SOME PROPERTIES OF CHEBYSHEV SYSTEMS

RICHARD A. ZALIK

Abstract. We study Chebyshev systems defined on an interval, whose constituent functions are either complex or real–valued, and focus on problems that may have have an application in the theory of differential equations and cannot be solved by a mere rewording of existing proofs, specifically those dealing with the existence of an adjoined function, the extension of the interval of definition, and the problem of embedding a set of functions into an Extended Complete Chebyshev System.

1. Introduction

A system of functions $F = (f_0, f_1, \dots, f_n)$ of complex–valued functions defined on a proper interval I is called a *Chebyshev system*, or *Tchebycheff system*, or *T–system*, if the determinant

$$D(f_0, \dots f_n; t_0, \dots t_n) := \det(f_j(t_k); 0 \le j, k \le n) \tag{1}$$

does not vanish for any choice of points $\{t_k; 0 \le k \le n\}$ in I. It is called a *Complete Chebyshev system* or *CT–system* or *Markov system*, if $(f_0, f_1, \dots, f_k)$ is a T–system for all $k = 0, \dots, n$.

If the functions f_j are sufficiently smooth, we can extend the definition of $D(f_0, \dots f_n; t_0, \dots t_n)$, so as to allow for equalities amongst the t_k: if $t_0 \le \cdots \le t_n$ is any set of points of I, then

$$D^*(f_0, \dots f_n; t_0, \dots t_n)$$

is defined to be the determinant on the right hand of (1), where for each set of consecutive t_k, the corresponding columns are replaced by the successive derivatives evaluated at the point. For example,

$$D^*(f_0, f_1, f_2; t_0, t_1, t_1) = \begin{vmatrix} f_0(t_0) & f_0(t_1) & f_0'(t_1) \\ f_1(t_0) & f_1(t_1) & f_1'(t_1) \\ f_2(t_0) & f_2(t_1) & f_2'(t_1) \end{vmatrix},$$

and $D^*(f_0, f_1, f_2; t, t, t) = W(f_0, f_1, f_2)(t)$.

With this definition, the system F is called an *Extended Chebyshev system* or ET–system on I, provided that for any set $t_0 \le \cdots \le t_n$ of points of I, $D^*(f_0, \dots f_n; t_0, \dots t_n)$ does not vanish, and it is called an *Extended Complete Chebyshev system* or *ECT–system* on I, if $(f_0, f_1, \dots, f_k)$ is an ET–system on I for all $k = 0, \dots, n$.

Chebyshev systems are of considerable importance in approximation theory, in particular in the study of spline functions, as well as in the theory of finite moments.

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Examples of T–systems include eigenfunctions of Sturm–Liouville operators. These topics are discussed, for example, in Karlin and Studden's classical monograph [2]. Results on spline functions have appeared in a plethora of later publications. For more recent results in the theory of real–valued T–systems, the reader is referred to the article by Carnicer, Peña and the author [1], and references thereof.

Lately, there has been renewed interest in Chebyshev systems because of their applications in the theory of differential equations. For example P. Marděsić in his memoir [3], which develops the theory of versal unfolding of cusps of order n, emphasizes the development of results on T–systems for the study of unfolding singularities of vector fields, whereas in [4] Mañosas and Villadelprat use ECT–systems in their study of the period functions of centers of potential systems. It is therefore useful to study properties of T-systems that may be applied in the study of differential equations, and that may have been previously overlooked.

The following theorem is well known, although it is usually stated for real–valued functions.

Theorem 1. *Let $F = (f_0, f_1, \dots, f_n)$ be a system of complex–valued functions defined on a proper interval I. Then*

(1) *$(f_0, f_1, \dots, f_n)$ is a T–system on I if and only if any nontrivial linear combination of the functions of F has at most n zeros.*
(2) *$(f_0, f_1, \dots, f_n)$ is an ET–system on I if and only if any nontrivial linear combination of the functions of F has at most n zeros counting multiplicities.*
(3) *$(f_0, f_1, \dots, f_n)$ is an ECT–system on I if and only if for any k, $0 \le k \le n$, $(f_0, f_1, \dots, f_k)$ is an ET– system.*

