ORTHONORMAL WAVELET SYSTEMS AND MULTIRESOLUTION ANALYSES

RICHARD A. ZALIK

ABSTRACT. Let $\mathbf{A} \in \mathbb{R}^{d\times d}$ be a matrix that preserves the lattice $\mathbb{Z}^d$ and $|a| := \det \mathbf{A}$. In [8], the author studied the properties of wavelet systems in $L^2(\mathbb{R}^d)$ of the form $\{|a|^{j/2}\psi_\ell(A^j\mathbf{t} + \mathbf{k}); j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le m\}$ that are associated with a multiresolution analysis of multiplicity n generated by $\mathbf{A}$. The purpose of the present paper is to extend those results to wavelet systems in $L^2(\mathbb{R}^d)$ of the form $\{|a|^{j/2}\psi_\ell(A^j\mathbf{t}+\mathbf{Bk}); j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le m\}$ that are associated with a multiresolution analysis of multiplicity n generated by $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A} \in \mathbb{R}^{d\times d}$ is a matrix that preserves the lattice $\mathbb{Z}^d$ and $\mathbf{B} \in \mathbb{R}^{d\times d}$ is a nonsingular matrix.

1. Introduction

In what follows, $\mathbb{Z}$ will denote the set of integers, $\mathbb{T} := [0, 1]$, and $\mathbb{T}^d$ will denote the d–dimensional torus. The underlying space will be $L^2(\mathbb{R}^d)$, where $d \ge 1$ is an integer and $\mathbb{R}$ is the set of real numbers, $\mathbf{I}$ will stand for the identity matrix, $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$, $a := \det \mathbf{A}$, $b := \det \mathbf{B}$, $\mathbf{C} := (A^{-1})^T$, and $\mathbf{D} := (B^{-1})^T$. Boldface lowcase letters will denote elements of $\mathbb{R}^d$, which will be represented as column vectors; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; $||\mathbf{x}||^2 := \mathbf{x} \cdot \mathbf{x}$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ will be denoted by $\langle f, g\rangle$, their bracket product with respect to $\mathbf{B}$ by $[f, g]^{\mathbf{B}}$, and the norm of f by $||f||$; thus,

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(\mathbf{t})\overline{g(\mathbf{t})}\, d\mathbf{t},$$

$$||f|| := \sqrt{\langle f, f\rangle},$$

and

$$[f, g]^{\mathbf{B}}(\mathbf{t}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} f(\mathbf{t} + \mathbf{k})\overline{g(\mathbf{t} + \mathbf{Bk})}.$$

In particular,

$$[f, g](\mathbf{t}) := [f, g]^{\mathbf{I}}(\mathbf{t}).$$

The Fourier transform of a function f will be denoted by $\widehat{f}$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-i2\pi\mathbf{x}\cdot\mathbf{t}} f(\mathbf{t})\, d\mathbf{t}.$$

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This file differs from the printed version in that a number of misprints have been corrected.

For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $D_j^{\mathbf{A}}$ and the translation operator $T_{\mathbf{k}}^{\mathbf{B}}$ are defined on $L^2(\mathbb{R}^d)$ by

$$D_j^{\mathbf{A}} f(\mathbf{t}) := |a|^{j/2} f(\mathbf{A}^j \mathbf{t})$$

and

$$T_{\mathbf{k}}^{\mathbf{B}} f(\mathbf{t}) := f(\mathbf{t} + \mathbf{B}\mathbf{k}).$$

In particular,

$$T_{\mathbf{k}} f(\mathbf{t}) := T_{\mathbf{k}}^{\mathbf{I}} f(\mathbf{t}).$$

A function f will be called $\mathbf{B}\mathbb{Z}^d$*–periodic* if it is defined on $\mathbb{R}^d$ and $T_{\mathbf{k}}^{\mathbf{B}} f = f$ for every $\mathbf{k} \in \mathbb{Z}^d$. A set $S \subset L^2(\mathbb{R}^d)$ is called $\mathbf{B}$ *shift–invariant* if $f \in S$ implies that $T_{\mathbf{k}}^{\mathbf{B}} f \in S$ for every $\mathbf{k} \in \mathbb{Z}^d$. If $\mathbf{B} = \mathbf{I}$, then we speak of a $\mathbb{Z}^d$ – periodic function f and of a shift – invariant space S, omitting mention of the matrix $\mathbf{I}$.

If f is a $\mathbf{B}\mathbb{Z}^d$–periodic function and $b := \det \mathbf{B}$, then

$$f^{\mathbf{B}}(\mathbf{t}) := D_1^{\mathbf{B}} f(\mathbf{t}) = |b|^{1/2} f(\mathbf{B}\mathbf{t}) \tag{1}$$

is $\mathbb{Z}^d$–periodic.

Let $\mathbf{u} \subset L^2(\mathbb{R}^d)$; then

$$T^{\mathbf{B}}(\mathbf{u}) := \{T_{\mathbf{k}}^{\mathbf{B}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}$$

and

$$S^{\mathbf{B}}(\mathbf{u}) := \overline{\operatorname{span}}\, T^{\mathbf{B}}(\mathbf{u}),$$

where the closure is in $L^2(\mathbb{R}^d)$. In particular,

$$T(\mathbf{u}) := T^{\mathbf{I}}(\mathbf{u})$$

and

$$S(\mathbf{u}) := S^{\mathbf{I}}(\mathbf{u})$$

If $\mathbf{u} = \{u_1, \cdots, u_m\}$ then $S^{\mathbf{B}}(\mathbf{u})$ is called a *finitely generated* $\mathbf{B}$ *shift–invariant space* and the functions u_ℓ are called the *generators* of $S^{\mathbf{B}}(\mathbf{u})$. In this case we will also use the symbols $T^{\mathbf{B}}(u_1, \cdots, u_n)$ and $S^{\mathbf{B}}(u_1, \cdots, u_n)$ to denote $S^{\mathbf{B}}(\mathbf{u})$ and $T^{\mathbf{B}}(\mathbf{u})$ respectively.

