}_\int}$$

$$(= \|\{f_n(x), n \in Z^+\}\|_{\mathcal{X}_\int}) \leq B_2\ \|Ux\|_{\mathcal{X}_\in} \leq B_2\ \|U\|\ \|f\|_{\mathcal{X}_\infty}.$$

This implies that

$$A_2\|U^{-1}\|^{-1} \leq A_1, \qquad B_1 \leq B_2\ \|U\|,$$

and the conclusion follows.

(b) Since the spaces $\mathcal{X}_\infty, \mathcal{X}_\in$ are reflexive, the canonical mappings are surjective, and therefore T is a linear homeomorphism from $\mathcal{X}_\infty$ onto $\mathcal{X}_\in$. Applying the conclusion of part (a), the assertion follows.

(c) Note that

$$SW^{-1}\{g_n(x), n \in Z^+\} = SW^{-1}W\{f_n(x), n \in Z^+\} = S\{f_n(x), n \in Z^+\} = x.$$

Moreover,

$$\|\{g_n(x), n \in Z^+\}\|_{\mathcal{X}_\sqcup} = \|W\{f_n(x), n \in Z^+\}\|_{\mathcal{X}_\sqcup} = \|\{f_n(x), n \in Z^+\}\|_{\mathcal{X}_\int},$$

and the conclusion follows. □

Let us now turn our attention to Hilbert spaces. The following consequence of Theorem 1(a) strengthens part of [24, Lemma 6.3.2]:

Corollary 1. *Let $U : \mathcal{H}_\infty \to \mathcal{H}_\in$ be a linear homeomorphism, and assume that $\{f_n, n \in Z^+\}$ is a frame in $\mathcal{H}_\infty$ with best bounds A_1 and B_1. Then $\{Uf_n, n \in Z^+\}$ is a frame in $\mathcal{H}_\in$, and its best bounds A_2, B_2 satisfy the inequalities*

$$A_1\|U^{-1}\|^{-2} \leq A_2 \leq A_1\|U\|^2 \quad \textit{and} \quad B_1\|U^{-1}\|^{-2} \leq B_2 \leq B_1\|U\|^2.$$

Proof. Let $V := (U^{-1})^*$. If T denotes the frame operator associated with $\{f_k, k \in Z^+\}$, v.i.z.

$$Tf := \sum_{k\in Z^+} \langle f, f_k\rangle f_k,$$

$\tilde{f}_n := T^{-1}f_n$, and $S : \ell^2 \to \mathcal{H}$ is defined by

$$S(\{c_n, n \in Z^+\}) := \sum_{n\in Z^+} c_n\tilde{f}_n$$

then, as remarked in [8], it is easy to see that $(\{f_n', n \in Z^+\}, S)$ is a Banach frame in $\mathcal{H}_\infty$ with respect to ℓ^2, with best bounds $A_1^{1/2}, B_1^{1/2}$. Applying Theorem 1(a) it follows that $(\{(V^{-1})'f_n', n \in Z^+\}, VS)$ is a Banach frame in $\mathcal{H}_\in$ with respect to ℓ^2, with best bounds $A_2^{1/2}, B_2^{1/2}$ that satisfy the inequalities

$$A_1^{1/2}\|V\|^{-1} \le A_2^{1/2} \le A_1^{1/2}\|V^{-1}\| \quad \text{and} \quad B_1^{1/2}\|V\|^{-1} \le B_2^{1/2} \le B_1^{1/2}\|V^{-1}\|,$$

However, $\|V\| = \|U^{-1}\|$, and $\|V^{-1}\| = \|U\|$. Moreover,

$$(V^{-1})' = ([(U^{-1})^*]^{-1})' = [U^*]' = e_2(U^*)^*e_1^{-1} = e_2Ue_1^{-1}.$$

Since $e_2Ue_1^{-1}f_n' = (Uf_n)'$, the conclusion follows. □

Example 1. Let $\{f_n, n \in Z^d\}$ be a frame in $L^2(\mathcal{R}^\lceil)$ with bounds A and B, and let $\mathcal{F}$ denote the Fourier transform operator. Since $\mathcal{F}$ is an isometry, $\|\mathcal{F}\| = \|\mathcal{F}^{-\infty}\| = \infty$; thus Corollary 1 implies that both $\{\mathcal{F}\{_\backslash, \backslash \in \mathcal{Z}^+\}$ and $\{\mathcal{F}^{-\infty}\{_\backslash, \backslash \in \mathcal{Z}^+\}$ are frames in $L^2(\mathcal{R}^\lceil)$ with bounds A and B. This, of course, also follows from Plancherel's formula. Since $\mathcal{F}$ and $\mathcal{F}^{-\infty}$ are linear isomorphisms from $L^2(\mathcal{R}^\lceil)$ onto $L^2(\mathcal{R}^\lceil)$, the sequences $\{f_n, n \in Z^d\}, \{\mathcal{F}\{_\backslash, \backslash \in \mathcal{Z}^+\}$, and $\{\mathcal{F}^{-\infty}\{_\backslash, \backslash \in \mathcal{Z}^+\}$ are *fully equivalent* (cf. [34, pp. 68-69]).

Example 2. Assume $\psi \in L^2(\mathcal{R})$ is such that

$$\int_{\mathcal{R}} |\xi|^{-1}\,|\hat{\psi}|^2\,d\xi = 1,$$

and let the wavelet transform $\mathcal{T}^{wav}$ be defined by

$$(\mathcal{T}^{wav}f)(x, y) := \int_{\mathcal{R}} |t|^{-1/2}f(t)\overline{\psi(\frac{t-y}{x})}\,dt.$$

Let $\mathcal{H}_\infty := \mathcal{L}^\in(\mathcal{R})$, and let $\mathcal{H}_\in := \mathcal{T}^{\sqsupseteq\dashv\sqsubseteq}(\mathcal{H}_\infty)$, endowed with the inner product

$$\langle f, g\rangle := \int_{\mathcal{R}^\in} x^{-2}\,f(x, y)\,\overline{g(x, y)}\,dx\,dy.$$

From e.g. [12, p. 24, Proposition 2.4.1 and p. 31] we know that $\mathcal{H}_\in$ is a Hilbert space, and that $\mathcal{T}^{wav}$ is an isometric linear homeomorphism from $\mathcal{H}_\infty$ onto $\mathcal{H}_\in$. Thus, as in the preceding example, it is clear that if $\{f_n, n \in Z^+\}$ is a frame in $\mathcal{H}_\infty$ with bounds A and B, then $\{\mathcal{T}^{wav}f_n, n \in Z^+\}$ is a frame in $\mathcal{H}_\in$ with bounds A and B.

The sequences described in the preceding examples are *unitarily equivalent*, i. e. there is a unitary linear isomorphism U from $\mathcal{H}_\infty$ onto $\mathcal{H}_\in$, such that $Uf_n = g_n$ for all n. This and other kinds of equivalence between frames are discussed in [1].

The following elementary result will play a pivotal role in the subsequent disscussion:

Theorem 2. *Let* $\{f_n, n \in Z^+\}$ *be a frame in* $\mathcal{H}$ *with bounds* A *and* B, *let* $\{g_n, n \in Z^+\}$ *be a Bessel sequence in* $\mathcal{H}$ *with bound* M, *and let* λ *be a complex number such that* $|\lambda| < (A/M)^{1/2}$. *Then* $\{f_n + \lambda g_n, n \in Z^+\}$ *is a frame in* $\mathcal{H}$ *with frame bounds* $[(A)^{1/2} - |\lambda|(M)^{1/2}]^2$ *and* $[(B)^{1/2} + |\lambda|(M)^{1/2}]^2$.

The proof of this theorem is a straightforward consequence of the triangle inequality for sequences in ℓ^2, and will therefore be omitted.

Setting $\lambda = -1$ and replacing g_n by $f_n - g_n$ in Theorem 2, we obtain a result of Christensen ([7, Corollary 2.7] or, more explicitly, [8, Corollary 6]). It was also obtained as a corollary of a more general statement on Banach frames (c.f. e.g., [8, Theorem 1]). (See also [6, Theorem 1]).

Theorem 3. *Let* $\{f_n, n \in Z^+\}$ *be a frame in* $\mathcal{H}$ *with bounds* A *and* B. *Assume* $\{g_n, n \in Z^+\} \subset \mathcal{H}$ *is such that* $\{f_n - g_n, n \in Z^+\}$ *is a Bessel sequence with bound* $M < A$. *Then* $\{g_n, n \in Z^+\}$ *is a frame with bounds* $[1 - (M/A)^{1/2}]^2 A$ *and* $[1 + (M/B)^{1/2}]^2 B$.