Note that if $F = (f_0, \dots, f_n)$ is a real–valued T–system on a proper interval I, a continuity argument shows that, multiplying if needed f_n by -1, there is no essential loss of generality if we assume that for any set $t_0 < \cdots < t_n$ of points of I the determinants $D(f_0, \dots f_n; t_0, \dots t_n)$ are strictly positive. Moreover, if F is an ET– system for which $D(f_0, \dots f_n; t_0, \dots t_n) > 0$ for any set $t_0 < \cdots < t_n$ of points of I then, proceeding as in [2, pp. 6–8], we deduce that for any set $t_0 \le \cdots \le t_n$ of points of $[a, b]$, the determinants $D^*(f_0, \dots f_n; t_0, \dots t_n)$ are strictly positive. This in turn implies that if F is an ECT–system for which $D(f_0, \dots f_k; t_0, \dots t_k) > 0$ for any $0 \le k \le n$ and any set $t_0 < \cdots < t_n$ of points of I then, for any $0 \le k \le n$ and any set $t_0 \le \cdots \le t_n$ of points of I, the determinants $D^*(f_0, \dots f_k; t_0, \dots t_k)$ are strictly positive for $0 \le k \le n$. We shall call such systems *positive*. Thus we may speak of positive T–systems, positive ET–systems, and positive ECT–systems. Positive ECT–systems, as we define them here, are the ECT–systems of Karlin and Studden [2]. They are called *full differentiable* ECT–systems by Marděsić [3]. We emphasize that all functions in a positive systems, and in particular positive ECT–systems, are assumed to be real–valued.

In the theory of real–valued ECT–systems defined on a closed interval $[a, b]$, the following theorem is of fundamental importance. A proof can be found in [2, pp. 376–379]. We have adapted the statement to our definition of T–systems.

Theorem 2. *Let $f_0, f_1, \dots, u_n$ be real–valued functions of class $C^n[a, b]$. The following two conditions are equivalent.*

(1) *$(f_0, \dots f_n)$ is a positive ECT–system on $[a, b]$.*

(2) *$W(f_0, \dots, f_k)(t) > 0$ on $[a, b]$ for $0 \le k \le n$.*

If, in addition, the functions f_k satisfy the initial conditions

$$f_k^{(p)}(a) = 0, \quad 0 \le p \le k-1; \quad 1 \le k \le n, \tag{2}$$

then (a) and (b) are equivalent to

(3) *There are functions w_k, strictly positive on $[a, b]$ and of continuity class $C^{n-k}[a, b]$, such that*

$$\begin{aligned} f_0(t) &= w_0(t) \\ f_1(t) &= w_0(t) \int_a^t w_1(s_1)\, ds_1 \\ f_2(t) &= w_0(t) \int_a^t w_1(s_1) \int_a^{s_1} w_2(s_2)\, ds_2\, ds_1 \\ &\vdots \\ f_n(t) &= w_0(t) \int_a^t w_1(s_1) \int_a^{s_1} w_2(s_2) \cdots \int_a^{s_{n-1}} w_n(s_n)\, ds_n \cdots ds_1. \end{aligned} \tag{3}$$

From [2, p. 380, (1.12) and (1.13)] we also know that if $(f_0, \dots f_n)$ has the representation (3), then

$$W(f_0, f_1, \cdots f_k) = w_0^{k+1} w_1^k \cdots w_k, \tag{4}$$

which implies that

$$\begin{aligned} &w_0 = f_0, \quad w_1 = \frac{W(f_0, f_1)}{f_0^2}, \\ &w_k = \frac{W(f_0, \cdots, f_k) W(f_0, \cdots f_{k-2})}{[W(f_0, \cdots f_{k-1})]^2}, \quad 2 \le k \le n. \end{aligned} \tag{5}$$

To prove that (c) implies (a) in Theorem 2, Rolle's theorem is used. Thus, the proof is not valid for complex–valued functions. The other parts of the statement still hold.

2. Existence of Adjoined Functions

In this section we discuss the existence of adjoined functions i.e., given a T–system $(f_0, \dots, f_n)$, whether there exists a function f_{n+1} such that $(f_0, \dots, f_n, f_{n+1})$ is a T–system. For dense subsets of open intervals this was answered in the affirmative by Zielke [7], and for any interval by the author [5]. The question has been raised of whether the same is true for complex–valued T–systems and whether to a T–system of analytic functions can be adjoined an analytic function. Although the methods usually used for real–valued functions cannot be applied in this setting, but we can still give an answer for real analytic functions.