Let $\mathbb{H}$ be a (separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| := \langle \cdot, \cdot \rangle^{1/2}$. A sequence $F = \{f_k, k \in \mathbb{Z}\} \subset \mathbb{H}$ is called a *Riesz sequence* if there are constants $0 < A \leq B$ such that for every sequence $\{c_k, k \in \mathbb{Z}\} \subset \ell^2$

$$A \sum_{\mathbf{k} \in \mathbb{Z}^d} |c_k|^2 \leq \left\| \sum_{\mathbf{k} \in \mathbb{Z}^d} c_k f_k \right\|^2 \leq B \sum_{\mathbf{k} \in \mathbb{Z}^d} |c_k|^2.$$

F is called a *Riesz basis* of $\mathbb{H}$ if it is a Riesz sequence and its linear span is dense in $\mathbb{H}$. The constants A and B are called (lower and upper) *bounds* of the Riesz basis. Clearly, every orthonormal basis is a Riesz basis with bounds $A = B = 1$. The theory of Riesz bases is discussed in, e.g., [3, 7].

Let $\Lambda \subset \mathbb{Z}$ and $\mathbf{u} = \{u_k; k \in \Lambda\} \subset S \subset L^2(\mathbb{R}^d)$. If S is a $\mathbf{B}$ shift–invariant space then $\mathbf{u}$ is called a *basis generator* of S, and we say that $\mathbf{u}$ *provides a basis* for S, if for every $f \in S$ there are $\mathbf{B}\mathbb{Z}^d$–periodic functions p_k, uniquely determined by f (up to a set of measure 0), such that

$$\widehat{f} = \sum_{k \in \Lambda} p_k \widehat{u_k}.$$

If $\mathbf{u}$ is a finite set, the uniqueness of the functions p_k is equivalent to $G^{\mathbf{B}}_{\mathbf{u}}(\mathbf{x})$ being nonsingular for almost every $\mathbf{x} \in \mathbb{T}^d$.

The theory of basis generators has been extensively developed by De Boor, DeVore, Ron and Shen in [1, 2, 5], under the assumption that $\mathbf{B} = \mathbf{I}$. In [8] we applied some of these results to the study of *Schauder basis generators*, *Riesz basis generators* and *orthonormal basis generators*, i.e. sets $\mathbf{u}$ such that $T(\mathbf{u})$ is either a Schauder basis, a Riesz basis, or an orthonormal basis of $S(\mathbf{u})$. Note that a Riesz basis generator is a basis generator. In the following section we will extend some of the results of [8] to the case of an arbitrary lattice $\mathbf{B}\mathbb{Z}^d$, where $\mathbf{B}$ is nonsingular.

2. Some theorems on Riesz bases of translates and Linear Transformations

Given a sequence of functions $\mathbf{u} := \{u_1, \cdots, u_m\}$ in $L^2(\mathbb{R}^d)$ and $\mathbf{B} \in \mathbb{R}^{d\times d}$, by $G^{\mathbf{B}}[u_1, \cdots, u_m](\mathbf{x})$ or $G^{\mathbf{B}}_{\mathbf{u}}(\mathbf{x})$ we will denote its $\mathbf{B}$ *Gramian matrix*, viz.

$$G^{\mathbf{B}}_{\mathbf{u}}(\mathbf{x}) := \Big([\widehat{u_\ell}, \widehat{u_j}]^{\mathbf{B}}(\mathbf{x})\Big)_{\ell,j=1}^{m}.$$

In particular,

$$G_{\mathbf{u}}(\mathbf{x}) := G^{\mathbf{I}}_{\mathbf{u}}(\mathbf{x}).$$

If $\mathbf{u} = \{u_1, \cdots u_n\}$ and the functions $u^{\mathbf{B}}_\ell$ are defined as in (1), then

$$\mathbf{u}^{\mathbf{B}} := \{u^{\mathbf{B}}_1, \cdots, \mathbf{u}^{\mathbf{B}}_n\}. \tag{2}$$

We begin with the following simple but important result:

Lemma 1. *Let $\mathbf{B} \in L^2(\mathbb{R}^{d\times d}$ be a nonsingular matrix, $\mathbf{u} \in L^2(\mathbb{R}^d)$, $\mathbf{D} := (\mathbf{B}^{-1})^T$, and let $u^{\mathbf{B}}(\mathbf{x})$ be given by (1). Then*

(a) *$T^{\mathbf{B}}(\mathbf{u})$ is an orthogonal basis of $S^{\mathbf{B}}(\mathbf{u})$ if and only if $T(\mathbf{u}^{\mathbf{B}})$ is an orthogonal basis of $S(\mathbf{u}^{\mathbf{B}})$.*

(b) *$T^{\mathbf{B}}(\mathbf{u})$ is a Riesz basis in $S^{\mathbf{B}}(\mathbf{u})$ if and only if $T(\mathbf{u}^{\mathbf{B}})$ is a Riesz basis in $S(\mathbf{u}^{\mathbf{B}})$. Moreover, $T^{\mathbf{B}}(\mathbf{u})$ and $T(\mathbf{u}^{\mathbf{B}})$ have the same Riesz bounds.*

Proof. Part (a) follows from a change of variables, whereas part (b) follows from the following computations:

$$||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} T^{\mathbf{B}}_{\mathbf{k}}(\mathbf{u})|| = ||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} e^{2\pi i \mathbf{x}\cdot\mathbf{B}\mathbf{k}} \widehat{u}(\mathbf{x})|| = |b|^{-1/2} ||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} e^{2\pi i \mathbf{s}\cdot\mathbf{k}} \widehat{u}(\mathbf{D}\mathbf{s})||,$$

and

$$||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{u}^{\mathbf{B}})|| = |b|^{-1/2} ||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} e^{2\pi i \mathbf{s}\cdot\mathbf{k}} \widehat{u}(\mathbf{D}\mathbf{s})||.$$

□

The remaining results in this section will follow from Lemma 1 and the corresponding results in [8].

