Representation of orthonormal multivariate wavelets

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Abstract. Given an orthonormal multivariate wavelet associated with a multiresolution analysis, we find a representation for all orthonormal multivariate wavelets associated with the same multiresolution analysis. This allows us to easily find orthonormal multivariate wavelets that are not the tensor products of one–dimensional wavelets.

§1. Introduction

In what follows $d > 1$ will be an integer, arbitrary but fixed; $\mathbb{Z}$ will denote the set of integers and $\mathbb{R}$ the set of real numbers; boldface lowcase letters will always denote elements of $\mathbb{R}^d$; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ will be denoted by $\langle f, g\rangle$, their bracket product by $[f, g]$, and the norm of f by $||f||$; thus,

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(\mathbf{t})\overline{g(\mathbf{t})}\, d\mathbf{t},$$

$$[f, g](\mathbf{t}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} f(\mathbf{t}+\mathbf{k})\overline{g(\mathbf{t}+\mathbf{k})},$$

and

$$||f|| := \sqrt{\langle f, f\rangle}.$$

The Fourier transform of a function f will be denoted by $\widehat{f}$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-i2\pi\mathbf{t}\cdot\mathbf{x}} f(\mathbf{t})\, d\mathbf{t}.$$

For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$, the operators D^j and $T_{\mathbf{k}}$ are defined in $L^2(\mathbb{R}^d)$ by

$$D^j f(\mathbf{t}) := 2^{dj/2}\psi(2^j\mathbf{t})$$

and

$$T_{\mathbf{k}} f(\mathbf{t}) := f(\mathbf{t}-\mathbf{k}).$$

A set of functions

$$\{\psi_1, \cdots, \psi_m\} \subset L^2(\mathbb{R}^d)$$

is called an orthonormal multivariate wavelet, if the affine sequence

$$\{D^j T_{\mathbf{k}} \psi^\ell; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le m\}$$

it generates, is an orthonormal basis of $L^2(\mathbb{R}^d)$.

A *multiresolution analysis* (MRA) in $L^2(\mathbb{R}^d)$ is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

(i)

$$V_j \subset V_{j+1} \text{ for every } j \in \mathbb{Z}.$$

(ii)

$$\text{For every } j \in \mathbb{Z}, f(\mathbf{t}) \in V_j \text{ if and only if } f(2\mathbf{t}) \in V_{j+1}.$$

(iii)

$$\bigcup_{j \in \mathbb{Z}} V_j \text{ is dense in } L^2(\mathbb{R}^d).$$

(iv)

There is a function u such that $\{T_{\mathbf{k}} u; \mathbf{k} \in \mathbb{Z}^d\}$ is an orthonormal basis of V_0.

Let $\mathbb{T} := [0, 1]$, and let $\mathbb{T}^d$ denote the d–dimensional torus. A function f will be called $\mathbb{Z}^d$*–periodic* if it is defined in $\mathbb{R}^d$, and for every $\mathbf{k} \in \mathbb{Z}^d$ and $\mathbf{x} \in \mathbb{R}^d$ we have $f(\mathbf{x} + \mathbf{k}) = f(\mathbf{x})$. It follows from the definition of MRA that there is a $\mathbb{Z}^d$–periodic function $p \in L^2(\mathbb{T}^d)$ such that

$$\widehat{u}(2\mathbf{x}) = p(\mathbf{x})\widehat{u}(\mathbf{x}) \qquad \text{a.e.}$$

The function u is called a *scaling function* for the MRA, and p is called the *low pass filter* associated with u.

By W_j we will denote the orthogonal complement of V_j in V_{j+1}. Thus, $V_{j+1} = V_j \oplus W_j$.

Let $\{\psi_1, \cdots, \psi_m\}$ be an orthonormal multivariate wavelet in $L^2(\mathbb{R}^d)$; for $j \in \mathbb{Z}$, let P_j denote the closure of the linear span of

$$\{D^j T_{\mathbf{k}} \psi_\ell; \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le m\},$$

and let $V_j := \sum_{r<j} P_r$. Note that $\psi_1, \cdots, \psi_m \in V_1$. We say that $\{\psi_1, \cdots, \psi_m\}$ is *associated* with an MRA, if $M := \{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis. If this is the case, we also say that $\{\psi_1, \cdots, \psi_m\}$ is associated with M. The definition implies that $\{\psi_1, \cdots, \psi_m\}$ is an orthonormal multivariate wavelet associated with M if and only if $\{T_{\mathbf{k}} \psi_\ell; \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le m\}$ is an orthonormal basis of W_0.