Heil and Christensen have remarked that, since $|\langle f, f_n - g_n\rangle| \leq \|f\| \, \|f_n - g_n\|$, Theorem 3 immediately yields another result of Christensen (cf.[5] and also [4, Proposition 2.4]):

Theorem 4. *Let* $\{f_n, n \in Z^+\}$ *be a frame in* $\mathcal{H}$ *with bounds* A *and* B, *and assume that*

$$M := \sum_{n \in Z^+} \|f_n - g_n\|^2 < A.$$

Then $\{g_n, n \in Z^+\}$ *is a frame in* $\mathcal{H}$ *with bounds* $A[1 - (M/A)^{1/2}]^2$ *and* $B[1 + (M/B)^{1/2}]^2$.

We will need the following:

Theorem 5. *Let* $\{f_n\}$ *be a Riesz basis in* $\mathcal{H}$ *with bounds* A *and* B. *Assume* $\{g_n, n \in Z^+\} \subset \mathcal{H}$ *is such that* $\{f_n - g_n, n \in Z^+\}$ *is a Bessel sequence with bound* $M < A$. *Then* $\{g_n, n \in Z^+\}$ *is a Riesz basis with bounds* $[1-(M/A)^{1/2}]^2 A$ *and* $[1+(M/B)^{1/2}]^2 B$. *Conversely, if* $\{f_n, n \in Z^+\}$ *and* $\{g_n, n \in Z^+\}$ *are Riesz bases in* $\mathcal{H}$ *with bounds* A_1, B_1 *and* A_2, B_2 *respectively, and* U *is a linear homeomorphism such that* $Uf_n = g_n, n \in Z^+$, *then* $\{f_n - g_n, n \in Z^+\}$ *is a Bessel sequence with bound* $M := min\,\{B_1 \|I - U\|^2, B_2 \|I - U^{-1}\|^2\}$.

Proof. Assume first that $\{f_n, n \in Z^+\}$ is a Riesz basis with frame bounds A and B, and $\{f_n - g_n, n \in Z^+\}$ a Bessel sequence with bound $M < A$. Thus, as remarked in Section 1,

$$\|\sum_{k=1}^{n} c_k(f_k - g_k)\|^2 \leq M \sum_{k=1}^{n} |c_k|^2 \tag{2}$$

for every finite sequence of scalars $\{c_1, \ldots, c_n\}$. Theorem 3 implies that $\{g_n, n \in Z^+\}$ is a frame with bounds $[1 - (M/A)^{1/2}]^2 A$ and $[1 + (M/B)^{1/2}]^2 B$. To prove that $\{g_n, n \in Z^+\}$ is a Riesz basis, it suffices to show that $\{f_n, n \in Z^+\}$ and $\{g_n, n \in Z^+\}$ in $\mathcal{H}$ are fully equivalent.

From [39, p. 188, Theorem 12 and p.189, Problem 2] we see that the right-hand side of (2) is bounded by $MA^{-1}\| \sum_{k=1}^{n} c_k f_k\|^2$. If f is an element of $\mathcal{H}$, then $f = \sum_{k \in Z^+} d_k f_k$ for some unique sequence of scalars $\{d_k, k \in Z^+\} \in \ell^2$. Thus, if

$$G(f) := \sum_{k \in Z^+} d_k(f_k - g_k),$$

we conclude that $G : \mathcal{H} \to \mathcal{H}$ is a linear operator bounded by $(MA^{-1})^{1/2} < 1$. Setting $U := I - G$ (where I denotes the identity operator), it is clear that $U : \mathcal{H} \to \mathcal{H}$ is a linear homeomorphism, and $Uf_n = g_n, n \in Z^+$, whence the conclusion follows.

To prove the converse, note first that

$$\langle f, f_n - g_n \rangle = \langle f, f_n \rangle - \langle f, Uf_n \rangle = \langle f, f_n \rangle - \langle U^* f, f_n \rangle = \langle (I - U^*)f, f_n \rangle,$$

and therefore

$$\sum_{k \in Z^+} |\langle f, f_n - g_n \rangle|^2 = \sum_{k \in Z^+} |\langle (I - U^*)f, f_n \rangle|^2 \le B_1 \|(I - U^*)f\|^2 =$$

$$B_1 \|(I - U)^* f\|^2 \le B_1 \|(I - U)^*\|^2 \, \|f\|^2.$$

Since also

$$\langle f, f_n - g_n \rangle = \langle f, U^{-1} g_n - g_n \rangle = \langle f, U^{-1} g_n \rangle - \langle f, g_n \rangle =$$
$$\langle (U^{-1})^* f, g_n \rangle - \langle f, g_n \rangle = \langle [(U^{-1})^* - I] f, g_n \rangle = \langle (U^{-1} - I)^* f, g_n \rangle,$$

a similar argument yields

$$\sum_{k \in Z^+} |\langle f, f_n - g_n \rangle|^2 \le B_2 \, \|I - U^{-1}\|^2,$$

and the conclusion follows. □

Applying the Cauchy-Schwarz inequality and Theorem 5 we obtain:

Theorem 6. *Let $\{f_n, n \in Z^+\}$ be a Riesz basis in $\mathcal{H}$ with bounds A and B, and assume that*

$$M := \sum_{n \in Z^+} \|f_n - g_n\|^2 < A.$$

Then $\{g_n, n \in Z^+\}$ is a Riesz basis in $\mathcal{H}$ with bounds $A[1 - (M/A)^{1/2}]^2$ and $B[1 + (M/B)^{1/2}]^2$.

The condition $M < A$ in the preceding four theorems cannot be improved. This follows from the following:

Example 3. Let $\{f_n, n \in Z^+\}$ be an orthonormal basis in $\mathcal{H}$, and let

$$g_n := \begin{cases} 0, & \text{if } n = 1, \\ f_n, & \text{otherwise.} \end{cases}$$

Then $\{f_n, n \in Z^+\}$ is a frame with bounds $A = B = 1$, $\{f_n - g_n, n \in Z^+\}$ is a Bessel sequence with bound 1, and

$$\sum_{n \in Z^+} \|f_n - g_n\|^2 = \|f_1\|^2 = 1.$$

However, $\{g_n, n \in Z^+\}$ is not dense in $\mathcal{H}$, and therefore it can be neither a frame nor a Riesz basis.

The statements of most of the remaining theorems have two parts, dealing with frame stability and Riesz basis stability respectively. The proof of the first part always follows from one of Christensen's results, (i.e. Theorem 3 and Theorem 4), whereas the proof of the second part always follows from either Theorem 5 or Theorem 6.

4. Exponential Frames and Bases

In this section $r = (r_1, r_2, \cdots, r_d) \in Z^d$ is arbitrary but fixed, $k = (k_1, k_2, \cdots, k_d) \in Z^d$, $t = (t_1, t_2, \cdots, t_d)$ and $\lambda_k = (\lambda_{k_1}, \lambda_{k_2}, \cdots, \lambda_{k_d})$ are elements of $\mathcal{R}^\lceil$, and $\| \cdot \|_r$ denotes the norm of $L^2(I_r)$, where $I_r := [\pi r, \pi(r+\mathbf{2})]$.

Theorem 7. *Assume that $|k_\ell - \lambda_{k_\ell}| \le L, \ell = 1, \cdots, d$. If $L < 1/4$, then $\{e^{i\langle k,t\rangle} - e^{i\langle \lambda_k,t\rangle}, k \in Z^d\}$ is a Bessel sequence in $L^2(I_r)$ with bound $B_d(L)$.*

Proof. Since L will remain fixed throughout the proof, the abbreviation B_d will be used for $B_d(L)$. The assertion will be proved by induction on d.

For $d = 1$, the assertion follows from [39, pp. 42-44] by a change of variable of the form $x \to x - (r+1)\pi$.