Theorem 2. *Let $\mathbf{u} := \{u_1, \cdots, u_n\}$ and $\mathbf{v} := \{v_1, \cdots, v_m\}$, and let $\mathbf{B} \in \mathbb{R}^{d\times d}$ be nonsingular. Then*

(a) *If $T^{\mathbf{B}}(\mathbf{u})$ and $T^{\mathbf{B}}(\mathbf{v})$ are Riesz bases of the same shift–invariant space $S \subset L^2(\mathbb{R}^d)$, then $n = m$.*
(b) *If $T^{\mathbf{B}}(\mathbf{u})$ and $T^{\mathbf{B}}(\mathbf{v})$ are Riesz sequences such that $n = m$ and $S^{\mathbf{B}}(\mathbf{u}) \subset S^{\mathbf{B}}(\mathbf{v})$, then $T^{\mathbf{B}}(\mathbf{u})$ is a Riesz basis of $S^{\mathbf{B}}(\mathbf{v})$.*

(c) *Let* $T^{\mathbf{B}}(\mathbf{u})$ *and* $T^{\mathbf{B}}(\mathbf{v})$ *be Riesz sequences in* $L^2(\mathbb{R}^d)$*, and assume that* $S^{\mathbf{B}}(\mathbf{u})$ *is a proper subset of* $S^{\mathbf{B}}(\mathbf{v})$*. Then* $n < m$ *and there are functions* $w_1, \cdots, w_{m-n}$ *such that*

$$T^{\mathbf{B}}(w_1, \cdots, w_{m-n})$$

is an orthonormal basis of the orthogonal complement $S^{\mathbf{B}}(\mathbf{u})^{\perp}$ *of* $S^{\mathbf{B}}(\mathbf{u})$ *in* $S^{\mathbf{B}}(\mathbf{v})$*, and*

$$T^{\mathbf{B}}(u_1, \cdots, u_n, w_1, \cdots, w_{m-n})$$

is a Riesz basis of $S^{\mathbf{B}}(v_1, \cdots, v_m)$*.*

Proof. The assertion follows from [8, Theorem 1] applied to $\mathbf{u}^{\mathbf{B}}$ and $\mathbf{v}^{\mathbf{B}}$. □

From the identity

$$\widehat{u^{\mathbf{B}}}(\mathbf{x}) = |b|^{-1/2}\widehat{u}(\mathbf{D}\mathbf{x}), \tag{3}$$

we obtain

Theorem 3. *Let* $\mathbf{u} := \{u_1, \cdots, u_n\} \subset L^2(\mathbb{R}^d)$*,* $\mathbf{h} := \{h_1, \cdots, h_m\} \subset L^2(\mathbb{R}^d)$*, let* $\mathbf{B} \in \mathbb{R}^{d\times d}$ *be a nonsingular matrix,* $b := \det \mathbf{B}$*,* $\mathbf{D} := (\mathbf{B}^{-1})^T$*, and assume that*

$$S^{\mathbf{B}}(\mathbf{u}) \subset S^{\mathbf{B}}(\mathbf{h}).$$

Then there are $\mathbf{D}\mathbb{Z}^d$*–periodic functions* $q_{\ell,j}(\mathbf{x})$ *such that*

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{m} q_{\ell,j}(\mathbf{x})\widehat{h_j}(\mathbf{x}) \quad a.e.; \quad \ell = 1, \cdots, n.$$

Proof. The hypotheses imply that

$$S(\mathbf{u}^{\mathbf{B}}) \subset S(\mathbf{h}^{\mathbf{B}})$$

Applying [8, Theorem F] we see that there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j}(\mathbf{x})$ such that

$$\widehat{u_\ell^{\mathbf{B}}}(\mathbf{x}) = \sum_{j=1}^{m} p_{\ell,j}(\mathbf{x})\widehat{h_j^{\mathbf{B}}}(\mathbf{x}) \quad a.e.; \quad \ell = 1, \cdots, n.$$

Setting $q_{\ell,j}(\mathbf{x}) := p_{\ell,j}(\mathbf{B}^T\mathbf{x})$ and applying (3) to u_j and h_j, the assertion follows. □

The $\mathbf{D}\mathbb{Z}^d$–periodic matrix

$$\mathbf{Q}^{\mathbf{D}}(\mathbf{x}) := \Big(q_{\ell,j}(\mathbf{x})\Big)_{\ell,j=1}^{n,m}$$

will be called a *transition matrix* from the sequence $T^{\mathbf{B}}(\mathbf{h})$ to the sequence $T^{\mathbf{B}}(\mathbf{u})$. If $\mathbf{h}$ is a basis generator of $S^{\mathbf{B}}(\mathbf{h})$, then $Q^{\mathbf{D}}(\mathbf{x})$ is unique (up to a set of measure 0).

Lemma 4. *Let* $\mathbf{B} \in \mathbb{R}^{d\times d}$ *be nonsingular and* $\mathbf{u} = \{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)$*. Then* $T^{\mathbf{B}}(\mathbf{u})$ *is an orthonormal sequence in* $L^2(\mathbb{R}^d)$ *if and only if* $G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{x}) = |b|\mathbf{I}$ *a.e. In particular if* $n = 1$*, then* $T^{\mathbf{B}}(u)$ *is an orthonormal sequence in* $L^2(\mathbb{R}^d)$ *if and only if*

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{u}(\mathbf{x} + \mathbf{D}\mathbf{k})|^2 = |b| \quad a.e.$$

Proof. From (3) we deduce that

$$G_{\mathbf{u}^{\mathbf{B}}}(\mathbf{x}) = |b|^{-1}G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{D}\mathbf{x}), \tag{4}$$

and the assertion follows from, e.g. [8, Lemma D]. □

Lemma 5. *Let $\mathbf{B} \in \mathbb{R}^{d\times d}$ be nonsingular, $\mathbf{D} := (\mathbf{B}^{-1})^T$, and assume that $T^{\mathbf{B}}(u)$ and $T^{\mathbf{B}}(v_1, \cdots, v_m)$ are orthonormal sequences in $L^2(\mathbb{R}^d)$, and that there are $\mathbf{D}\mathbb{Z}^d$–periodic functions p_ℓ such that*

$$\widehat{u}(\mathbf{x}) = \sum_{\ell=1}^{m} p_\ell(\mathbf{x})\widehat{v_\ell}(\mathbf{x}) \quad a.e.$$

Then

$$\sum_{\ell=1}^{m} |p_\ell(\mathbf{x})|^2 = 1 \quad a.e.$$

Proof. The hypotheses imply that

$$\widehat{u}(\mathbf{D}\mathbf{x}) = \sum_{\ell=1}^{m} p_\ell(\mathbf{D}\mathbf{x})\widehat{v_\ell}(\mathbf{D}\mathbf{x}) \quad a.e.$$