Theorem 3. *Let $(f_0, \dots, f_n)$ be an ECT–system on a proper interval I. Assume, moreover, that the functions f_k are analytic on an open region D that contains I, and that they are real–valued on I. Then there is a function f_{n+1}, analytic on an open region D_1 that contains I and real–valued on I, such that $(f_0, \dots, f_n, f_{n+1})$ is an ECT–system on I.*

Proof. The hypotheses imply that the Wronskians $W(f_0, \ldots, f_k)$, $1 \leq k \leq n$ do not vanish on I. Multiplying the functions f_k by -1 if necessary, we may assume without essential loss of generality that these Wronskians are strictly positive on I. Let $a < b$ be points in I. Subtracting if necessary from each function f_k a suitable linear combination of its predecessors we obtain a system $(u_0, \ldots, u_n)$ that satisfies the initial conditions (2). Thus, from Theorem 2 we know that $(u_0, \ldots, u_n)$ has a representation of the form (3) on $[a, b]$. It follows from (5) that the functions w_k are strictly positive on I and analytic on some open region D_1 that contains I. Let w_{n+1} be an entire function strictly positive on I (eg. e^{-t^2}), and define

$$u_{n+1}(t) := w_0(t) \int_a^t w_1(s_1) \cdots \int_a^{s_{n-1}} w_n(s_n) \int_a^{s_n} w_{n+1}(s_{n+1})\, ds_{n+1} ds_n \cdots ds_1.$$

Clearly u_{n+1} is analytic on D_1. From (4) we deduce that

$$W(u_0, u_1, \cdots u_{n+1}) = w_0^{n+2} w_1^{n+1} \cdots w_{n+1} > 0$$

on I, and by another application of Theorem 2 we deduce that $(u_0, \ldots, u_n, u_{n+1})$ is an ECT system on $[a, b]$. Since a and b are arbitrary, the assertion follows. □

3. Extending the Domain of Definition

The problem of extending the domain of definition of a T– system has been studied extensively (see [1]). Here we consider the problem of extending the domain of definition of an ECT–system of complex–valued functions. Theorem 2 cannot be applied in this case, which makes the arguments more involved.

Theorem 4. *Let $F = (f_0, \ldots, f_n)$ be an ECT–system of complex–valued functions defined on a proper interval I with endpoints a and b. Assume, moreover, that the functions f_k are of class $C^n(\alpha, \beta)$, where $\alpha < a < b < \beta$. If $a \in I$ there is a $c < a$ such that F is an ECT–system on $(c, a) \cup I$, whereas if $b \in I$ there is a $d > b$ such that F is ECT–system on $I \cup (b, d)$.*

Proof. It suffices to assume that $a \in I$: the other case readily follows by the change of variables $t \to -t$.

Let I_k denote the set of integers from 0 to k. A *partition* of I_k is a family $\{S_r; 0 \leq m\}$ of sets of integers such that

(1) $\bigcup_{r=0}^m S_r = I_k$.
(2) If α is the largest number in S_r and β is the smallest number in S_{r+1},then $\beta = \alpha + 1$.

The preceding definition implies that the S_r are sets of consecutive integers. A simple inductive argument shows that there are 2^{k+1} different partitions of I_k.

If P is a partition of I_k and S is a set in P, then S is called a *component* of P. A set of integers $t_0 \leq t_1 \cdots \leq t_k$ is called a *configuration* associated with P if, whenever α and β belong to the same component of P, $t_\alpha = t_\beta$, and whenever α and β belong to different components, then $t_\alpha \neq t_\beta$. Thus, any set $t_0 \leq t_1 \cdots \leq t_k$ of points of I_k belongs to one of 2^{k+1} configurations.

For each configuration, $D^*(f_0, \ldots f_k; t_0, \ldots t_k)$ is a continuous function of the free variables involved. For example, if $t_0 < t_1 < t_2$, then $D^*(f_0, f_1, f_2; t_0, t_1, t_2)$ is a continuous function of t_1, t_2 and t_3, whereas if $t_0 < t_1 = t_2$, then $D^*(f_0, f_1, f_2; t_0, t_1, t_1)$ is a continuous function of t_0 and t_1. It follows that for an arbitrary k, $0 \le k \le n$, if P is a partition of I_k having m sets and S is a configuration associated with P, then $D^*(f_0,\dots f_k;t_0,\dots t_k)$ is a continuous nonvanishing function in the m–fold cartesian product of I with itself. Therefore there is a number $c_k(P) < a$ such that $D^*(f_0,\dots f_k;t_0,\dots t_k) \neq 0$ whenever $t_0 \le t_1 \le \cdots \le t_k$ is a configuration associated with P and the points t_k are in $(c_k(P), a) \cup I$. Setting c_k to be the largest of the $c_k(P)$ and c to be the largest of the c_k, the assertion follows. □

