SOME SMOOTH COMPACTLY SUPPORTED TIGHT FRAMELETS ASSOCIATED TO THE QUINCUNX MATRIX

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ABSTRACT. We construct several families of tight wavelet frames in $L^2(\mathbb{R}^2)$ associated to the quincunx matrix. A couple of those families has five generators. Moreover, we construct a family of tight wavelet frames with three generators. Finally, we show families with only two generators. The generators have compact support, any given degree of regularity, and any fixed number of vanishing moments. Our construction is made in Fourier space and involves some refinable functions, the Oblique Extension Principle and a slight generalization of a theorem of Lai and Stöckler. In addition, we will use well known results on construction of tight wavelet frames with two generators on $\mathbb{R}$ with the dyadic dilation. The refinable functions we use are constructed from the Daubechies low pass filters and are compactly supported. The main difference between these families is that while the refinable functions associated to the five generators have many symmetries, the refinable functions used in the construction of the others families are merely even.

1. INTRODUCTION

Compactly supported wavelets and wavelets frames constructed by univariate tensor product of wavelets (i.e., separable wavelets) have been used widely, but have some drawbacks. As remarked by Cohen and Daubechies [6] in the two–dimensional setting, this choice is restrictive and it gives a particular importance to the horizontal and vertical directions. Referring specifically to image processing in $\mathbb{R}^2$, Belogay and Wang [2] point out that such wavelets have very little design freedom, and that separability imposes an unnecessary product structure on the plane, which is artificial for natural images. These considerations provide a justification for this paper, whose purpose is to construct wavelet frames obtained by non–tensor product methods.

For the quincunx matrix, we construct several families of tight wavelet frames in $L^2(\mathbb{R}^2)$ with five, three and two generators respectively, with compact support, any given degree of regularity and any fixed number of vanishing moments. The main difference between these families is that while the refinable functions associated to the five generators in the first family have many symmetries, the refinable functions used in the construction of the other families are merely even.

We now introduce the notation and definitions that we shall use in what follows. The sets of nonnegative integers, strictly positive integers, integers, rational numbers and real numbers will be denoted by $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively.

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Given a matrix M, its transpose will be denoted by M^T and the conjugate of its transpose will be denoted by M^*. The $n \times n$ identity matrix will be denoted by $\mathbf{I}_{n\times n}$. We say that $A \in \mathbb{R}^{d\times d}$, $d \geq 1$, is a dilation matrix preserving the lattice $\mathbb{Z}^d$ if all eigenvalues of A have modulus greater than 1 and $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$. The set of all $d \times d$ dilation matrices preserving the lattice $\mathbb{Z}^d$ will be denoted by $\mathbf{E}_d(\mathbb{Z})$. Note that if $A \in \mathbf{E}_d(\mathbb{Z})$ then $a := |\det A|$ is an integer greater than 1, and the quotient groups $\mathbb{Z}^d/A\mathbb{Z}^d$ and $A^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ are well defined. By $\boldsymbol{\Delta}_A$ and $\boldsymbol{\Gamma}_A$ we will denote full collections of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$ and $A^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ respectively. From [10, Lemma 2] we know that $\mathbb{Z}^d/A\mathbb{Z}^d$ has exactly a cosets, which readily implies that also $A^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ has exactly a cosets.

Let $\widehat{f}$ denote the Fourier transform of the function f. Thus, if $f \in L^1(\mathbb{R}^d)$ and $\mathbf{x}, \mathbf{t} \in \mathbb{R}^d$, then

$$\widehat{f}(\mathbf{t}) := \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\mathbf{t}} d\mathbf{x},$$

where $\mathbf{x}\cdot\mathbf{t}$ denotes the dot product of vectors $\mathbf{x}$ and $\mathbf{t}$. The Fourier transform is extended to $L^2(\mathbb{R}^d)$ in the usual way.

A sequence $\{\phi_n\}_{n=1}^{\infty}$ of elements in a separable Hilbert space $\mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist constants $C_1, C_2 > 0$ (called *frame bounds*) such that

$$C_1\|h\|^2 \leq \sum_{n=1}^{\infty} |\langle h, \phi_n\rangle|^2 \leq C_2\|h\|^2, \qquad \forall h \in \mathbb{H},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{H}$. A frame is *tight* if we may choose $C_1 = C_2$.