The inductive step is proved as follows: Let $\{c_k\}$ be an arbitrary sequence in ℓ^2, let $J(d)$ denote the set of all $k \in Z^d$ for which these inequalities are satisfied, and let $H_1, H_2 \subset \mathcal{R}^{\lceil-\infty}$ be such that $I_r = [\pi r_1, \pi(r_1+2)] \times H_1 = H_2 \times [\pi r_d, \pi(r_d+2)]$. The projection of $t \in \mathcal{R}^\lceil$ onto $\mathcal{R}^{\lceil-\infty}$ will be denoted by $\tilde{t}$, i.e. $\tilde{t} := (t_1, t_2, \cdots, t_{d-1})$. Also, $\tilde{k}k_d$ will be used instead of k. Since

$$\sum_{k\in Z^d} c_k[e^{i\langle k,t\rangle} - e^{i\langle \lambda_k,t\rangle}] =$$

$$\sum_{k\in Z^d} c_k e^{ik_dt_d}[e^{i\langle \tilde{k},\tilde{t}\rangle} - e^{i\langle \tilde{\lambda}_k,\tilde{t}\rangle}] + \sum_{k\in Z^d} c_k e^{i\langle \tilde{\lambda}_k,\tilde{t}\rangle}[e^{ik_dt_d} - e^{i\lambda_{k_d}t_d}],$$

it follows that

$$\| \sum_{k\in Z^d} c_k[e^{i\langle k,t\rangle} - e^{i\langle \lambda_k,t\rangle}]\|_r \le J_1 + J_2,$$

where

$$J_1 := \| \sum_{k\in Z^d} c_k e^{ik_dt_d}[e^{i\langle \tilde{k},\tilde{t}\rangle} - e^{i\langle \tilde{\lambda}_k,\tilde{t}\rangle}]\|_r,$$

and

$$J_2 := \| \sum_{k\in Z^d} c_k e^{i\langle \tilde{\lambda}_k,\tilde{t}\rangle}[e^{ik_dt_d} - e^{i\lambda_{k_d}t_d}]\|_r.$$

Thus, the inductive hypothesis and Bessel's identity imply that

$$J_1^2 = \int_{\pi r_1}^{\pi(r_1+2)} \int_{H_2} | \sum_{\tilde{k}\in Z^{d-1}} \sum_{k_d\in Z} [c_{\tilde{k}k_d}\, e^{ik_dt_d}][e^{i\langle \tilde{k},\tilde{t}\rangle} - e^{i\langle \tilde{\lambda}_k,\tilde{t}\rangle}]|^2\, d\tilde{t}\, dt_d \le$$

$$B_{d-1}\int_{\pi r_1}^{\pi(r_1+2)}\sum_{\tilde{k}\in Z^{d-1}}\sum_{k_d\in Z}|c_{\tilde{k}k_d}e^{ik_dt_d}|^2\,dt_d = B_{d-1}\sum_{\tilde{k}\in Z^{d-1}}\sum_{k_d\in Z}|c_{\tilde{k}k_d}|^2 = B_{d-1}\sum_{k\in Z^d}|c_k|^2.$$

On the other hand the inductive hypothesis yields

$$J_2^2 = \int_{H_2}\int_{\pi r_d}^{\pi(r_d+2)}|\sum_{k_d\in Z}\sum_{\tilde{k}\in Z^{d-1}}c_k e^{i\langle\tilde{\lambda}_k,\tilde{t}\rangle}[e^{ik_dt_d}-e^{i\lambda_{k_d}t_d}]|^2\,dt_d\,d\tilde{t} \le$$

$$B_1\int_{H_2}\sum_{k_d\in Z}|\sum_{\tilde{k}\in Z^{d-1}}c_{\tilde{k}k_d}e^{i\langle\tilde{\lambda}_k,\tilde{t}\rangle}|^2\,d\tilde{t}.$$

An application of the triangle inequality, Bessel's identity, and the inductive hypothesis shows that

$$\left(\int_{H_2}|\sum_{\tilde{k}\in Z^{d-1}}c_{\tilde{k}k_d}e^{i\langle\tilde{\lambda}_k,\tilde{t}\rangle}|^2\,d\tilde{t}\right)^{1/2} \le$$

$$\left(\int_{H_2}|\sum_{\tilde{k}\in Z^{d-1}}c_{\tilde{k}k_d}e^{i\langle\tilde{k},\tilde{t}\rangle}|^2\,d\tilde{t}\right)^{1/2} + \left(\int_{H_2}|\sum_{\tilde{k}\in Z^{d-1}}c_{\tilde{k}k_d}[e^{i\langle\tilde{k},\tilde{t}\rangle}-e^{i\langle\tilde{\lambda}_k,\tilde{t}\rangle}]|^2\,d\tilde{t}\right)^{1/2} \le$$

$$\left(\sum_{\tilde{k}\in Z^{d-1}}|c_{\tilde{k}k_d}|^2\right)^{1/2} + \left(B_{d-1}\sum_{\tilde{k}\in Z^{d-1}}|c_{\tilde{k}k_d}|^2\right)^{1/2} = (1+B_{d-1}^{1/2})\left(\sum_{\tilde{k}\in Z^{d-1}}|c_{\tilde{k}k_d}|^2\right)^{1/2}.$$

Thus

$$J_2^2 \le B_1(1+B_{d-1}^{1/2})^2\sum_{k\in Z^d}|c_k|^2,$$

whence the conclusion follows. □

Applying Theorem 5 we obtain:

Corollary 2. *Assume that* $|k_\ell - \lambda_{k_\ell}| \le L, \ell = 1,\cdots,d,$ *and that* $L < 1/4$. *If* $B_d(L) < 1$, *then* $\{e^{i\langle\lambda_k,t\rangle}, k\in Z^d\}$ *is a Riesz basis in* $L^2(I_r)$ *with frame bounds* $[1-B_d(L)^{1/2}]^2$ *and* $[1+B_d(L)^{1/2}]^2$.

The requirement $B_d(L) < 1$ in the statement of the preceding corollary is somewhat cumbersome. The following proposition, which will be useful in the sequel, gives upper and lower bounds for $B_d(L)$, as well as a sufficient (but not necessary) condition for $B_d(L) < 1$ that is easier to verify:

Lemma 1. *We have:*

(a) If $0 < B_1(L) < 1$, *then* $B_1(L)\,[1+B_1(L)^{1/2}]^{2(d-1)} \le B_d(L) \le 9^{d-1}B_1(L)$.

(b) Let $0 < \alpha \le 1$. *If*

$$0 < L < \pi^{-1}\,cos^{-1}\left(\frac{1-\alpha\,9^{1-d}}{\sqrt{2}}\right) - \frac{1}{4},$$

then $B_d(L) < \alpha$.

Proof. (a) We proceed by induction on d. The inductive steps are proved as follows:

$$B_{d+1}(L) \geq [B_d(L)^{1/2} + B_1(L)^{1/2} B_d(L)^{1/2}]^2 \geq$$

$$B_1(L)\,[\{1 + B_1(L)^{1/2}\}^{d-1} + B_1(L)^{1/2}\{1 + B_1(L)^{1/2}\}^{d-1}]^2 = B_1(L)\,[1 + B_1(L)^{1/2}]^{2d},$$

and

$$B_{d+1}(L) \leq [3^{d-1} B_1(L)^{1/2} + B_1(L)^{1/2}(1 + 3^{d-1})]^2 \leq [3^d B_1(L)^{1/2}]^2.$$

(b) The hypothesis implies that $cos(\pi L + \pi/4) > 2^{-1/2}(1 - \alpha\, 9^{1-d})$. Thus $cos(\pi L) - sin(\pi L) > 1 - \alpha\, 9^{1-d}$, i. e. $B_1(L) < \alpha\, 9^{1-d}$, and the conclusion follows from (a). □

Remarks:

Note that for $\alpha = d = 1$, the condition in Lemma 1(b) reduces to $0 < L < 1/4$.

It is easy to see that the inequality $A + B + C \leq B_1(L)$ mentioned in the proof of Theorem 7 can be established under the weaker assumption that $L < 1/2$. However, since $0 < L < 1/2$ and $B_1(L) < 1$ imply that $L < 1/4$, and, conversely, $0 < L < 1/4$ implies that $B_1(L) < 1$, nothing is gained by it. It is also clear that, for $d = 1$, the condition $B_d(L) < 1$ is redundant.

In [37, Lemma 3.1.6], it is shown that given $\varepsilon > 0$ there is a $\delta > 0$ such that if $|k - \lambda_k| \leq \delta$ for every $k \in Z^d$, then $\{e^{i\langle k,t\rangle} - e^{i\langle \lambda_k,t\rangle}, k \in Z^d\}$ is a Bessel sequence in $L^2(-\boldsymbol{\pi}, \boldsymbol{\pi})$ with bound ε.

For $d = 1$, Corollary 2 is variously known as Kadec's 1/4-theorem ([39]), or the theorem of Paley-Wiener and Kadec ([19]). The latter also contains a multivariate version of the theorem for tensor product spaces. Note, however, that none of these results give frame bounds.

The condition $L < 1/4$ is known as the Kadec-Levinson condition ([2]). The study of the stability of $\{e^{int}, n \in Z\}$ was initiated by Paley and Wiener who showed that $\{e^{i\lambda_n t}, n \in Z\}$ is a Riesz basis in $L^2(-\pi, \pi)$ provided that $|\lambda_n - n| \leq L < \pi^{-2}$. Eventually, Kadec showed that π^{-2} could be replaced by 1/4.

It is well known that for $d = 1, L$ cannot equal 1/4. Whether this remains true if $d > 1$ is still unknown: The existing one-dimensional proof ([39, p. 122]) is based on growth properties of entire functions of one complex variable, and in particular on the fact that, with the exception of the zero function, the zeros of such functions do not hace a finite limit point. This is not necessarily the case for entire functions of more than one complex variable (cf. e. g. [27]), making a straightforward generalization of the one-dimensional proof impossible.