4. Embedding

Given a finite set of functions, the embedding problem consists in finding necessary and sufficient conditions for the existence of a T–system whose linear span contains them. For a single real–valued function, this problem was solved by the author in [6], whereas in [4, Proposition 2.2 and Proposition 2.3] Mañosas and Villadelprat show how to embed an analytic function into an ECT–system of analytic functions defined on an interval. The problem in its full generality remains unsolved. We can give an answer in a particular case, but first we need to prove some auxiliary propositions.

Lemma 5. *Let $(f_0,\cdots,f_n)$, $n \ge 1$, be a positive ECT–system on a closed interval $[a,b]$ such that the functions f_k satisfy (3), and let $(c(k); 0 \le k \le m)$ be a strictly increasing sequence with $0 \le c(0) < c(n) \le n$. Then $(f_{c(k)}; 0 \le k \le m)$ is a positive ECT–system on $(a,b]$.*

Proof. Let $D-0 = f/w_0$,

$$D_r f(t) := \frac{d}{dt}\left(\frac{f(t)}{w_r(t)}\right), \quad 1 \le r \le k,$$

and

$$L_r f(t) := D_r D_{r-1} \cdots D_0 f(t).$$

We proceed by induction. The assertion is obvious for $n = 1$. To prove the inductive step we proceed as follows: Let $\alpha = c_0$. Since $(L_\alpha f_k; \alpha+1 \le k \le n)$ has a representation of the form (3), the inductive hypothesis implies that $(L_\alpha f_{c(k)}; c_1 \le k \le m)$ is a positive ECT–system on $(a,b]$. By repeated application of the inverse operators D_α^{-1}, $D_{\alpha-1}^{-1} \dots D_0^{-1}$ to $(L_\alpha f_{c(k)}; c_1 \le k \le m)$ and using Rolle's theorem at each step, we deduce that $(f_{c(k)}; 0 \le k \le m)$ is a positive ECT–system on $(a,b]$. □

Lemma 6. *Let $(f_0,\cdots,f_n)$, $n \ge 1$, be a positive ECT– system on an interval $I(a,b)$ having endpoints $a < b$, let $c \in I(a,b)$, assume that the functions f_k satisfy initial conditions of the form (2) at the point c, that $f_0'(t) > 0$ on $I(a,b)$, and let $g_k(t) := (t-c)f_k(t)$. Then $(g_0',\cdots,g_n')$ is a positive ECT– system on $I(a,b]$.*

Proof. We may assume, without loss of generality, that $I(a,b)$ is a closed interval. Assume first that $c = a$.

Let k be arbitrary but fixed and $\mathbf{f} := (f_0,\cdots f_k)^T$. Then $W(g_0')(t) > 0$, and for $k > 0$

$$W(g_0',\cdots,g_k')(t) = \det((t-a)\mathbf{f}'(t) + \mathbf{f}(t),\cdots,(t-a)\mathbf{f}^{(k)}(t) + \mathbf{f}^{(k-1)}(t)).$$

Let $\mathbf{c} = (c(r); 0 \le r \le k)$ be a sequence of zeros and ones, let Λ denote the set of all such sequences, and for $0 \le r \le k$ let $\mathbf{q}_r(1,t) := (t-a)\mathbf{f}^{(r+1)}(t)$, $\mathbf{q}_r(2,t) := (r+1)\mathbf{f}^{(r)}(t)$. Then

$$W_k(t) := \det\left(\mathbf{q}_0(1,t) + \mathbf{q}_0(2,t),\cdots,\mathbf{q}_k(1,t) + \mathbf{q}_k(2,t)\right)(t) =$$

$$\sum_{\mathbf{c}\in\mathbf{\Lambda}} \det\left(\mathbf{q}_\ell(c(r),t); 0\le \ell\le k\right) = \sum_{\mathbf{c}\in\Lambda} D_{\mathbf{c}}(t).$$