Setting $q_\ell(\mathbf{x}) := p_\ell(\mathbf{D}\mathbf{x})$ and applying (3) to u and the functions v_ℓ we see that

$$\widehat{u}^{\mathbf{B}}(\mathbf{x}) = \sum_{\ell=1}^{m} q_\ell(\mathbf{x})\widehat{v_\ell}^{\mathbf{B}}(\mathbf{x}) \quad a.e.$$

Since $T(u^{\mathbf{B}})$ and $T(v_1^{\mathbf{B}}, \cdots, v_m^{\mathbf{B}})$ are orthonormal sequences in $L^2(\mathbb{R}^d)$, and the functions $q_\ell(\mathbf{x})$ are $\mathbb{Z}^d$–periodic, the assertion follows by an application of [8, Lemma E]. □

Recall that if $\mathbf{u}$ is a finite set of functions such that $T(\mathbf{u})$ is a Riesz basis, then $G_{\mathbf{u}}$ is positive definite for almost every $\mathbf{x} \in \mathbb{T}^d$. Thus the square root of $G_{\mathbf{u}}$ (i.e. the unique positive definite matrix $H_{\mathbf{u}}$ such that $H_{\mathbf{u}}^2 = G_{\mathbf{u}}$) exists for almost every $\mathbf{x} \in \mathbb{T}^d$. Thus, (4) implies that also the square root of $G_{\mathbf{u}}^{\mathbf{D}}$ exists.

Proposition 6. *Let $\mathbf{B} \in \mathbb{R}^{d\times d}$ be nonsingular, and assume that $\mathbf{u} := \{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)$ is such that $T^{\mathbf{B}}(\mathbf{u})$ is a Riesz sequence. Let*

$$\mathbf{R}^{\mathbf{D}}(\mathbf{x}) := \left[G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{x})\right]^{-1/2}$$

and

$$(\widehat{h_1}(\mathbf{x}), \cdots, \widehat{h_m}(\mathbf{x}))^T := |b|^{1/2}\mathbf{R}^{\mathbf{D}}(\mathbf{x})(\widehat{u_1}(\mathbf{x}), \cdots, \widehat{u_m}(\mathbf{x}))^T.$$

Then $T^{\mathbf{B}}(\mathbf{h})$ is an orthonormal basis of $S^{\mathbf{B}}(\mathbf{u})$.

Proof. Lemma 1 implies that $T(\mathbf{u}^{\mathbf{B}})$ is a Riesz sequence. The hypotheses and (3) imply that

$$(\widehat{h_1^{\mathbf{B}}}(\mathbf{x}), \cdots, \widehat{h_m^{\mathbf{B}}}(\mathbf{x}))^T = |b|^{1/2}\mathbf{R}^{\mathbf{D}}(\mathbf{D}\mathbf{x})(\widehat{u}_1^{\mathbf{B}}(\mathbf{x}), \cdots, \widehat{u}_m^{\mathbf{B}}(\mathbf{x}))^T.$$

But (4) implies that

$$|b|^{1/2}\mathbf{R}^{\mathbf{D}}(\mathbf{D}\mathbf{x}) = \left[G_{\mathbf{u}^{\mathbf{B}}}(\mathbf{x})\right]^{-1/2}.$$

Thus, [8, Proposition G] implies that $T(\mathbf{h}^{\mathbf{B}})$ is an orthonormal basis of $S(\mathbf{h}^{\mathbf{B}})$, and the assertion follows from Lemma 1. □

Theorem 7. *Let $\mathbf{B} \in \mathbb{R}^{d\times d}$ be nonsingular, let $\mathbf{D} := (\mathbf{B}^{-1})^T$, assume that $T^{\mathbf{B}}(\mathbf{h})$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, that $\mathbf{u}$ is a set of functions such that $S^{\mathbf{B}}(\mathbf{u}) \subset S^{\mathbf{B}}(\mathbf{h})$, and let $\mathbf{Q}^{\mathbf{D}}(\mathbf{x})$ denote the transition matrix from $T^{\mathbf{B}}(\mathbf{h})$ to $T^{\mathbf{B}}(\mathbf{u})$. Then*

$$G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{x}) = |b|\mathbf{Q}^{\mathbf{D}}(\mathbf{D}\mathbf{x})(\mathbf{Q}^{\mathbf{D}}(\mathbf{D}\mathbf{x}))^* \quad a.e., \tag{5}$$

and the following statements are equivalent:

(a) $T^{\mathbf{B}}(\mathbf{u})$ *is a Riesz basis of* $S^{\mathbf{B}}(\mathbf{h})$ *with bounds* $0 < A \leq B$.

(b) $r = m$ *and for almost every* $\mathbf{x} \in \mathbb{T}^d$

$$||G^{\mathbf{D}}_{\mathbf{u}}(\mathbf{x})|| \leq |b|B \quad \textit{and} \quad || \left(G^{\mathbf{D}}_{\mathbf{u}}(\mathbf{x})\right)^{-1} || \leq |b|^{-1}A^{-1}.$$

(c) $r = m$ *and for almost every* $\mathbf{x} \in \mathbb{T}^d$

$$||\mathbf{Q}^{\mathbf{D}}(\mathbf{x})|| \leq B^{1/2} \quad \textit{and} \quad || \left(\mathbf{Q}^{\mathbf{D}}(\mathbf{x})\right)^{-1} || \leq A^{-1/2}.$$

In particular, $T^{\mathbf{B}}(\mathbf{u})$ *is an orthonormal basis of* $S^{\mathbf{B}}(\mathbf{h})$ *if and only if* $r = m$ *and* $G^{\mathbf{D}}_{\boldsymbol{\psi}}(\mathbf{x}) = |b|\mathbf{I}$ *for almost every* $\mathbf{x} \in \mathbb{T}^d$*, or, equivalently, if and only if* $r = m$ *and* $\mathbf{Q}^{\mathbf{D}}(\mathbf{x})$ *is a unitary matrix for almost every* $\mathbf{x} \in \mathbb{T}^d$.