There are a number of conditions guaranteeing that a sequence of exponentials is a frame in $L^2(-\boldsymbol{\pi}, \boldsymbol{\pi})$. For example Jaffard [26] gives a complete characterization of such sequences in terms of properties of the sequence $\{\lambda_n\}$, thus generalizing earlier work of Duffin and Schaeffer [15]. However, none of these results gives any information about frame bounds either. Others do ([19, 20, 22, 23]), but are not applicable in this context.

5. Wavelet Frames and Bases.

In this section a will be a fixed positive real number, whereas b will be a positive real number which will occassionally be allowed to vary. Given a function $\phi : \mathcal{R}^\lceil \to \mathcal{C}$, and $j, k \in Z^d$, $\phi_{b,j,k}(x) := a^{d\,j/2}\phi(a^j x - bk)$, and $\Phi_b := \{\phi_{b,j,k}, j, k \in Z^d\}$. Given a fixed sequence $\{\lambda_{j,k}, j, k \in Z^d\} \subset \mathcal{R}^\lceil$, $\phi^{\{p\}}_{b,j,k}(x) := a^{d\,j/2}\phi(a^j x - b\lambda_{j,k})$, and $\Phi^{\{p\}}_b := \{\phi^{\{p\}}_{b,j,k}, j, k \in Z^d\}$. If there is no danger of ambiguity, the subscript "b" will be omitted. By abuse of notation, ϕ will be called the mother wavelet.

The first two theorems in this section study the effect of a perturbation to the original sampling sequence, replacing the sequence of integers by a double sequence $\{\lambda_{j,k}\}$. Roughly speaking these theorems say that, given a smooth ϕ, the frame condition and the Riesz basis condition are preserved under perturbation of the sampling points, if the accuracy of the sampling is increased as the resolution is increased or decreased (i.e. as $|j|$ increases). At first glance one might think that this is not a very satisfactory state of affairs, since in order to implement the frame algorithm for a frame $\{f_k, k \in Z^+\}$ in any Hilbert space, one must know the frame operator, and therefore all the elements f_k. However, as pointed out by Gröchenig, in practice one uses a truncation of the frame operator. In [22, Lemma 2] he presents a modified frame algorithm using these truncations. (A generalization to certain operators in Banach spaces, with error estimates, can be found in [19, p. 308].) Thus, knowledge of an arbitrary finite number of the f_k suffices for practical purposes.

Theorem 8. *Let* $\|\phi(x+h) - \phi(x)\| \leq C|h|^\alpha$, *where* $0 < \alpha \leq 1$, *and let*

$$\delta := b^{2\alpha} C^2 \sum_{j,k \in Z^d} |k - \lambda_{j,k}|^{2\alpha}.$$

If Φ *is a wavelet frame (wavelet Riesz basis) in* $L^2(\mathcal{R}^\lceil)$ *with bounds* A *and* B, *and* $\delta < A$, *then* $\Phi^{\{p\}}$ *is a wavelet frame (wavelet Riesz basis) in* $L^2(R)$ *with bounds* $[1-(\delta/A)^{1/2}]^2 A$ *and* $[1+(\delta/B)^{1/2}]^2 B$.

Proof.

$$\|\phi_{j,k}(x) - \phi^{\{p\}}_{j,k}(x)\|^2 = a^{dj} \|\phi(a^j x - bk) - \phi(a^j x - b\lambda_{j,k})\|^2 =$$
$$\|\phi(x) - \phi(x + bk - b\lambda_{j,k})\|^2 \leq b^{2\alpha} C^2 \, |k - \lambda_{j,k}|^{2\alpha}.$$

Thus

$$\sum_{j,k \in Z^d} \|\phi_{j,k}(x) - \phi^{\{p\}}_{j,k}(x)\|^2 \leq b^{2\alpha} C^2 \sum_{j,k \in Z^d} |k - \lambda_{j,k}|^{2\alpha} = \delta,$$

and the assertion follows from Theorem 4 and Theorem 6. ☐

In the sequel, "$|\cdot|_1$"denotes the taxicab norm (which was defined in Section 2). The next theorem does not require any knowledge of the constant b:

Theorem 9. *Let* $\|\phi(x+h) - \phi(x)\| \leq C|h|^\alpha$, *where* $0 < \alpha \leq 1$, *let*

$$\delta_1 := 2\,C\,[2 + (2\pi)^{d/2}], \qquad \delta_2 := \pi^{d/4} 2^{(d+1)/2} \|\phi\|,$$

and

$$M := (2\pi)^{-d} \{\delta_1\, [\sum_{j,k \in Z^d} |k - \lambda_{j,k}|_1^{(4\alpha)/(4+d)}]^{1/2} + \delta_2\, [\sum_{j,k \in Z^d} |k - \lambda_{j,k}|_1^{4/(4+d)}]^{1/2}\}^2,$$

and assume that $|k - \lambda_{j,k}|_1 \leq 1$ *for* $j,k \in Z$. *If* $M < A$ *and* Φ *is a wavelet frame (wavelet Riesz basis) in* $L^2(\mathcal{R}^\lceil)$ *with bounds* A *and* B, *then* $\Phi^{\{p\}}$ *is a wavelet frame (wavelet Riesz basis) in* $L^2(\mathcal{R}^\lceil)$ *with bounds* $[1-(M/A)^{1/2}]^2 A$ *and* $[1+(M/B)^{1/2}]^2 B$.

The proof of this theorem is based on the following proposition:

Lemma 2. *Let* $f \in L^2(\mathcal{R}^\lceil)$, $\S, \langle \in \mathcal{R}^\lceil$, $\| \in \mathcal{Z}^+$, $\rceil(\S) := (\in\pi)^{-\lceil/\in\rceil - |\S|^\in/\in}$, $\rceil_\|(\S) := \|^\lceil \rceil(\|\S)$, *and* $A_k f := f * e_k$, *where "*" denotes the convolution product. Then*

$$\|A_k f(x+h) - A_k f(x)\| \le k^{(d+2)/2} 2^{-(d+1)/2} \pi^{-d/4} \|f\| \, |h|_1.$$

Proof. Let k be arbitrary but fixed. Since both

$$\int_{\mathcal{R}^\lceil} f(t) e^{-k^2|x-t|^2/2} \, dt,$$

and

$$\int_{\mathcal{R}^\lceil} f(t) e^{-k^2|x-t|^2/2} (t_j - s_j) \, dt, \qquad 1 \le j \le d,$$

are uniformly convergent in $\mathcal{R}^\lceil$, differentiation under the integral sign is allowed. (cf. e.g. [38, p.350, Thm.8]). Thus, by the Mean Value Theorem there is an element $s \in \mathcal{R}^\lceil$ such that

$$A_k f(x+h) - A_k f(x) = k^d \, (2\pi)^{-d/2} \sum_{j=1}^{d} k^2 h_j \int_{\mathcal{R}^\lceil} f(t) e^{-k^2|s-t|^2/2} (t_j - s_j) dt.$$

But

$$\Big| \int_{\mathcal{R}^\lceil} f(t) e^{-k^2|s-t|^2/2} (t_j - s_j) dt \Big| \le \|f\| \left[\int_{\mathcal{R}^\lceil} e^{-k^2|s-t|^2} (t_j - s_j)^2 dt \right]^{1/2} =$$

$$\|f\| \left[k^{-(d-1)} \pi^{(d-1)/2} \frac{\sqrt{\pi}}{2k^3} \right]^{1/2},$$

and the conclusion follows. □

Proof of Theorem 9. Assume first that $\lambda_{j,k} \ne k$, and let $n(j,k) := \lfloor 1 + |k - \lambda_{j,k}|_1^{-2/(4+d)} \rfloor$. (Here "$\lfloor . \rfloor$" means the integral part). Then

$$|k - \lambda_{j,k}|_1^{-2/(4+d)} \le n(j,k) \le 2|k - \lambda_{j,k}|_1^{-2/(4+d)}.$$

If the convolution operators A_n are defined as in Lemma 2, then [35, p.205 Lemma 9.2.4] and the hypotheses imply that

$$\|f - A_n f\| \le \frac{C \, [2 + (2\pi)^{d/2}]}{n^\alpha (2\pi)^{d/2}}. \tag{3}$$