Let $A \in \mathbf{E}_d(\mathbb{Z})$. A set of functions $\Psi = \{\psi_1, \ldots, \psi_N\} \subset L^2(\mathbb{R}^d)$ is called a *wavelet frame* or *framelet* with dilation A, if the system

$$\{a^{j/2}\psi_\ell(A^j\mathbf{x} + \mathbf{k}); j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, 1 \leq \ell \leq N\}$$

is a frame for $L^2(\mathbb{R}^d)$. If this system is a tight frame for $L^2(\mathbb{R}^d)$ then Ψ is called a tight framelet.

A wavelet frame $\Psi = \{\psi_1, \ldots, \psi_N\} \subset L^2(\mathbb{R}^d)$ has vanishing moments of order $m \in \{0, 1, \cdots, \}$, if $\widehat{\psi_\ell}$, $\ell = 1, \cdots, N$, the Fourier transform of ψ_ℓ, has a zero of order m at the origin.

A function $\theta \in L^2(\mathbb{R}^d)$ is said to be refinable if it satisfies the following refinement equation

$$\theta(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} a_{\mathbf{k}}\theta(A\mathbf{x} - \mathbf{k}), \qquad a_{\mathbf{k}} \in \mathbb{C}, \tag{1}$$

where the convergence is in $L^2(\mathbb{R}^d)$ and $\sum_{\mathbf{k}\in\mathbb{Z}^d} |a_{\mathbf{k}}|^2 < \infty$. Taking the Fourier transform, we obtain

$$\widehat{\phi}(A^*\mathbf{t}) = H(\mathbf{t})\widehat{\phi}(\mathbf{t}) \quad \text{ a.e. on } \mathbb{R}^n \tag{2}$$

where

$$H(\mathbf{t}) = \frac{1}{a}\sum_{\mathbf{k}\in\mathbb{Z}^n} a_{\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{t}}$$

is a function in $L^2(\mathbb{T}^n)$.

A main method for constructing wavelet frames is the Oblique Extension Principle (OEP). It was developed by Chui, He and Stöckler [5], and independently by Daubechies, Han, Ron, and Shen [8], who gave the method its name. The OEP may be formulated as follows:

Theorem A. *Let $A \in \mathbf{E}_d(\mathbb{Z})$. Let $\phi \in L^2(\mathbb{R}^d)$ be compactly supported and refinable, i.e.*

$$\widehat{\phi}(A^*\mathbf{t}) = P(\mathbf{t})\widehat{\phi}(\mathbf{t}),$$

where $P(\mathbf{t})$ is a trigonometric polynomial. Assume moreover that $|\widehat{\phi}(\mathbf{0})| = 1$. Let $S(\mathbf{t})$ be another trigonometric polynomial such that $S(\mathbf{t}) \geq 0$ and $S(\mathbf{0}) = 1$. Assume there are trigonometric polynomials or rational functions Q_ℓ, $\ell = 1, \cdots, N$, that satisfy the OEP condition

$$S(A^*\mathbf{t})P(\mathbf{t})\overline{P(\mathbf{t}+\mathbf{j})} + \sum_{\ell=1}^{N} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} = \begin{cases} S(\mathbf{t}) & if \quad \mathbf{j} \in \mathbb{Z}^d, \\ 0 & if \quad \mathbf{j} \in \Big((A^*)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d\Big) \setminus \mathbb{Z}^d \end{cases} . \tag{3}$$

If

$$\widehat{\psi_\ell}(A^*\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad \ell = 1, \ldots, N,$$

then $\Psi = \{\psi_1, \ldots, \psi_N\}$ is a tight framelet in $L^2(\mathbb{R}^d)$ with dilation A and frame constant 1.

With an additional decay condition, Theorem A follows from [8, Proposition 1.11], except for the value of the frame constant, which follows from e.g. [19, Theorem 6.5]. However, recent results of Han imply that this decay condition is redundant. Indeed, Theorem A in its present formulation is a consequence of Proposition 4, Corollary 12 and Theorem 17 in [11] (for a simpler version of Han's results in dimension one see [12]). See also the version proved by Atreas, Melas and Stavropoulos [1].

In [21], San Antolín and Zalik developed a method to generate wavelet frames using the Oblique Extension Principle (see eg. [11]), and a slight generalization of a theorem of Lai and Stöckler [16]. This method was used in [22] to construct, for a 2×2 expansive dilation matrix with integer entries and determinant ± 2, families of compactly supported tight framelets with three generators and with any desired degree of smoothness. The same method was used in [23] to construct compactly supported tight framelets having the following additional properties: both the framelets and the refinable functions that generate them can be made as smooth as desired. Moreover, these refinable functions are nonseparable, in the sense that they cannot be expressed as the product of two functions defined on lower dimensions.