Proof. The hypotheses imply that $T(\mathbf{h}^{\mathbf{B}})$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ and that $S(\mathbf{u}^{\mathbf{B}}) \subset S(\mathbf{h}^{\mathbf{B}})$. If $Q(\mathbf{x})$ denotes the transition matrix from $T(\mathbf{h}^{\mathbf{B}})$ to $T(\mathbf{u}^{\mathbf{B}})$ then [8, Theorem 5] implies that

$$G_{\mathbf{u}^{\mathbf{B}}}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\left(\mathbf{Q}(\mathbf{x})\right)^* \quad a.e.$$

By definition,

$$(\widehat{u^{\mathbf{B}}_1}(\mathbf{x}), \cdots, \widehat{u^{\mathbf{B}}}_m(\mathbf{x}))^T = \mathbf{Q}(\mathbf{x})(\widehat{h^{\mathbf{B}}_1}(\mathbf{x}), \cdots, \widehat{h^{\mathbf{B}}_m}(\mathbf{x}))^T$$

or, from (3),

$$(\widehat{u_1}(\mathbf{Dx}), \cdots, \widehat{u_m}(\mathbf{Dx}))^T = \mathbf{Q}(\mathbf{x})(\widehat{h_1}(\mathbf{Dx}), \cdots, \widehat{h_m}(\mathbf{Dx}))^T.$$

This implies that

$$\mathbf{Q}(\mathbf{x}) = Q^{\mathbf{D}}(\mathbf{Dx}) \tag{6}$$

and (5) follows from (4) and (6).

Assume now that (a) holds; then $T(\mathbf{u}^{\mathbf{B}})$ is a Riesz basis of $S(\mathbf{u}^{\mathbf{B}})$ with bounds $0 < A \leq B$, and [8, Theorem 5] implies that $r = m$ and for almost every $x \in T^d$

$$||G_{\mathbf{u}^{\mathbf{B}}}(\mathbf{x})|| \leq B \quad \text{and} \quad || \left(G_{\mathbf{u}^{\mathbf{B}}}(\mathbf{x})\right)^{-1} || \leq A^{-1}, \tag{7}$$

and (b) follows from (4).

If (b) is satisfied, then (4) implies (7), and (c) follows from [8, Theorem 5] and (6).

Finally, if (c) is satisfied then (6) implies that $||Q(\mathbf{x})|| \leq \mathbf{B}^{1/2}$ and $||(Q(\mathbf{x}))^{-1}|| \leq A^{-1/2}$; thus [8, Theorem 5] implies that $T(u^{\mathbf{B}})$ is a Riesz basis of $S(\mathbf{h}^{\mathbf{B}})$ with bounds A and B, and (a) follows from Lemma 1.

Let us now prove the last paragraph in the statement of the theorem: $T^{\mathbf{B}}(\mathbf{u})$ is an orthonormal basis of $S(\mathbf{h}^{\mathbf{B}})$ if and only if $T(\mathbf{u}^{\mathbf{B}})$ is an orthonormal basis of $S(\mathbf{h}^{\mathbf{B}})$, and [8, Theorem 5] and (4) imply that this is equivalent to $G^{\mathbf{D}}_{\mathbf{u}}(\mathbf{Dx}) = |b|\mathbf{I}$. Finally, (4) and (5) imply that $T(\mathbf{u}^{\mathbf{B}})$ is an orthonormal basis of $S(\mathbf{h}^{\mathbf{B}})$ if and only if $Q^{\mathbf{D}}(\mathbf{x})$ is unitary. ∎

3. Wavelets

$\mathbf{A} \in \mathbb{R}^{d\times d}$ is called a dilation matrix preserving the lattice $\mathbb{Z}^d$ if $\mathbf{A}\mathbb{Z}^d \subset \mathbb{Z}^d$ and all its eigenvalues have modulus greater than 1. These conditions imply that $\mathbf{A} \in \mathbb{Z}^{d\times d}$, and that if $a := \det \mathbf{A}$ then $|a|$ is an integer larger than 1 (cf. Madych [4]).

Assume that $\mathbf{A} \in \mathbb{R}^{d\times d}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$. A coset of $\mathbf{A}\mathbb{Z}^d$ is a set of the form

$$\mathbf{j} + \mathbf{A}\mathbb{Z}^d = \{\mathbf{j} + \mathbf{A}\mathbf{r}; \mathbf{r} \in \mathbb{Z}^d\},$$

where $\mathbf{j} \in \mathbb{Z}^d$. An element of a coset is called a *representative* of the coset. Any pair of cosets are either identical or disjoint, and the union of all disjoint cosets equals $\mathbb{Z}^d$. There are exactly $|a|$ disjoint cosets. (cf. Wojtaszczyk [6]). The collection of all disjoint cosets is denoted by $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$. A set $\mathbf{J} \subset \mathbb{Z}^d$ is said to be a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$ if it contains exactly $|a|$ elements and

$$\bigcup_{\mathbf{j}\in\mathbf{J}} (\mathbf{j} + \mathbf{A}\mathbb{Z}^d) = \mathbb{Z}^d.$$

Theorem 8. *Let $\mathbf{B} \in \mathbb{R}^{d\times d}$ be a nonsingular matrix, $\mathbf{u} = \{u_1, \cdots, u_n\} \subset L^2(\mathbb{R}^d)$, and assume that $T^{\mathbf{B}}(\mathbf{u})$ is an orthonormal sequence. Let $\mathbf{A}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, $a := \det \mathbf{A}$, $m := |a|n$, let $\mathbf{J}$ be a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$. For $x > 0$ define $I(x) := [1, x] \cap \mathbb{Z}$, and let*

$$p = (p_1, p_2) : I(m) \longrightarrow I(n) \times \mathbf{J}$$

be a bijection. If

$$v_\ell(t) := |a|^{1/2} u_{p_1(\ell)}(\mathbf{A}t + \mathbf{B}p_2(\ell))$$

and $\mathbf{v} := \{v_1, \cdots, v_m\}$, then $T^{\mathbf{B}}(\mathbf{v})$ is an orthonormal basis of $S^{\mathbf{B}}(\mathbf{A}; \mathbf{u})$, and every Riesz basis generator of $S^{\mathbf{B}}(\mathbf{A}; \mathbf{u})$ has exactly m functions.