Since $\|\phi_{j,k}\| = \|\phi\| = \|\phi_{j,k}^{\{p\}}\|$, applying (3) and then Lemma 2, we see that

$$\|\phi_{j,k} - \phi_{j,k}^{\{p\}}\| \le \|\phi_{j,k} - A_{n(j,k)}\phi_{j,k}\| + \|A_{n(j,k)}\phi_{j,k} - A_{n(j,k)}\phi_{j,k}^{\{p\}}\| + \|A_{n(j,k)}\phi_{j,k}^{\{p\}} - \phi_{j,k}^{\{p\}}\| \le$$

$$\frac{\delta_1}{\{n(j,k)\}^\alpha (2\pi)^{d/2}} + \|A_{n(j,k)}\phi(x) - A_{n(j,k)}\phi(x + k - \lambda_{j,k})\| \le$$

$$\frac{\delta_1}{\{n(j,k)\}^\alpha (2\pi)^{d/2}} + \frac{[n(j,k)]^{(d+2)/2} \pi^{d/4} 2^{-1/2}}{(2\pi)^{d/2}} \|\phi\| |k - \lambda_{j,k}|_1 \le$$

$$\frac{\delta_1\,|k-\lambda_{j,k}|_1^{(2\alpha)/(4+d)}}{(2\pi)^{d/2}}+\frac{\pi^{d/4}2^{(d+1)/2}|k-\lambda_{j,k}|_1^{2/(4+d)}\|\phi\|}{(2\pi)^{d/2}}=$$

$$(2\pi)^{-d/2}[\delta_1\,|k-\lambda_{j,k}|_1^{(2\alpha)/(4+d)}+\delta_2\,|k-\lambda_{j,k}|_1^{2/(4+d)}].$$

Since this inequality holds trivially if $\lambda_{j,k}=k$, the assertion follows from Theorem 4, Theorem 6, and the triangle inequality. □

The next theorem studies the effect of perturbing the mother wavelet. This requires some care, since even a small perturbation may destroy the frame. Indeed, if a function ψ generates a wavelet frame, then $x^{-1}\hat{\psi}(x)^2$ must be in $L(\mathcal{R}^\lceil)$ (cf. [11, 12]). Thus, if ϕ generates a wavelet frame in $L^2(\mathcal{R}^\lceil)$ and if, for instance, $\psi(x) := \phi(x)+\varepsilon x^{-1}\prod_{\ell=1}^{d} sin\, x_\ell$, then ψ will not generate a wavelet frame.

A double sequence $\{f_{j,k}, j,k\in Z^d\}\subset L^2(\mathcal{R}^\lceil)$ will be called *semiorthogonal* if for every pair of $j,m\in Z^d$ with $j\neq m$, and any arbitrary choice of k and n in Z^d, the functions $f_{j,k}$ and $f_{m,n}$ are orthogonal. (In other words, for every pair of $j,m\in Z^d$ with $j\neq m$, and any arbitrary choice of k and n in Z^d, the functions $exp(-a^{-j}2\pi i\langle x,bk\rangle)\hat{f}(a^{-j}x)$ and $exp(-a^{-m}2\pi i\langle x,bn\rangle)\hat{f}(a^{-m}x)$ are orthogonal.) This is consistent with the definition of semiorthogonality for wavelets introduced by Auscher and by Chui and Wang (cf. [11]).

Theorem 10. *Let Φ be a semiorthogonal sequence in $L^2(\mathcal{R}^\lceil)$, and let ψ be any function in the closure of the linear span of $\{\phi_{0,k}, k\in Z^d\}$. If Φ is a wavelet frame (wavelet Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds A and B and $\|\psi-\phi\|<A^{3/2}/B$, then $\Psi:=\{\psi_{j,k}, j,k\in Z^d\}$ is a semiorthogonal wavelet frame (wavelet Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds*

$$([1-(B/A^{3/2})]\|\psi-\phi\|)^2A \qquad \textit{and} \qquad ([1+(B^{1/2}/A)]\|\psi-\phi\|)^2B.$$

Proof. The semiorthogonality is trivial. Let $g:=\phi-\psi$, and let T be the frame operator, i.e.

$$Tf:=\sum_{j\in Z^d}\sum_{k\in Z^d}\langle f,\phi_{j,k}\rangle\phi_{j,k}.$$

The hypotheses imply that, if $j\neq 0$, then $\langle g,\phi_{j,m}\rangle=0$. Thus, $g=\sum_{m\in Z^d}a_m\phi_{0,m}(x)$, where $a_m=\langle T^{-1}g,\phi_{0,m}\rangle$. Let $\{c_{j,k}\}$ be any finite sequence of scalars. Then:

$$\|\sum_j\sum_k c_{j,k}g_{j,k}\|^2=\|\sum_j\sum_k\sum_{m\in Z^d}a_mc_{j,k}\phi_{j,m+k}\|^2\leq$$

$$B\sum_j\sum_k\sum_{m\in Z^d}|a_mc_{j,k}|^2=B\sum_{m\in Z^d}|a_m|^2\sum_j\sum_k|c_{j,k}|^2.$$

But

$$\sum_{m\in Z^d}|a_m|^2=\sum_{m\in Z^d}|\langle T^{-1}g,\phi_{0,m}\rangle|^2\leq B\|T^{-1}g\|^2\leq BA^{-2}\|g\|^2,$$

where the last step follows from, e. g. [3, p. 100, Theorem 3.2]. We have therefore shown that $\{\phi_{j,k}-\psi_{j,k}, j,k\in Z^d\}$ is a Bessel sequence with bound $(B/A)^2\|\phi-\psi\|^2$, and the conclusion follows from Theorem 3 and Theorem 5. □

A function $\phi(x) \in L^2(\mathcal{R}^\lceil)$ will be called *orthogonal with respect to a*, if for every pair of $j, m \in Z^d$ with $j \neq m$ and any arbitrary choice of δ and λ in $\mathcal{R}^\lceil$, the functions $\phi(a^j x + \delta)$ and $\phi(a^m x + \lambda)$ are orthogonal in $L^2(\mathcal{R}^\lceil)$. This is the same as saying that for every pair of $j, m \in Z^d$ with $j \neq m$, and any arbitrary choice of δ and λ in $\mathcal{R}^\lceil$, the functions $exp(a^{-j} 2\pi i \langle x, \delta \rangle)\hat{\phi}(a^{-j}x)$ and $exp(a^{-m} 2\pi i \langle x, \lambda \rangle)\hat{\phi}(a^{-m}x)$ are orthogonal in $L^2(\mathcal{R}^\lceil)$.

Theorem 11. *If $\Phi := \{\phi_{j,k}, j, k \in Z^d\}$ is a semiorthogonal sequence in $L^2(\mathcal{R}^\lceil)$, such that $\hat{\phi}$ is essentially bounded and $supp\{\hat{\phi}\}$ is contained in an interval I of the form $[\mathbf{0}, 1/b] + h$, then ϕ is orthogonal with respect to a.*

Proof. Let $\lambda \in R^d$ be arbitrary. If $\{c_k, k \in Z^d\}$ is the sequence of Fourier coefficients of $e^{2\pi i \langle t, \lambda \rangle}$, then

$$\lim_{N \to \infty} \int |\hat{\phi}(t) e^{2\pi i \langle t, \lambda \rangle} - \hat{\phi}(t) \sum_{|k| \leq N} c_k e^{-2\pi i \langle t, bk \rangle}|^2 \, dt = \lim_{N \to \infty} \int_I |\hat{\phi}(t) e^{2\pi i \langle t, \lambda \rangle} - \hat{\phi}(t) \sum_{|k| \leq N} c_k e^{-2\pi b i \langle k, t \rangle}|^2 \, dt \leq$$

$$\|\hat{\phi}\|_\infty^2 \lim_{N \to \infty} \int_I |e^{2\pi i \langle t, \lambda \rangle} - \sum_{|k| \leq N} c_k e^{-2\pi b i \langle k, t \rangle}|^2 \, dt = 0.$$

This implies that $\hat{\phi}(t) e^{2\pi i \langle t, \lambda \rangle}$ is in the closure of the linear span of $\{\hat{\phi}_{0,k}, k \in Z^d\}$, and therefore that for any $j \in \mathcal{R}^\lceil$, $\hat{\phi}(\sqcup)]^{\dashv^{-|}\in\pi\rangle\langle\sqcup,\lambda\rangle}$ is in the closure of the linear span of $\{\hat{\phi}_{j,k}, k \in Z^d\}$. Since Φ is semiorthogonal, the conclusion follows. □

Using Theorem 7 we can show that we may perturb a frame into a frame or a Riesz basis into a Riesz basis, without necessarily increasing the accuracy of the sampling sequence as $|j|$ increases: The next theorem shows that it is possible to perturb a coherent state into another coherent state.