In this paper we will use the same method to construct smooth compactly supported tight wavelet frames and refinable functions with good approximation properties in $L^2(\mathbb{R}^2)$, associated to either of the following two dilation matrices:

$$R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4}$$

In order to show other constructions of tight wavelet frames associated to R, we will use well known results on construction of tight wavelet frames with two generators on $\mathbb{R}$ with the dyadic dilation

In Section 2 we will construct two families of tight wavelet frames associated to the dilation matrix R in (4) with five generators. Similar results hold for the dilation matrix B. Section 3 is devoted to the construction of another family of tight wavelet frames that has three generators. In Section 4 we will construct two families of tight wavelet frames associated to the dilation matrix R with only two generators. All the generators have compact support and any desired degree of smoothness and vanishing moments.

2. Tight framelets with five generators

For our construction we will use the following trigonometric polynomials in $\mathbb{R}$. For $n = 1, 2, 3, \cdots$, let

$$g_n(t) := 1 - c_n \int_0^t (\sin 2\pi\xi)^{2n+1} d\xi, \tag{5}$$

where $c_n = (\int_0^{1/2} (\sin 2\pi\xi)^{2n+1} d\xi)^{-1}$.

Some properties of the functions g_n are summarized in [24, Lemma 4.8].

It is easy to see that the set

$$\mathbf{\Delta} := \{\mathbf{q}_0 = (0,0)^T, \mathbf{q}_1 = (1,0)^T\}, \tag{6}$$

is a full collection of representatives of the cosets of $\mathbb{Z}^2/R\mathbb{Z}^2$, and that

$$\mathbf{\Gamma} := \{\mathbf{r}_0 = (0,0)^T, \mathbf{r}_1 = (1/2,1/2)^T\}. \tag{7}$$

is a full collection of representatives of the cosets of $R^{-1}\mathbb{Z}^2/\mathbb{Z}^2$.

In the remainder of this article $\mathbf{t}$ will stand for $(t_1, t_2) \in \mathbb{R}^2$. With this convention the matrix $\mathcal{M}(\mathbf{t})$, defined in [21, Theorem 1] or [22, Theorem B] with $A = R$, is

$$\mathcal{M}(\mathbf{t}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i2\pi t_1} \\ 1 & -e^{i2\pi t_1} \end{pmatrix}. \tag{8}$$

We now construct a family of smooth compactly supported refinable functions in $L^2(\mathbb{R}^2)$ with dilation matrix R.

For $n \in \mathbb{N}$, let the trigonometric polynomials $P(\mathbf{t})$ in $\mathbb{R}^2$ be defined by

$$P(\mathbf{t}) := g_n(t_1) g_n(t_2). \tag{9}$$

where g_n is defined in (5). From [24, Lemma 4.8] we know that $P(\mathbf{0}) = 1$, that

$$|P(\mathbf{t})|^2 + |P(\mathbf{t} + \mathbf{r}_1)|^2 \le 1, \tag{10}$$

and that the equality holds only if $\mathbf{t} \in \mathbb{Z}^2 \cup \big(\mathbb{Z}^2 + (1/2, 1/2)\big)$.

We need the following technical lemma:

Lemma 1. *Let $j, n \in \mathbb{N}$. If g_n are the trigonometric polynomials defined in (5) and P is defined by (9), then $P((R^T)^{-(2j-1)}\mathbf{t}) = g_n(2^{-j}(t_1 - t_2)) g_n(2^{-j}(t_1 + t_2))$ and $P((R^T)^{-2j}\mathbf{t}) = g_n(2^{-j}t_1) g_n(2^{-j}t_2)$.*

Proof. Since $(R^T)^{-1} = (1/2)R$, and R represents a clockwise rotation by $\pi/4$ multiplied by $\sqrt{2}$, it will suffice to prove the assertion for the first eight natural numbers. We have:

$$(R^T)^{-2} = \frac{1}{2^2} \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad (R^T)^{-3} = \frac{1}{2^3} \begin{pmatrix} -2 & -2 \\ 2 & -2 \end{pmatrix},$$

$$(R^T)^{-4} = \frac{1}{2^4} \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}, \quad (R^T)^{-5} = \frac{1}{2^5} \begin{pmatrix} -4 & 4 \\ -4 & -4 \end{pmatrix},$$

$$(R^T)^{-6} = \frac{1}{2^6} \begin{pmatrix} 0 & 8 \\ -8 & 0 \end{pmatrix}, \quad (R^T)^{-7} = \frac{1}{2^7} \begin{pmatrix} 8 & 8 \\ -8 & 8 \end{pmatrix}.$$

$$(R^T)^{-8} = \frac{1}{2^8}\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}.$$