Proof. The hypotheses imply that $T(u^{\mathbf{B}})$ is an orthonormal basis of $S(u^{\mathbf{B}})$, and from [8, Theorem 3] we conclude that if

$$w_\ell(t) := |a|^{1/2} u^{\mathbf{B}}_{p_1(\ell)}(\mathbf{A}t + p_2(\ell))$$

and $\mathbf{w} := \{w_1, \cdots, w_m\}$, then $T(\mathbf{w})$ is an orthonormal basis of $S(\mathbf{A}; \mathbf{u}^{\mathbf{B}})$. Let $b := \det \mathbf{B}$ and

$$L : S^{\mathbf{B}}(A, u) \longrightarrow S(A; u^{\mathbf{B}}); \qquad Lf := f^{\mathbf{B}}.$$

Since $LT_{\mathbf{k}}v_\ell = T_{\mathbf{k}}w_\ell$, proceeding as in the proof of Lemma 1 we see that L is an isometry from $S(A, u^{\mathbf{B}})$ onto $S^{\mathbf{B}}(A; u)$, and the assertion follows. □

Note. There is a typographical error in the statement of [8, Theorem 3]: The range of the function p described in that theorem is $I(n) \times \mathbf{J}$.

Let $\mathbf{A} \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, and assume that $\mathbf{B} \in \mathbb{R}^{d\times d}$ is nonsingular. A *multiresolution analysis* (MRA) of multiplicity n in $L^2(\mathbb{R}^d)$ (generated by $\mathbf{A}$ and $\mathbf{B}$) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.

(ii) For every $j \in \mathbb{Z}, f(\mathbf{t}) \in V_j$ if and only if $f(\mathbf{A}\mathbf{t}) \in V_{j+1}$.

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

(iv) There are functions $\mathbf{u} := \{u_1, \cdots, u_n\}$ such that $T^{\mathbf{B}}(\mathbf{u})$ is an orthonormal basis of V_0.

From Proposition 6 we deduce that the condition that $T^{\mathbf{B}}(\mathbf{u})$ be an orthonormal basis may be replaced by the condition that $T^{\mathbf{B}}(\mathbf{u})$ be a Riesz basis.

It follows from the definition of multiresolution analysis that there are $\mathbf{D}\mathbb{Z}^d$–periodic functions $p_{\ell,j} \in L^2(\mathbb{T}^d)$ such that the functions u_ℓ satisfy the *scaling identity*

$$\widehat{u_\ell}(\mathbf{A}^T\mathbf{x}) = \sum_{j=1}^{n} p_{\ell,j}(\mathbf{x})\widehat{u_j}(\mathbf{x}), \quad j, \ell = 1, \cdots, n \quad a.e.,$$

The functions u_ℓ are called *scaling functions* for the multiresolution analysis, and the functions $p_{\ell,j}$ are called the *low pass filters* associated with $\mathbf{u}$.

Assume that $\mathbf{A}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$ and that $\mathbf{B} \in \mathbb{Z}^{d\times d}$ is nonsingular. A finite set of functions $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_m\} \in L^2(\mathbb{R}^d)$ will be called an orthonormal or Riesz wavelet system if the affine sequence

$$\bigcup_{j\in\mathbb{Z}} T^{\mathbf{B}}(\mathbf{A}^j; \boldsymbol{\psi}) = \{D_j^{\mathbf{A}} T_{\mathbf{k}}^{\mathbf{B}} \psi_\ell; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \ell = 1, \cdots, m\}$$

is respectively an orthonormal basis or a Riesz basis of $L^2(\mathbb{R}^d)$. If $d = 1$ we omit the word "system". If we need to emphasize the connection with the matrices $\mathbf{A}$ and $\mathbf{B}$ we will say that the wavelet system is *generated* by $\mathbf{A}$ and $\mathbf{B}$.

Let $\boldsymbol{\psi} := \{\psi_1, \cdots, \psi_m\}$ be a Riesz wavelet system in $L^2(\mathbb{R}^d)$ generated by matrices $\mathbf{A}$ and $\mathbf{B}$; for $j \in \mathbb{Z}$ we define $P_j := S^{\mathbf{B}}(\mathbf{A}^j; \boldsymbol{\psi})$ and $V_j := \sum_{r<j} P_r$, i.e.,

$$V_j = \sum r < jS^{\mathbf{B}}(\mathbf{A}^r; \boldsymbol{\psi}).$$

We say that $\boldsymbol{\psi}$ is *associated* with an MRA, if $M := \{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis. If this is the case, we also say that $\boldsymbol{\psi}$ is associated with M. Let W_j denote the orthogonal complement of V_j in V_{j+1}. Then $\boldsymbol{\psi}$ is an orthonormal wavelet system associated with M if and only if $P_j = W_j$ for every $j \in \mathbb{Z}$, and $T(\boldsymbol{\psi})$ is an orthonormal basis of W_0. This implies that $\boldsymbol{\phi}$ is another orthonormal wavelet system associated with the same multiresolution analysis M if and only if $T^{\mathbf{B}}(\boldsymbol{\psi})$ is an orthonormal basis of W_0.

Theorem 9. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n, generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ and a nonsingular matrix $\mathbf{B}$, and having scaling functions $u_1, \cdots, u_n$. Let $a := \det \mathbf{A}$, $m := |a|n$, $\mathbf{D} := (B^{-1})^T$, and let $\{v_1, \cdots, v_m\}$ be an orthonormal basis generator of V_1 (such as the one given in Theorem 8). The following propositions are equivalent:*

(a) *$\{w_1, \cdots w_{m-n}\}$ is an orthonormal wavelet system associated with M.*

(b) *There is an $m\times m$ matrix $\mathbf{Q}(\mathbf{x})$ of $\mathbf{D}\mathbb{Z}^d$–periodic and measurable functions, a.e. unitary on $\mathbf{D}T^d$, such that, if*

$$(\widehat{y_1}(\mathbf{x}), \cdots, \widehat{y_m}(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})(\widehat{v_1}(\mathbf{x}), \cdots, \widehat{v_m}(\mathbf{x}))^T,$$

then

$$y_{(\ell-1)|a|+1} = u_\ell; \quad 1 \le \ell \le n$$

and

$$y_{(\ell-1)|a|+k+1} = w_{(\ell-1)|a|+k-\ell+1}; \quad 1 \le \ell \le n, \quad 1 \le k \le |a| - 1.$$

Proof. Let $\mathbf{v} := \{v_1, \cdots, v_n\}$ be such that $T^{\mathbf{B}}(\mathbf{v})$ is an orthonormal basis of V_0, $v_\ell^{\mathbf{B}}(\mathbf{t}) := |b|^{1/2} v_\ell(\mathbf{B}\mathbf{t})$, $w_\ell^{\mathbf{B}}(\mathbf{t}) := |b|^{1/2} w_\ell(\mathbf{B}\mathbf{t})$,

$$U_j := \{f : f(\mathbf{B}^{-1}\mathbf{t}) \in V_j\},$$

and let W_0^* be the orthogonal complement of U_0 in U_1. From Lemma 1 we deduce that $N := \{U_j; j \in \mathbb{Z}\}$ is a multiresolution analysis of multiplicity n generated by $\mathbf{A}$ and $\mathbf{I}$, with scaling functions $\mathbf{v^B} := \{v_1^{\mathbf{B}}, \cdots, v_n^{\mathbf{B}}\}$.