Theorem 12. *Let $\Phi := \{\phi_{j,k}, j, k \in Z^d\}$ be a wavelet frame (wavelet Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds A and B such that $\phi \in L^2(\mathcal{R}^\lceil)$ is orthogonal with respect to a. For $r \in Z^d$, let*

$$I_r := [\pi r, \pi(r + \mathbf{2})], \qquad J_r := [r/(2b), (r + \mathbf{2})/(2b)], \qquad L := sup\{|k - \lambda_{j,k}|, j, k \in Z^d\} < 1/4,$$

$$S_r := ess\, sup\{|\hat{\phi}(t)|, t \in J_r\} < \infty, \qquad M := \frac{B_d(L)}{(2\pi b)^d} \sum_{r \in Z} S_r^2, \text{and} \qquad M_1 := \frac{9^{d-1} B_1(L)}{(2\pi b)^d} \sum_{r \in Z} S_r^2.$$

If Φ is a wavelet frame (wavelet Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds A and B, and $M < A$, (in particular, if $M_1 < A$,) then $\Phi^{\{p\}}$ is a wavelet frame (wavelet Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds $[1 - (M/A)^{1/2}]^2 A$ and $[1 + (M/B)^{1/2}]^2 B$.

Proof. Lemma 1(a) implies that $M < M_1$. Let $\|\cdot\|_r$ denote the norm of I_r and let $\{c_{j,k}\}$ be an arbitrary finite sequence of scalars. The hypotheses and the isometry of the Fourier transform imply that

$$\|\sum_j \sum_k c_{j,k}(\phi_{j,k} - \phi_{j,k}^{\{p\}})\|^2 = \sum_j \|\sum_k c_{j,k}(\phi_{j,k} - \phi_{j,k}^{\{p\}})\|^2 = \sum_j \|\sum_k c_{j,k}(\hat{\phi}_{j,k} - \hat{\phi}_{j,k}^{\{p\}})\|^2.$$

But

$$\|\sum_k c_{j,k}(\hat\phi_{j,k}-\hat\phi^{\{p\}}_{j,k})\|^2 = a^{dj}\int|\sum_k c_{j,k}\hat\phi(a^jt)\{e^{-2\pi i a^{-j}\langle bk,t\rangle}-e^{-2\pi i a^{-j}\langle b\lambda_{j,k},t\rangle}\}|^2\,dt =$$

$$\int|\sum_k c_{j,k}\hat\phi(t)\{e^{-2\pi i\langle bk,t\rangle}-e^{-2\pi i\langle b\lambda_{j,k},t\rangle}\}|^2\,dt = \int|\hat\phi(t)|^2\,|\sum_k c_{j,k}\{e^{-2\pi i\langle bk,t\rangle}-e^{-2\pi i\langle b\lambda_{j,k},t\rangle}\}|^2\,dt =$$

$$\sum_{r\in Z}\int_{J_r}|\hat\phi(t)|^2\,|\sum_k c_{j,k}\{e^{-2\pi i\langle bk,t\rangle}-e^{-2\pi i\langle b\lambda_{j,k},t\rangle}\}|^2\,dt \le$$

$$(2\pi b)^{-d}\sum_{r\in Z}S_r^2\|\sum_k c_{j,k}\{e^{-i\langle k,t\rangle}-e^{-i\langle\lambda_{j,k},t\rangle}\}\|_r^2 =$$

$$(2\pi b)^{-d}\sum_{r\in Z}S_r^2\|\sum_k c_{j,k}\{e^{i\langle k,t\rangle}-e^{i\langle-\lambda_{j,-k},t\rangle}\}\|_r^2.$$

Since $|k-(-\lambda_{j,-k})| = |-k-\lambda_{j,-k}| < \frac{1}{4}$, setting $S := (\sum_{r\in Z} S_r^2)^{1/2}$ and applying Theorem 7, we see that

$$\|\sum_k c_{j,k}(\hat\phi_{j,k}-\hat\phi^{\{p\}}_{j,k})\|^2 \le (2\pi b)^{-d}S^2B_d(L)\sum_k|c_{j,k}|^2.$$

Therefore

$$\|\sum_j\sum_k c_{j,k}(\phi_{j,k}-p_{j,k})\|^2 \le (2\pi b)^{-d}S^2B_d(L)\sum_j\sum_k|c_{j,k}|^2,$$

and the conclusion follows from Theorem 3 and Theorem 5. □

Example 4. Let $a \ge 2$ be an integer, and let $b > 0$ be given. Assume that $\Phi_b := \{\phi_{j,k}, j,k \in Z\}$ is an orthonormal basis in $L^2(\mathcal{R})$ such that $\hat\phi$ is essentially bounded and supp$\{\hat\phi\}$ is contained in an interval of the form $[\mathbf{0}, 1/b]+h$. Thus, Theorem 11 implies that $\phi(x)$ is orthogonal with respect to a. Let $n > 0$ be an integer relatively prime to a. By the multivariate version of the Second Oversampling Theorem [9, Theorem 8] (see also [10]), $\Phi_{b/n} := \{\phi_{b/n,j,k}, j,k \in Z^d\}$ is a wavelet frame with bounds $A = B = n^d$. Thus $\Phi_{b/n}$ and ϕ satisfy the hypotheses of Theorem 12.

The hypotheses of Theorem 10 seem to be applicable to a rather small class of mother wavelets. The situation improves if a is an integer, for then the Second Oversampling Theorem can be used:

Theorem 13. *Let $a \ge 2$ be an integer, and let $\phi \in L^2(\mathcal{R}^\lceil)$ be such that $\hat\phi$ is essentially bounded and supp$\{\hat\phi\}$ is contained in an interval of the form $I_n := [-(n/2)(1/b), (n/2)(1/b)]$, where n is relatively prime to a. Let $\psi \in L^2(\mathcal{R})$ be such that there exists a real number $\lambda > 0$ for which*

$$|\hat\phi(t)-\hat\psi(t)| \le \lambda|\hat\phi(t)| \quad a.e.$$

If $\lambda < (A/Bn^d)^{1/2}$ and Φ is a wavelet frame (wavelet Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds A and B, then Ψ is a wavelet frame (wavelet Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds $[1-\lambda(n^dB/A)^{1/2}]^2A$ and $[1+\lambda(n^d/B)^{1/2}]^2B$.

Proof. Let f be an arbitrary element of $L^2(\mathcal{R}^\lceil)$, and let $p(x)$ be any function in $L^2(\mathcal{R}^\lceil)$ such that $\text{supp}(\hat{p}) \subset I_n$ and $\hat{p}$ is essentially bounded. Since $\{b^{d/2}(2/n)^{d/2}e^{-2\pi i n^{-1}\langle bj,x\rangle}, j \in Z^d\}$ is an orthonormal basis of $L^2(I_n)$ and $f(a^j t)\hat{p}(t)$ is in $L^2(I_n)$, Plancherel's formula and Bessel's identity yield:

$$\sum_{j,k\in Z^d} |\langle f, p_{b/n,j,k}\rangle|^2 = \sum_{j,k\in Z^d} |\langle \mathcal{F}\{, \mathcal{F}(\sqrt{_{\lfloor/\backslash,|,\|}}))\rangle|^\in =$$

$$\sum_{j\in Z^d} a^{-dj} \sum_{k\in Z^d} |\int f(t)\overline{\hat{p}(a^{-j}t)}e^{-2\pi i a^{-j}n^{-1}\langle bk,t\rangle}\,dt|^2 =$$

$$\sum_{j\in Z^d} a^{dj} \sum_{k\in Z^d} |\int_{I_n} f(a^j t)\overline{\hat{p}(t)}e^{-2\pi i n^{-1}\langle bk,t\rangle}\,dt|^2 = \sum_{j\in Z^d} a^{dj}b^{-d}(n/2)^d \int_{I_n} |f(a^j t)\overline{\hat{p}(t)}|^2\,dt.$$

By the multivariate version of the Second Oversampling Theorem $\{\phi_{b/n,j,k}, j,k \in Z^d\}$ is a wavelet frame with bounds $n^d A$ and $n^d B$. Thus, applying the preceding identity from right to left with $p = \phi$ we obtain:

$$\sum_{j\in Z^d} a^{dj}b^{-d}(n/2)^d \int_{I_n} |f(a^j t)\overline{\hat{\phi}(t)}|^2\,dt \le n^d B\|f\|^2.$$

Applying now the same identity from left to right, but with $p = \phi - \psi$, we obtain:

$$\sum_{j,k\in Z^d} |\langle f, \phi_{b,j,k} - \psi_{b,j,k}\rangle|^2 \le \sum_{j,k\in Z^d} |\langle f, \phi_{b/n,j,k} - \psi_{b/n,j,k}\rangle|^2 =$$

$$\sum_{j\in Z^d} a^{dj}b^{-d}(n/2)^d \int_{I_n} |f(a^j t)[\overline{\hat{\phi}(t) - \hat{\psi}(t)}]|^2\,dt \le$$

$$\lambda^2 \sum_{j\in Z^d} a^{dj}b^{-d}(n/2)^d \int_{I_n} |f(a^j t)\overline{\hat{\phi}(t)}|^2\,dt \le \lambda^2 n^d B\|f\|^2.$$