Since g_n is even, this implies that

$$\begin{aligned}
P((R^T)^{-1}\mathbf{t}) &= P(\frac{1}{2}(t_1-t_2), \frac{1}{2}(t_1+t_2)) = g_n(2^{-1}(t_1-t_2))g_n(2^{-1}(t_1+t_2)),\\
P((R^T)^{-2}\mathbf{t}) &= P(-2^{-1}t_2, 2^{-1}t_1) = g_n(2^{-1}t_2)g_n(2^{-1}t_1),\\
P((R^T)^{-3}\mathbf{t}) &= P(-2^{-2}(t_1+t_2), 2^{-2}(t_1-t_2)) = g_n(2^{-2}(t_1+t_2))g_n(2^{-2}(t_1-t_2)),\\
P((R^T)^{-4}\mathbf{t}) &= P(-2^{-2}t_1, -2^{-2}t_2) = g_n(2^{-2}t_1)g_n(2^{-2}t_2),\\
P((R^T)^{-5}\mathbf{t}) &= P(-2^{-3}(t_1-t_2), -2^{-3}(t_1+t_2)) = g_n(2^{-3}(t_1-t_2))g_n(2^{-3}(t_1+t_2)),\\
P((R^T)^{-6}\mathbf{t}) &= P(2^{-3}t_2, -2^{-3}t_1) = g_n(2^{-3}t_2)g_n(2^{-3}t_1),\\
P((R^T)^{-7}\mathbf{t}) &= P(2^{-4}(t_1+t_2), -2^{-4}(t_1-t_2)) = g_n(2^{-4}(t_1+t_2))g_n(2^{-4}(t_1-t_2)),\\
P((R^T)^{-8}\mathbf{t}) &= P(2^{-4}t_1, 2^{-4}t_2) = g_n(2^{-4}t_1)g_n(2^{-4}t_2).
\end{aligned}$$

□

We have

Proposition 1. *Let $n \in \mathbb{N}$ and let $P(\mathbf{t})$ be defined by* (9)*. Then the infinite product*

$$\prod_{j=1}^{\infty} P((R^T)^{-j}\mathbf{t})$$

converges to a nonnegative continuous function $\widehat{\phi}$ in $L^2(\mathbb{R}^2)$ such that $\|\widehat{\phi}\|_{L^2(\mathbb{R}^2)} \leq 1$, $\widehat{\phi}(\mathbf{0}) = 1$, has the symmetries $\widehat{\phi}(t_1,t_2) = \widehat{\phi}(t_2,t_1)$ and $\widehat{\phi}(t_1,t_2) = \widehat{\phi}(\epsilon_1 t_1, \epsilon_2 t_2)$ where $\epsilon_i = \pm 1$, $i = 1,2$, and satisfies the refinement equation

$$\widehat{\phi}(R^T\mathbf{t}) = P(\mathbf{t})\widehat{\phi}(\mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}^2.$$

Proof. We only need to prove the symmetries of $\widehat{\phi}$, since the other statements follow from (10) proceeding as in the proof of [3, Theorem 1] (see also [13, Lemma 2.1]).

Applying Lemma 1, we have:

$$\begin{aligned}
\widehat{\phi}(\mathbf{t}) &= \widehat{\phi}(t_1,t_2) = \prod_{j=1}^{\infty} P((R^T)^{-j}\mathbf{t})\\
&= \prod_{j=1}^{\infty} g_n(2^{-j}(t_1-t_2))g_n(2^{-j}(t_1+t_2))g_n(2^{-j}t_1)g_n(2^{-j}t_2). \qquad (11)
\end{aligned}$$

Thus

$$\begin{aligned}
\widehat{\phi}(\mathbf{t}) &= \prod_{j=1}^{\infty} g_n(2^{-j}(\epsilon_1 t_1-\epsilon_2 t_2))g_n(2^{-j}(\epsilon_1 t_1+\epsilon_2 t_2))g_n(2^{-j}\epsilon_1 t_1)g_n(2^{-j}\epsilon_2 t_2)\\
&= \widehat{\phi}(\epsilon_1 t_1, \epsilon_2 t_2),
\end{aligned}$$

where $\epsilon_1, \epsilon_2 = \pm 1$. Finally, applying (11) and using again the hypothesis that g_n is even, we obtain $\widehat{\phi}(t_1,t_2) = \widehat{\phi}(t_2,t_1)$. □