Given $\{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)$, we will adopt the following notation:

$$T(u_1, \cdots, u_m) := \{T_{\mathbf{k}} u_\ell; \mathbf{k} \in \mathbb{Z}^d, 1 \leq \ell \leq m\},$$

and

$$S(u_1, \cdots, u_m) := \overline{\operatorname{span}}\, T(u_1, \cdots, u_m).$$

In [11] Wilson and Weiss showed that if $\{\psi_1, \cdots, \psi_m\}$ is an orthonormal multivariate wavelet in $L^d(\mathbb{R}^d)$ associated with a multiresolution analysis, then $m = 2^d - 1$. Another proof of this statement follows from a more general result recently obtained by Guo, Labate et al. [5, Proposition 1]; for the special case we are considering, it can be stated as follows:

Theorem 1. *Assume that $T(u_1, \cdots, u_m)$ and $T(v_1, \cdots, v_n)$ are orthonormal sequences in $L^2(\mathbb{R}^d)$ such that $S(u_1, \cdots, u_m) = S(v_1, \cdots, v_n)$. Then $m = n$.*

As discussed in [5], this proposition readily implies the Wilson–Weiss theorem.

For the characterization of orthonormal multivariate wavelets see [2, 3, 4]. The construction of orthonormal multivariate wavelets associated with an MRA by means of tensor products of unidimensional wavelets is described in e.g. [7, 10, 11]. These wavelets are often called *separable*. The construction of nonseparable wavelets may be challenging (see, e.g. [1] for an example).

Every multiresolution analysis in $L^2(\mathbb{R})$ has orthonormal wavelets associated with it, and their characterization is summarized in Theorem 2 below. The existence of orthonormal multivariate wavelets associated with multiresolution analyses is studied in, e.g., [9]. In this paper we will just assume that we have an orthonormal multivariate wavelet associated with an MRA, and we will use it to represent every other orthonormal multivariate wavelet associated with the same MRA. Our starting point and motivation is the characterization of orthonormal MRA wavelets in $L^2(\mathbb{R})$ in Fourier domain which, with the definition of Fourier transform we have adopted, can be stated as follows:

Theorem 2. *([6, p. 57]) If φ is a scaling function for an MRA $\{V_j; j \in \mathbb{Z}\} \in L^2(\mathbb{R})$ and p is the associated low pass filter, then h is an orthonormal wavelet associated with this MRA if and only if there is a measurable unimodular and $\mathbb{Z}$–periodic function $\nu(x)$, such that*

$$\widehat{h}(2x) = e^{i2\pi x}\, \nu(2x)\, \overline{p(x + 1/2)}\, \widehat{\varphi}(x) \qquad \text{a.e.}$$

Corollary 1. *If h is an orthonormal wavelet associated with an MRA, then ψ is an orthonormal wavelet associated with the same MRA if and* *only if there is a measurable unimodular and $\mathbb{Z}$–periodic function $q(x)$ such that*

$$\widehat{\psi}(x) = q(x)\widehat{h}(x) \quad \text{a.e.}$$

The purpose of this paper is to generalize Corollary 1 to multivariate wavelets.

§2. Main Results

We begin with a number of auxiliary propositions of some independent interest. The first result is well known (cf. e.g. [5, Proposition 6]), although perhaps its proof has not been explicitly given in the literature. It is standard, and we include it for the sake of completeness.

Lemma 1.
(a) *$T(u_1, \cdots, u_m)$ is an orthogonal sequence in $L^2(\mathbb{R}^d)$ if and only if*

$$[\widehat{u_\ell}, \widehat{u_j}](\mathbf{x}) = 0 \quad \text{a.e.}, \quad \ell, j = 1, \cdots m, \quad \ell \neq j.$$

(b) *$T(u_1, \cdots, u_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ if and only if*

$$[\widehat{u_\ell}, \widehat{u_j}](\mathbf{x}) = \delta_{\ell,j} \quad \text{a.e.}, \quad \ell, j = 1, \cdots m.$$