Applying Theorem 3 and Theorem 5, the conclusion follows. □

Finally, the method of proof used in the preceding theorem also allows us to prove a theorem on the stability of wavelet frames and Riesz bases under perturbations of the sampling sequence, that can be used to perturb a coherent state into another coherent state:

Theorem 14. *Let $a \ge 2$ be an integer, and let $\phi \in L^2(\mathcal{R}^\lceil)$ be such that $\hat{\phi}$ is essentially bounded and $supp\{\hat{\phi}\}$ is contained in an interval of the form $I_n := [-(n/2)(1/b), (n/2)(1/b)]$, where n is relatively prime to a. Let $|\lambda_k - k| \le L, \alpha := (\pi/n)^d(A/B)$ and $M := B(n/\pi)^d B_d(L)$. Assume that, either $L < 1/4$ and $B_d(L) < \alpha$, or $\alpha \le 1$ and*

$$L < \pi^{-1}\,cos^{-1}\left(\frac{1 - \alpha\, 9^{1-d}}{\sqrt{2}}\right) - \frac{1}{4}.$$

If Φ is a wavelet frame (wavelet Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds A and B, then $\Phi^{\{p\}}$ is a wavelet frame (wavelet Riesz basis) in $L^2(\mathcal{R})$ with bounds $(1 - M/A)^2 A$ and $(1 + M/B)^2 B$.

Proof. If $\alpha \leq 1$ and

$$L < \pi^{-1}\, cos^{-1}(\frac{1-\alpha\, 9^{1-d}}{\sqrt{2}}) - \frac{1}{4},$$

then Lemma 1(b) implies that $B_d(L) < \alpha$. Let f be an arbitrary element of $L^2(\mathcal{R}^\lceil)$. Proceeding as in the proof of the preceding theorem, we see that

$$\sum_{j \in Z^d} a^{dj} b^{-d} 2^{-d} \int_{I_n} |f(a^j t)\overline{\hat{\phi}(t)}|^2\, dt \leq B\|f\|^2. \tag{4}$$

Also

$$\sum_{j,k \in Z^d} |\langle f, \phi_{b,j,k} - \phi^{\{p\}}_{b,j,k}\rangle|^2 \leq \sum_{j,k \in Z^d} |\langle f, \phi_{b/n,j,k} - \phi^{\{p\}}_{b/n,j,k}]\rangle|^2 =$$

$$\sum_{j \in Z^d} a^{dj} \sum_{k \in Z^d} |\int_{I_n} f(a^j t)\overline{\hat{\phi}(t)}[e^{-2\pi i n^{-1}\langle bk,t\rangle} - e^{-2\pi i n^{-1}\langle b\lambda_k,t\rangle}\, dt|^2.$$

But Theorem 7 implies that $\{e^{-2\pi i n^{-1}\langle bk,t\rangle} - e^{-2\pi i n^{-1}\langle b\lambda_k,t\rangle}, k \in Z^d\}$ is a Bessel sequence in $L^2(I_n)$ with bound $(n/2\pi)^d b^{-d} B_d(L)$. Thus

$$\sum_{k \in Z^d} |\int_{I_n} f(a^j t)\overline{\hat{\phi}(t)}[e^{-2\pi i n^{-1}\langle bk,t\rangle} - e^{-2\pi i n^{-1}\langle b\lambda_k,t\rangle}]\, dt|^2 \leq (n/2\pi)^d b^{-d} B_d(L) \int_{I_n} |f(a^j t)\overline{\hat{\phi}(t)}|^2\, dt,$$

and (4) implies that

$$\sum_{j,k \in Z^d} |\langle f, \phi_{b,j,k} - \phi^{\{p\}}_{b,j,k}\rangle|^2 \leq M\|f\|^2.$$

The conclusion now follows from Theorem 3 and Theorem 5. □

Remark:
The preceding theorem is essentially a quantitative version of a theorem of Seip ([33, Theorem 5]), for the particular case in which n is relatively prime with respect to a. Seip's result is that for functions ϕ satisfying certain hypotheses, if Φ is a Riesz basis in $L^2(\mathcal{R})$ then there is a number $\varepsilon > 0$ such that, if $\lambda_k = k + \varepsilon$, then also $\Phi^{\{p\}}$ is a Riesz basis in $L^2(\mathcal{R})$. This result in turn generalizes a theorem of Daubechies [13, Theorem 2.10, p. 985], who assumes that the unperturbed sequence Φ is an orthonormal basis of $L^2(\mathcal{R})$.

6. Gabor Frames and Bases.

In this section, $a > \mathbf{0}$ and $b > \mathbf{0}$ are elements of R^d, $j, k \in Z^d$, and $r \in Z^+$. Given $\phi : \mathcal{R}^\lceil \to \mathcal{C}^\lceil$, and $j, k \in \mathcal{R}^\lceil$,

$$\phi_{a,b,j,k}(x) := e^{2\pi i\langle j\,b,x\rangle}\phi(x - ka), \quad \text{and} \quad \Phi_{a,b} := \{\phi_{a,b,j,k},\, j,k \in \mathcal{R}^\lceil\},$$

where

$$ka := (k_1a_1, k_2a_2, \ldots, k_da_d), \quad \text{and} \quad jb := (j_1b_1, j_2b_2, \ldots, j_db_d).$$

Given a fixed sequence $\{\lambda_k, k \in Z^d\} \subset \mathcal{R}^\lceil$,

$$\phi^{\{p\}}_{a,b,j,k}(x) := e^{2\pi i\langle jb,x\rangle}\phi(x - \lambda_k a), \quad \phi^{\{q\}}_{a,b,j,k}(x) := e^{2\pi i\langle \lambda_j b,x\rangle}\phi(x - ka),$$

$$\Phi_{a,b}^{\{p\}} := \{\phi_{a,b,j,k}^{\{p\}},\ j,k \in \mathcal{R}^\lceil\}, \quad \Phi_{\dashv,\lfloor}^{\{\mathrm{II}\}} := \{\phi_{\dashv,\lfloor,|,\|}^{\{\mathrm{II}\}},\ |,\| \in \mathcal{R}^\lceil\}.$$

When there is no danger of ambiguity either or both of the indices a and b will be omitted.

The first theorem in this section studies the effect of perturbing the mother wavelet.

Theorem 15. *Let* $\{h_r, \mathbf{1} \leq r \leq n\} \subset \mathcal{R}^\lceil$ *and* $J_r := \bigcup_{\ell=1}^{r}([\mathbf{0}, 1/b] + h_\ell)$. *Let* $\phi, \psi \in L^2(\mathcal{R}^\lceil)$ *be such that supp*$(\phi - \psi) \in J_n$. *Let* C_r *denote the characteristic function of* $J_r, \psi^{\{0\}} := \phi$, *and, for* $r > 0$, $\psi^{\{r\}}(x) := \phi(x) + C_r(x)[\psi(x) - \phi(x)]$. *Let*

$$M_r := ess\, sup\{\sum_{n\in Z^d} |\psi^{\{r\}}(t - na) - \psi^{\{r-1\}}(t - na)|^2, t \in \mathcal{R}^\lceil\} < \infty,$$

and assume that Φ *is a Gabor frame (Gabor Riesz basis) in* $L^2(\mathcal{R}^\lceil)$ *with bounds* A *and* B. *If*

$$\delta := \sum_{r=1}^{n} M_r^{1/2} < b^{d/2} A^{1/2},$$

then Ψ *is a Gabor frame (Gabor Riesz basis) in* $L^2(\mathcal{R}^\lceil)$ *with bounds* $[1 - (\delta^2/A)^{1/2} b^{-d/2}]^2 A$ *and* $[1 + (\delta^2/B)^{1/2} b^{-d/2}]^2 B$.