Clearly (a) is equivalent to $\mathbf{w^B} := \{w_1^{\mathbf{B}}, \cdots, w_{m-n}^{\mathbf{B}}\}$ being an orthonormal wavelet system associated with N. On the other hand, we see from (3) that (b) is equivalent to the existence of an $m \times m$ matrix $R(\mathbf{x})$ (i.e., $Q(\mathbf{D}^{-1})(\mathbf{x})$) of $\mathbb{Z}^d$–periodic and measurable functions, and a.e. unitary on $\mathbb{T}^d$, such that if

$$(\widehat{y_1^{\mathbf{B}}}(\mathbf{x}), \cdots, \widehat{y_m^{\mathbf{B}}}(\mathbf{x}))^T := R(\mathbf{x})(\widehat{v_1^{\mathbf{B}}}(\mathbf{x}), \cdots, \widehat{v_m^{\mathbf{B}}}(\mathbf{x}))^T,$$

then

$$y^{\mathbf{B}}_{(\ell-1)|a|+1} = u_\ell^{\mathbf{B}}, \qquad 1 \leq \ell \leq n$$

and

$$y^{\mathbf{B}}_{(\ell-1)|a|+k+1} = w^{\mathbf{B}}_{(\ell-1)|a|+k-\ell+1}; \qquad 1 \leq \ell \leq n, 1 \leq k \leq |a|-1,$$

whence the assertion follows by an application of [8, Theorem 8]. □

Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n with scaling functions $\mathbf{u} := \{u_1, \cdots, u_n\}$, generated by a matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ and a nonsingular matrix $\mathbf{B}$. By orthogonality we know that

$$V_1 = S^{\mathbf{B}}(A, u_1) \oplus S^{\mathbf{B}}(A, u_2) \oplus \cdots \oplus S^{\mathbf{B}}(A, u_n).$$

Theorem 8 implies that there are functions $v_{\ell,k}$ such that

$$\{v_{\ell,1}, \cdots, v_{\ell,|a|}\} \tag{8}$$

is an orthonormal basis generator of $S^{\mathbf{B}}(\mathbf{A}, u_\ell)$. It follows that

$$\{v_{\ell,k}; 1 \leq \ell \leq n,\ 1 \leq k \leq |a|\}$$

is an orthonormal basis generator of V_1.

For $k > 1$ let $\mathrm{diag}\,\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{k}}$ denote the $k \times k$ diagonal matrix with $-e^{i\omega}, 1, \cdots, 1$ as its diagonal entries.

Theorem 10. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n, generated by a matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ and a nonsingular matrix $\mathbf{B}$, with scaling functions $\mathbf{u} := \{u_1, \cdots, u_n\}$. Let $a := \det \mathbf{A}$, $b := \det \mathbf{B}$, $m := |a|n$, $\mathbf{e} := (1, 0, \cdots, 0) \in \mathbb{R}^{|a|}$, and for $1 \leq \ell \leq n$ let (8) be an orthonormal basis generator of $S^{\mathbf{B}}(\mathbf{A}, u_\ell)$. Let*

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{|a|} c_{\ell,j}(\mathbf{x})\widehat{v_{\ell,j}}(\mathbf{x}), \tag{9}$$

and define

$$w_{\ell,j}(\mathbf{t}) := |b|^{1/2} v_{\ell,j}(\mathbf{Bt}); \quad 1 \leq \ell \leq n, \qquad b_{\ell,j}(\mathbf{t}) := c_{\ell,j}(\mathbf{Bt}),$$

$$\mathbf{b}_\ell(\mathbf{x}) := (b_{\ell,1}(\mathbf{x}), \cdots, b_{\ell,|a|}(\mathbf{x}))^T, \quad \delta_\ell(\mathbf{x}) := e^{iArg\, b_{\ell,1}(\mathbf{x})}, \quad \mathbf{q}_\ell(\mathbf{x}) := \mathbf{b}_\ell(\mathbf{x}) + \delta_\ell(\mathbf{x})\mathbf{e},$$

$$\widehat{\mathbf{w}}(\mathbf{x}) := (\widehat{w}_{1,1}(\mathbf{x}), \cdots, \widehat{w}_{1,|a|}(\mathbf{x}), \cdots, \widehat{w}_{n,1}(\mathbf{x}), \cdots, \widehat{w}_{n,|a|}(\mathbf{x}))^T,$$

and

$$\mathbf{Q}_\ell(\mathbf{x}) := \mathrm{diag}\,\{-\delta_\ell(\mathbf{x}), 1, \cdots, 1\}_{|\mathrm{a}|} \left[\overline{\mathbf{I} - 2\mathbf{q}_\ell(\mathbf{x})\mathbf{q}_\ell(\mathbf{x})^*/\mathbf{q}_\ell(\mathbf{x})^*\mathbf{q}_\ell(\mathbf{x})}\right].$$

Let

$$\mathbf{Q}(\mathbf{x}) = \Big(q_{\ell,k}(\mathbf{x})\Big)_{\ell,k=1}^{m}$$

be the $m \times m$ block diagonal matrix

$$\mathbf{Q}_1(\mathbf{x}) \oplus \mathbf{Q}_2(\mathbf{x}) \oplus \cdots \oplus \mathbf{Q}_n(\mathbf{x}) = \begin{pmatrix} \mathbf{Q}_1(\mathbf{x}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_n(\mathbf{x}) \end{pmatrix}.$$

If

$$(\widehat{y_1}(\mathbf{x}), \cdots, \widehat{y_m}(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})\widehat{bw}(\mathbf{x})$$

and

$$z_\ell(\mathbf{t}) := |b|^{-1/2} y_\ell(\mathbf{B}^{-1}\mathbf{t}), \quad 1 \le \ell \le m,$$

then

$$z_{(\ell-1)|a|+1} = u_\ell; \quad 1 \le \ell \le n, \tag{10}$$

and

$$\{z_{(\ell-1)|a|+k}; 1 \le \ell \le n,\ 2 \le k \le |a|\} \tag{11}$$

is an orthonormal wavelet system associated with M.