Proof: It suffices to prove (b). The case $\ell = j$ is proved in, e.g., [7, p. 27] or [8, pp. 26–27]. Assume therefore that $\ell \neq j$. Then

$$\begin{aligned}\langle u_\ell, T_{\mathbf{r}} u_j \rangle &= \int_{\mathbb{R}^d} u_\ell(\mathbf{t}) \overline{u_j(\mathbf{t} - \mathbf{r})} d\mathbf{t} \\ &= \int_{\mathbb{R}^d} \widehat{u_\ell}(\mathbf{x}) e^{i2\pi \mathbf{r}\cdot\mathbf{x}} \overline{\widehat{u_j}(\mathbf{x})} d\mathbf{x} \\ &= \int_{\mathbb{T}^d} [\widehat{u_\ell}, \widehat{u_j}](\mathbf{x}) e^{i2\pi \mathbf{r}\cdot\mathbf{x}} d\mathbf{x}.\end{aligned}$$

Thus, by Bessel's identity,

$$\sum_{\mathbf{r}\in\mathbb{Z}^d} |\langle u_\ell, T_{\mathbf{r}} u_j \rangle|^2 = ||[\widehat{u_\ell}, \widehat{u_j}]||^2_{L^2(\mathbb{T}^d)},$$

whence the assertion follows. □

In the next lemma we will assume that $T(h_1, \cdots, h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, and that $S(u_1, \cdots, u_m) \subset S(h_1, \cdots, h_m)$. This implies that there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j}(\mathbf{x}) \in L^2(\mathbb{T}^d)$, uniquely defined a.e., such that

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{m} p_{\ell,j}(\mathbf{x}) \widehat{h_j}(\mathbf{x}) \quad \text{a.e.}, \quad \ell, j = 1, \cdots, m. \tag{1}$$

Lemma 2. *Assume that $T(h_1,\cdots,h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ and that*

$$S(u_1,\cdots,u_m)\subset S(h_1,\cdots,h_m),$$

and assume there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j}(\mathbf{x})$ such that (1) is satisfied. Then $T(u_1,\cdots,u_m)$ is an orthonormal sequence if and only if

$$\sum_{j=1}^{m} p_{\ell,j}(\mathbf{x})\overline{p_{r,j}(\mathbf{x})}=\delta_{\ell,r}\quad \text{a.e.},\quad \ell,r=1,\cdots,m. \tag{2}$$

Proof: Let $u_{\ell,j}$ denote the orthogonal projection of u_ℓ onto $S(h_j)$. Then

$$\widehat{u_{\ell,j}}(\mathbf{x})=p_{\ell,j}(\mathbf{x})\widehat{h_j}(\mathbf{x})\quad \text{a.e.},\quad \ell,j=1,\cdots,m.$$

Since $T(u_{\ell,1},\cdots,u_{\ell,m})$ is an orthogonal sequence for each ℓ, Lemma 1 implies that if $r\neq j$, then $[\widehat{u_{\ell,r}},\widehat{u_{j,r}}](\mathbf{x})=0$. Thus

$$[\widehat{u_\ell},\widehat{u_\ell}](\mathbf{x})=\sum_{\mathbf{q}\in\mathbb{Z}^d}\Big(\sum_{r=1}^{m}\widehat{u_{\ell,r}}(\mathbf{x}+\mathbf{q})\sum_{s=1}^{m}\overline{\widehat{u_{\ell,s}}(\mathbf{x}+\mathbf{q})}\,\Big)=\sum_{r=1}^{m}[\widehat{u_{\ell,r}},\widehat{u_{\ell,r}}](\mathbf{x}). \tag{3}$$

Moreover,

$$\begin{aligned}\langle u_\ell,T_{\mathbf{k}}u_r\rangle &=\int_{\mathbb{R}^d}\sum_{j=1}^{m}u_{\ell,j}(t)\sum_{s=1}^{m}\overline{T_{\mathbf{k}}u_{r,s}(\mathbf{t})}\,d\mathbf{t}\\ &=\int_{\mathbb{R}^d}\sum_{j=1}^{m}u_{\ell,j}(t)\overline{T_{\mathbf{k}}u_{r,j}(\mathbf{t})}d\mathbf{t}\quad \text{(by orthogonality)}\\ &=\int_{\mathbb{R}^d}\sum_{j=1}^{m}\Big(\widehat{u_{\ell,j}}(\mathbf{x})\overline{\widehat{u_{r,j}}(\mathbf{x})}\Big)e^{i2\pi\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\\ &=\int_{\mathbb{T}^d}\left(\sum_{j=1}^{m}[\widehat{u_{\ell,j}},\widehat{u_{r,j}}](\mathbf{x})\right)e^{i2\pi\mathbf{k}\cdot\mathbf{x}}d\mathbf{x},\end{aligned}$$

and by Bessel's identity we conclude that

$$\sum_{\mathbf{k}\in\mathbb{Z}^d}|\langle u_\ell,T_{\mathbf{k}}u_r\rangle|^2=||\sum_{j=1}^{m}[\widehat{u_{\ell,j}},\widehat{u_{r,j}}]||^2_{L^2(\mathbb{T}^d)}. \tag{4}$$