Proof. Let $I_r := [\mathbf{0}, 1/b] + h_r$,

$$G_r(t) := \sum_{n\in Z^d} |\psi^{\{r\}}(t - na) - \psi^{\{r-1\}}(t - na)|^2,$$

and let $f \in L^2(\mathcal{R}^\lceil)$. Since $\mathrm{supp}(\psi^{\{r\}} - \psi^{\{r-1\}}) \subset J_r$, proceeding as in the proof of [3, p. 111, Theorem 3.13] it is readily seen that

$$\sum_{j,k\in Z^d} |\langle f, \psi_{j,k}^{\{r\}} - \psi_{j,k}^{\{r-1\}}\rangle|^2 = b^{-d} \int_{I_r} |f(t)|^2 G_r(t)\, dt \leq b^{-d} M_r \|f\|^2.$$

Thus,

$$\left(\sum_{j,k\in Z^d} |\langle f, \phi_{j,k} - \psi_{j,k}\rangle|^2\right)^{1/2} = \left(\sum_{j,k\in Z^d} |\langle f, \psi_{j,k}^{\{0\}} - \psi_{j,k}^{\{n\}}\rangle|^2\right)^{1/2} \leq$$

$$\sum_{r=1}^{n} \left(\sum_{j,k\in Z^d} |\langle f, \psi_{j,k}^{\{r\}} - \psi_{j,k}^{\{r-1\}}\rangle|^2\right)^{1/2} \leq b^{-d/2} \delta \|f\|,$$

and the conclusion follows from Theorem 3 and Theorem 5. $\square$

Example 5. Let $I := [0, 1]$, and assume that $\phi \in L^2(\mathcal{R})$ and $\mathrm{supp}(\phi) \subset I$. Let $p(t) := \lambda^{1/2} e^{-\lambda^2\pi^2 t^2/2}$,

$$q(t) := \begin{cases} \lambda^{1/2} e^{-\lambda^2\pi^2 t^2/2}, & \text{if } 0 < t < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $\psi(t) := \phi(t) + q(t)$. By an application of the integral test as in e.g. [12, p. 65], it is readily seen that

$$\sum_{n\in Z} |\phi(t - na) - \psi(t - na)|^2 = \sum_{n\in Z} |q(t - na)|^2 \leq \sum_{n\in Z} |p(t - na)|^2 \leq$$

$$(\lambda/a)\int_{-\infty}^{\infty} e^{-\lambda^2\pi^2 t^2/2}dt + \lambda = (a\sqrt{\pi})^{-1} + \lambda.$$

Thus, Theorem 15 implies that, if $\Phi := \{\phi_{j,k}, j,k \in Z\}$ is a Gabor frame (Gabor Riesz basis) in $L^2(\mathcal{R})$ with bounds A and B, and $\delta := (a\sqrt{\pi})^{-1} + \lambda < A$, then Ψ is a Gabor frame (Gabor Riesz basis) in $L^2(\mathcal{R})$ with bounds $[1-(\delta/A)^{1/2}]^2 A$ and $[1+(\delta/B)^{1/2}]^2 B$.

It is now time to study the effect of perturbing the sampling sequence.

Theorem 16. *Let $h, \beta, \gamma \in \mathcal{R}^\lceil$ and $\{\lambda_k, k \in Z^d\} \subset \mathcal{R}^\lceil$ be such that*

$$\mathbf{0} < \gamma - \beta \le 1/b, \quad \mathbf{0} < h < (1/2)(1/b + \beta - \gamma), \quad -h(1/a) \le \lambda_k - k \le h(1/a).$$

Moreover, let ϕ be such that supp$(\phi) \subset [\beta,\gamma]$, and assume there is a point $c \in \mathcal{R}^\lceil$ such that

$$M_k := sup\{|\phi(t-ka) - \phi(t-(c+\lambda_k)a)|^2, t \in \mathcal{R}^\lceil\} < \infty.$$

If Φ is a Gabor frame (Gabor Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds A and B, and $M := \sum_{k\in Z^d} M_k < b^d A$, then $\Phi^{\{p\}}$ is a Gabor frame (Gabor Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds $[1-(M/A)^{1/2}b^{-d/2}]^2 A$ and $[1+(M/B)^{1/2}b^{-d/2}]^2 B$.

Proof. Making, if necessary, the change of variable $x \to x + c\,a$ we may assume, without essential loss of generality, that $c = 0$. Let $f(t) \in L^2(\mathcal{R}^\lceil)$ and $I := [\beta, \beta+1/b]$, $I_r := I + r(1/b)$, Since $\{b^{d/2}e^{2\pi i\langle bj,x\rangle}, j \in Z^d\}$ is an orthonormal basis in $L^2(I)$, setting

$$F_k(t) := \sum_{r\in Z^d} f[t-r(1/b)]\{\phi[t-\lambda_k a - r(1/b)] - \phi[t-ka-r(1/b)]\},$$

we have

$$\sum_{j,k\in Z^d} |\langle f, \phi_{j,k} - \phi_{j,k}^{\{p\}}\rangle|^2 = \sum_{k\in Z^d}\sum_{j\in Z^d} |\int f(t)\overline{[\phi_{j,k}(t) - \phi_{j,k}^{\{p\}}(t)]}\,dt|^2 =$$

$$\sum_{k\in Z^d}\sum_{j\in Z^d} |\sum_{r\in Z^d}\int_{I_r} f(t)\overline{[\phi_{j,k}(t) - \phi_{j,k}^{\{p\}}(t)]}\,dt|^2 = \sum_{k\in Z^d}\sum_{j\in Z^d} |\int_I e^{-2\pi i\langle bj,t\rangle}\overline{F_k(t)}\,dt|^2.$$

However, assuming that $F_k \in L^2(\mathcal{R}^\lceil)$ for each k, Plancherel's formula implies that

$$\sum_{k\in Z^d}\sum_{j\in Z^d} |\int_I e^{-2\pi i\langle bj,t\rangle}\overline{F_k(t)}\,dt|^2 = b^{-d}\sum_{k\in Z^d}\int_I |F_k(t)|^2\,dt.$$

Moreover, the hypotheses guarantee that the support of $\phi[t-ka-r(1/b)] - \phi[t-\lambda_k - r(1/b)]$ is contained in $[\beta-h, \gamma+h] + ka + r(1/b)$, and we therefore conclude that, for $k \in Z^d$, the supports of the functions $\phi[t-\lambda_k a - r(1/b)] - \phi[t-ka-r(1/b))]$ are mutually disjoint. Thus,

$$b^{-d}\sum_{k\in Z^d}\int_I |F_k(t)|^2\,dt = b^{-d}\sum_{k\in Z^d}\sum_{r\in Z^d}\int_I |f(t-r(1/b))[\phi(t-\lambda_k a - r(1/b)) - \phi(t-ka-r(1/b))]|^2\,dt =$$

$$b^{-d}\sum_{k\in Z^d}\int |f(t)[\phi(t-\lambda_k a) - \phi(t-ka)]|^2\,dt \le b^{-d}\sum_{k\in Z^d} M_k\|f\|^2,$$

and the conclusion follows from Theorem 3 and Theorem 5. □

We also have:

Theorem 17. *Let $h, \beta, \gamma \in \mathcal{R}^\lceil$ and $\{\lambda_k, \in Z^d\} \subset \mathcal{R}^\lceil$ be such that*

$\mathbf{0} < \gamma - \beta \leq 1/a, \quad \mathbf{0} < h < (1/2)(1/a + \beta - \gamma), \quad (\gamma - \beta + h - 1/a)(1/b) \leq \lambda_k - k \leq h(1/b).$

Moreover, let ϕ be such that $supp(\phi) \in [\beta, \gamma]$, and assume there is a point $c \in \mathcal{R}^\lceil$ such that

$$M_k := sup\{|\phi(t - kb) - \phi(t - (c + \lambda_k)b)|^2, t \in \mathcal{R}^\lceil\} < \infty.$$

Let Φ be a Gabor frame (Gabor Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds A and B, and assume that $M := \sum_{k \in Z^d} M_k < a^d A$. Then $\{\phi^{\{q\}}_{j,k}, j, k \in Z^d\}$ is a Gabor frame (Gabor Riesz basis) in $L^2(\mathcal{R}^\lceil)$ with bounds $[1 - (M/A)^{1/2} a^{-d/2}]^2 A$ and $[1 + (M/B)^{1/2} a^{-d/2}]^2 B$.

Proof. Note that

$$\mathcal{F}\phi_{\dashv,\lfloor,|,||} = \hat{\phi}_{\lfloor,\dashv,-||,|} \rceil^{\in\pi\rangle\langle|\lfloor,||\dashv\rangle}.$$

Since $|2\pi i \langle jb, ka\rangle| = 1$, from Plancherel's formula we deduce that $\{\hat{\phi}_{b,a,-k,j}, k, j \in Z^d\}$ is a Gabor frame in $L^2(\mathcal{R}^\lceil)$ with bounds A and B, and Theorem 16 implies that $\{\hat{\phi}^{\{p\}}_{b,a,-k,j}, k, j \in Z^d\}$ is a Gabor frame in $L^2(\mathcal{R}^\lceil)$ with bounds $[1-(M/A)^{1/2}a^{-d/2}]^2 A$ and $[1+(M/B)^{1/2}a^{-d/2}]^2 B$, or a Gabor Riesz basis with the same bounds. However,

$$\hat{\phi}^{\{p\}}_{b,a,-k,j} = \mathcal{F}\phi^{\{\amalg\}}_{\dashv,\lfloor,|,||} \rceil^{-\in\pi\rangle\langle(\rfloor+\lambda_|)\lfloor,||\dashv\rangle}.$$

Since $|e^{-2\pi i \langle \lambda_j b, ka\rangle}| = 1$, the conclusion follows by another application of Plancherel's formula. $\square$

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