For a given $0\le \ell\le k-1$ there are four possibilities:
If $c(\ell)=1$ and $c(\ell+1)=2$ then $q_\ell(c(\ell),t)=(t-a)f^{(\ell+1)}(t)$ and $q_{\ell+1}(c(\ell+1),t)=(\ell+2)f^{(\ell+1)}(t)$. (This implies that $D_{\mathbf{c}}(t)=0$).
If $c(\ell)=2$ and $c(\ell+1)=1$ then $q_\ell(c(\ell),t)=(\ell+1)f^{(\ell)}(t)$ and $q_{\ell+1}(c(\ell+1),t)=(t-a)f^{(\ell+2)}(t)$.
If $c(\ell)=1$ and $c(\ell+1)=1$ then $q_\ell(c(\ell),t)=(t-a)f^{(\ell+1)}(t)$ and $q_{\ell+1}(c(\ell+1),t)=(t-a)f^{(\ell+2)}(t)$.
If $c(\ell)=2$ and $c(\ell+1)=2$ then $q_\ell(c(\ell),t)=(\ell)f^{(\ell)}(t)$ and $q_{\ell+1}(c(\ell+1)=(\ell+1)f^{(\ell+1)}(t)$.

In summation, there are constants $\alpha>0$ and $\beta\ge 0$, and a sequence $a_0\le a_1\le \cdots\le a_k$ such that

$$D_{\mathbf{c}}(t)=\alpha(t-a)^\beta \det(f^{(a_0)}(t), f^{(a_1)}(t), \cdots f^{(a_k)}(t)),$$

whence Lemma 5 implies that $D_{\mathbf{c}}(t)\ge 0$. In particular, if $c(\ell)=2$ for all ℓ, then

$$D_{\mathbf{c}}(t)=\det\Big((\ell+1)f^{(\ell)}(t); 0\le\ell\le k\Big)=(k+1)!W(f_0,\cdots,f_k)(t)>0.$$

Thus $W(g_0',\cdots,g_k')(t)>0$ on I for $0\le k\le n$.

If $c=b$, the assertion follows by the change of variable $t\to -t$. In the general case, whether $t<c$ or $t\ge c$, the preceding discussion insures that $W(g_0',\cdots,g_k')(t)>0$ on I for $0\le k\le n$.

□

Theorem 7. *Let $0\le k\le n$ and assume that the functions $f_{k+1},\ldots,f_n$ are in $C^n(a,b)$, that for $k+1\le r\le n$, any linear combination of the functions $f_{k+1},\ldots,f_r$ has at most r zeros counting multiplicities and there is at least one linear combination of these functions having exactly r zeros counting multiplicities. Assume, moreover, that*

$$f_r^{(p)}(a)=0,\quad k+1\le r\le n,\quad 0\le p\le r-1.$$

Then there are functions $f_0,\ldots,f_k$ such that $(f_0,\ldots,f_n)$ is an ECT system on $[a,b]$.

Proof. Without essential loss of generality we may assume that each function f_r has exactly r zeros counting multiplicities. We proceed by induction. Assume first that $n=1$. The hypotheses imply that $f_1(t)=(t-a)g(t)$, where $g(t)$ is nonvanishing and continuously differentiable on $[a,b]$. Setting $f_0(t):=g(t)$, the assertion follows.

To prove the inductive step, let $g_{r-1}(t):=(t-a)^{-1}f_r(t)$. Then for $k\le r\le n-1$ every nontrivial linear combination of the functions $g_k,\ldots,g_r$ has at most r zeros counting multiplicities. By inductive hypothesis there are functions $g_0,\ldots g_{k-1}$ such that $(g_0,\cdots,g_{n-1})$ is an ECT–system on I. Since

$$W(exp(\alpha\ \cdot)f_0,\ldots,exp(\alpha\ \cdot)f_r)(t)=exp(r\ \alpha\ t)W(f_0,\ldots,f_r)(t),$$

multiplying if necessary the functions f_r by $exp(\alpha\ t)$ with α sufficiently large we may assume, without loss of generality that g_0' is strictly positive on I. If $f_\ell(t):=(t-a)g_{\ell+1}$, $1\le\ell\le k$, we see that $(f_1',\cdots,f_n')$ is a positive ECT–system on I. Defining $f_0(t):=1$ we see that for $1\le k\le n$,

$$W(f_0,\cdots,f_k)(t)=W(f_1',\cdots,f_k')(t)>0,$$

whence the assertion follows from Theorem 2. □

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Department of Mathematics and Statistics, Auburn University, AL 36849–5310.
E-mail address: zalik@auburn.edu