Combining (3), (4) and Lemma 1, we conclude that $T(u_1,\cdots,u_m)$ is an orthonormal sequence if and only if

$$A_{\ell,r}(\mathbf{x}):=\sum_{j=1}^{m}[\widehat{u_{\ell,j}},\widehat{u_{r,j}}](\mathbf{x})=\delta_{\ell,r},\quad \text{a.e.},\quad \ell,r=1,\cdots m.$$

Applying Lemma 1 again, we have:

$$A_{\ell,r}(\mathbf{x}) = \sum_{j=1}^{m} p_{\ell,j}(\mathbf{x})\overline{p_{r,j}(\mathbf{x})} \sum_{j=1}^{m} [\widehat{h_j}, \widehat{h_j}](\mathbf{x}) = \sum_{j=1}^{m} p_{\ell,j}(\mathbf{x})\overline{p_{r,j}(\mathbf{x})},$$

whence the assertion follows. □

Lemma 3. *Assume that $T(u_1, \cdots, u_m)$ and $T(h_1, \cdots, h_m)$ are orthonormal sequences in $L^2(\mathbb{R}^d)$. Then $S(u_1, \cdots, u_m) = S(h_1, \cdots, h_m)$ if and only if there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j}(\mathbf{x}) \in L^2(\mathbb{T}^d)$ that satisfy (1), and the matrix*

$$P(\mathbf{x}) := \Big(p_{\ell,j}(\mathbf{x})\Big)_{\ell,j=1}^{m} \tag{5}$$

is nonsingular almost everywhere.

Proof: If

$$U(\mathbf{x}) := \begin{pmatrix} \widehat{u_1}(\mathbf{x}) \\ \vdots \\ \widehat{u_m}(\mathbf{x}) \end{pmatrix} \quad \text{and} \quad H(\mathbf{x}) := \begin{pmatrix} \widehat{h_1}(\mathbf{x}) \\ \vdots \\ \widehat{h_m}(\mathbf{x}) \end{pmatrix},$$

then

$$U(\mathbf{x}) = P(\mathbf{x})H(\mathbf{x}) \quad \text{a.e.} \tag{6}$$

If $P(\mathbf{x})$ is nonsingular almost everywhere, setting

$$Q(\mathbf{x}) := \begin{cases} [P(\mathbf{x})]^{-1} & \text{if } P(\mathbf{x}) \text{ is nonsingular} \\ 0 & \text{if } P(\mathbf{x}) \text{ is singular} \end{cases},$$

we readily see that $Q(\mathbf{x})$ is $\mathbb{Z}^d$–periodic and that

$$H(\mathbf{x}) = Q(\mathbf{x})U(\mathbf{x}) \quad \text{a.e.}$$

Moreover, if

$$Q(\mathbf{x}) := \Big(q_{\ell,j}(\mathbf{x})\Big)_{\ell,j=1}^{m},$$

then

$$\widehat{h_\ell}(\mathbf{x}) = \sum_{j=1}^{m} q_{\ell,j}(\mathbf{x})\widehat{u_j}(\mathbf{x}).$$

By orthonormality and Lemma 1 we thus have

$$\begin{aligned} 1 = ||\widehat{h_\ell}||^2 = \\ \sum_{j=1}^{m} ||q_{\ell,j}\widehat{u_j}||^2 \geq ||q_{\ell,k}\widehat{u_k}||^2 = ||q_{\ell,k}[\widehat{u_k}, \widehat{u_k}]||^2_{L^2(\mathbb{T}^d)} \\ = ||q_{\ell,k}||^2_{L^2(\mathbb{T}^d)}, \end{aligned}$$

and therefore $q_{\ell,k} \in L^2(\mathbb{T}^d)$ for $\ell, k = 1, \cdots, m$.