Proof. Let $u_\ell^{\mathbf{B}}(\mathbf{t}) := |b|^{1/2} u_\ell(\mathbf{B}\mathbf{t})$ and

$$U_j := \{f : f(\mathbf{B}^{-1}\mathbf{t}) \in V_j\}.$$

Then $N := \{U_j; j \in \mathbb{Z}\}$ is a multiresolution analysis of multiplicity n generated by $\mathbf{A}$ and $\mathbf{I}$, with scaling functions $\mathbf{u}^{\mathbf{B}} := \{u_1^{\mathbf{B}}, \cdots, u_n^{\mathbf{B}}\}$. Moreover, the hypotheses imply that $\{w_{\ell,j}; 1 \le \ell \le |a|\}$ is an orthonormal basis generator of $S(\mathbf{A}, u_\ell^{\mathbf{B}})$, whereas (9) implies that

$$\widehat{u_\ell^{\mathbf{B}}}(\mathbf{x}) = \sum_{j=1}^{|a|} b_{\ell,j}(\mathbf{x})\widehat{w_j}(\mathbf{x}).$$

Applying [8, Theorem 9] we conclude that

$$y_{(\ell-1)|a|+1} = u_\ell^{\mathbf{B}}; \quad 1 \le \ell \le n, \tag{12}$$

and that

$$\{y_{(\ell-1)|a|+k}; 1 \le \ell \le n,\ 2 \le k \le |a|\}$$

is an orthonormal wavelet system associated with N. The definitions of U_0 and $\mathbf{u}^{\mathbf{B}}$ together with (12) imply (10). Finally, if $W_0^{\mathbf{B}}$ denotes the orthogonal complement of U_0 in U_1 and W_0 denotes the orthogonal complement of V_0 in V_1, it is clear that

$$W_0^{\mathbf{B}} = \{f : f(\mathbf{B}^{-1}\cdot) \in W_0\},$$

and we conclude that (11) is an orthonormal wavelet system associated with M. □

Corollary 11. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity 1, generated by a matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ and a nonsingular matrix $\mathbf{B}$, with scaling functions u. Let $a := \det \mathbf{A}$, $b := \det \mathbf{B}$, $m := |a|$, and let $\{v_k; 1 \le k \le m\}$ be an orthonormal basis generator of $S^{\mathbf{B}}(\mathbf{A}, u)$. Let*

$$\widehat{u}(\mathbf{x}) = \sum_{j=1}^{m} c_j(\mathbf{x})\widehat{v_j}(\mathbf{x}),$$

and define

$$w_j(\mathbf{t}) := |b|^{1/2} v_j(\mathbf{B}\mathbf{t}), \quad \delta(\mathbf{x}) := e^{i\,Arg\, c_1(\mathbf{B}\mathbf{x})},$$

$$\mathbf{q}(\mathbf{x}) := (c_1(\mathbf{B}\mathbf{x}), \cdots, c_{m-1}(\mathbf{B}\mathbf{x}), c_m(\mathbf{B}\mathbf{x}) + \delta(\mathbf{x}))^T,$$
$$\mathbf{W}(\mathbf{x}) := (w_1(\mathbf{x}), \cdots, w_m(\mathbf{x}))^T,$$

and

$$\mathbf{Q}(\mathbf{x}) := \operatorname{diag}\{-\delta(\mathbf{x}), 1, \cdots, 1\}_{\mathrm{m}} \left[\overline{\mathbf{I} - 2\mathbf{q}(\mathbf{x})\mathbf{q}(\mathbf{x})^*/\mathbf{q}(\mathbf{x})^*\mathbf{q}(\mathbf{x})}\right].$$

If

$$(\widehat{y_1}(\mathbf{x}), \cdots, \widehat{y_m}(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})\mathbf{W}(\mathbf{x})$$

and

$$z_k(\mathbf{t}) := |b|^{-1/2} y_k(\mathbf{B}^{-1}\mathbf{t}), \quad 1 \le k \le m,$$

then $z_1 = u$, *and*

$$\{z_k; 2 \le k \le m\}$$

is an orthonormal wavelet system associated with M.

Corollary 12. *Let* M *be a multiresolution analysis of multiplicity* n, *generated by a matrix* $\mathbf{A}$ *that preserves the lattice* $\mathbb{Z}^d$ *and a nonsingular matrix* $\mathbf{B}$. *Then there is an orthogonal wanelet system of* $(|a|-1)n$ *functions associated with* M.

Theorem 13. *Let* M *be a multiresolution analysis of order* n *generated by a matrix* $\mathbf{A}$ *that preserves the lattice* $\mathbb{Z}^d$ *and a nonsingular matrix* $\mathbf{B}$, *let* $k := (|a|-1)n$, *and assume that* $\mathbf{w} := \{w_1, \cdots .w_k\}$ *is an orthonormal wavelet system associated with* M. *Let* $\mathbf{y} := \{y_1, \cdots, y_r\} \subset L^2(\mathbb{R}^d)$. *Then* $\mathbf{y}$ *is an orthonormal wavelet system associated with* M *if and only if* $r = k$ *and there is a* $k \times k$ *matrix* $P(\mathbf{x})$ *of* $\mathbf{D}\mathbb{Z}^d$*–periodic and measurable functions, a.e. unitary on* $\mathbf{D}\mathbb{T}^d$, *such that*

$$(y_1(\mathbf{x}), \cdots, y_k(\mathbf{x}))^T = P(\mathbf{x})(w_1(\mathbf{x}), \cdots, w_k(\mathbf{x}))^T.$$

Proof. $\mathbf{y}$ is an orthonormal wavelet system associated with M if and only if $T^{\mathbf{B}}(\mathbf{y})$ is an orthonormal basis of W_0, and the assertion follows from Theorem 7. □

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Department of Mathematics and Statistics, Auburn University, AL 36849-5310.
E-mail address: zalik@auburn.edu