Conversely, assume that $S(u_1, \cdots, u_m) = S(h_1, \cdots, h_m)$. Then, there are $\mathbb{Z}^d$–periodic matrices

$$P(\mathbf{x}) = \Big(p_{\ell,j}(\mathbf{x})\Big)_{\ell,j=1}^{m} \quad \text{and} \quad Q(\mathbf{x}) = \Big(q_{\ell,j}(\mathbf{x})\Big)_{\ell,j=1}^{m}$$

such that

$$p_{\ell,j}, q_{\ell,j} \in L^2(\mathbb{T}^d), \quad \ell, j = 1, \cdots, m,$$

(6) is satisfied, and

$$U(\mathbf{x}) = Q(\mathbf{x})H(\mathbf{x}) \quad \text{a.e.},$$

Thus

$$U(\mathbf{x}) = Q(\mathbf{x})P(\mathbf{x})U(\mathbf{x}) \quad \text{a.e.}$$

Since $T(u_1, \cdots, u_m)$ is a basis, this implies *a fortiori* that

$$Q(\mathbf{x})P(\mathbf{x}) = I \quad \text{a.e.}$$

□

Combining Lemma 2 and Lemma 3 we have:

Theorem 3. *Assume that $T(h_1, \cdots, h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, and let $\{u_1, \cdots, u_n\}$ be a set of functions defined on $\mathbb{R}^d$. Then $T(u_1, \cdots, u_n)$ is an orthonormal sequence and*

$$S(h_1, \cdots, h_m) = S(u_1, \cdots, u_n)$$

if and only $m = n$, there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j}(\mathbf{x}) \in L^2(\mathbb{T}^d)$ such that

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{m} p_{\ell,j}(\mathbf{x})\widehat{h_j}(\mathbf{x}) \quad \text{a.e.}, \quad \ell = 1, \cdots, m, \tag{7}$$

and the matrix (5) is orthogonal a.e.

Proof: Assume that the sequences $T(h_1, \cdots, h_m)$ and $T(u_1, \cdots, u_m)$ are orthogonal and such that $S(h_1, \cdots, h_m) = S(u_1, \cdots, u_n)$. That $m = n$ follows from Theorem 1. If $u_{\ell,j}$ denotes the orthogonal projection of u_ℓ onto h_j, there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j}(\mathbf{x}) \in L^2(\mathbb{T}^d)$, such that

$$\widehat{u_{\ell,j}}(\mathbf{x}) = p_{\ell,j}(\mathbf{x})\widehat{h_j}(\mathbf{x}) \quad \text{a.e.}, \quad \ell, j = 1, \cdots, m,$$

whence (7) follows. Lemma 2 implies that (2) is satisfied, which is equivalent to the orthogonality of (5).

Conversely, assume there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j}(\mathbf{x}) \in L^2(\mathbb{T}^d)$ such that (7) is satisfied and (5) is orthogonal a.e. Then (2) is satisfied, and therefore

$$|p_{\ell,j}(x)| \leq 1 \quad \text{a.e.}, \quad \ell = 1, \cdots, m,$$

and (7) implies that

$$u_\ell \in L^2(\mathbb{R}^d), \quad \ell = 1, \cdots, m.$$

Moreover, (7) also implies that

$$S(u_1, \cdots, u_m) \subset S(h_1, \cdots, h_m).$$

Applying Lemma 3 we therefore conclude that

$$S(u_1, \cdots, u_m) = S(h_1, \cdots, h_m).$$

Finally, from Lemma 2 we deduce that $T(u_1, \cdots, u_m)$ is an orthonormal sequence. Thus $T(u_1, \cdots, u_m)$ is an orthonormal basis of $S(h_1, \cdots, h_m)$, and the assertion follows. □

As we remarked above, if $\{\phi_1, \cdots, \phi_m\}$ is an orthonormal multivariate wavelet in $L^2(\mathbb{R}^d)$ associated with an MRA, then $m = 2^d - 1$. Thus, an immediate consequence of Theorem 3 is

Theorem 4. *Assume that $\{\phi_1, \cdots, \phi_m\}$ is an orthonormal multivariate wavelet in $L^2(\mathbb{R}^d)$ associated with an MRA, and let $\{\psi_1, \cdots, \psi_n\}$ be a set of functions defined in $L^2(\mathbb{R}^d)$. Then $\{\psi_1, \cdots, \psi_n\}$ is an orthonormal multivariate wavelet associated with the same MRA as $\{\phi_1, \cdots, \phi_m\}$, if and only if $m = n = 2^d - 1$, and there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j}(\mathbf{x}) \in L^2(\mathbb{T}^d)$, such that*

$$\widehat{\psi_\ell}(\mathbf{x}) = \sum_{j=1}^{m} p_{\ell,j}(\mathbf{x})\widehat{\phi_j}(\mathbf{x}) \quad \text{a.e.}, \quad \ell = 1, \cdots, m,$$

and the matrix (5) is orthogonal a.e.

Note that for $d = 1$ Theorem 4 reduces to Corollary 1.

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