We now prove

Proposition 2. *Let $n \in \mathbb{N}$ and let $\widehat{\phi}$ be defined in Proposition 1. Then the function ϕ whose Fourier transform is $\widehat{\phi}$ is not identically zero and is real valued, compactly supported, $\|\phi\|_{L_2(\mathbb{R}^2)} \leq 1$ and has the symmetries $\phi(x_1, x_2) = \phi(x_2, x_1)$ and $\phi(\epsilon_1 x_1, \epsilon_2 x_2) = \phi(x_1, x_2)$, where $\epsilon_i = \pm 1$, $i = 1, 2$. Moreover, let $p := \log_2(\frac{8}{3\sqrt{3}}) \approx 0.6226$. Then for any $\beta > 0$ there are a constant C_n and an integer N, such that if $n \geq N$ then*

$$\prod_{j=1}^{\infty} g_n(2^{-j}t) \leq C_n |t|^{(p+\beta-1)n}. \tag{12}$$

In particular, if

$$(1 - p - \beta)n - 2 > r \geq 0 \tag{13}$$

and $n \geq N$, then ϕ is in continuity class C^r.

Proof. We remark that, since $1-p > 0$, then for any integer r and sufficiently large n there exist numbers β that satisfy (5). We also note that the estimate for p in [24, Remark 4.10] is too low.

We will first obtain a refinement of [24, Proposition 4.10]. As remarked in the proof of that proposition, $g_k(t)$ may be written in the form

$$g_k(t) = \left(\frac{1 + \cos t}{2}\right)^{k/2} M_k(t), \tag{14}$$

where $M_k(t) \geq 0$. Combining (4.38), (4.40) and (4.44) of [24] we obtain

$$M_k(t) \leq 6\sqrt{k}\ 2^{k/2} \left(\frac{4}{3}\sqrt{\frac{2}{3}}\right)^k = 6\sqrt{k} \left(\frac{8}{3\sqrt{3}}\right)^k = 6\sqrt{k}\ 2^{pk}.$$

Thus, for any $\beta > 0$ there is a integer N such that, if $n \geq N$ and $|t| > 1$ then $M_k(t) \leq 2^{(p+\beta)k}$. Replacing $2^{(p+\beta)k}$ for $2^{\alpha k r}$ in [24, (4.50)] we obtain

$$\prod_{j=1}^{\infty} |M_k(2^{-j}t)| \leq 2C_k |t|^{(p+\beta)k}. \tag{15}$$

Combining (4.46) and (4.48) of [24] with (15) we deduce that (12) holds.

Since $\widehat{\phi}$ is in $L^2(\mathbb{R}^2)$ and is not identically zero, it follows that also ϕ is in $L^2(\mathbb{R}^2)$ and is not identically zero. Moreover, $\|\phi\|_{L^2(\mathbb{R}^2)} = \|\widehat{\phi}\|_{L^2(\mathbb{R}^2)} \leq 1$. That ϕ is real valued follows because $\widehat{\phi}$ is real valued and even.

Replicating an argument of Wojtaszczyk [24, p. 79] it is easy to see that ϕ is compactly supported on $\mathbb{R}^2$.

The symmetry properties of ϕ follow from the symmetry properties of $\widehat{\phi}$ in Proposition 1.

It remains to prove the estimates for the degree of smoothness of ϕ. By (11) and bearing in mind that $0 \leq g_n(t) \leq 1$, we have

$$0 \leq \widehat{\phi}(\mathbf{t}) \leq \prod_{j=1}^{\infty} g_n(2^{-j}t_1) g_n(2^{-j}t_2).$$

Then, if $|t_1|, |t_2| > 1$, $n \geq N$, from (12) we obtain

$$0 \leq \widehat{\phi}(\mathbf{t}) \leq C_n^2 |t_1 t_2|^{(p+\beta-1)n}.$$

Since $\widehat{\phi}$ is continuous and $\widehat{\phi}(\mathbf{0}) = 1$,

$$0 \le \widehat{\phi}(\mathbf{t}) \le D_n (1 + |\mathbf{t}|)^{(p+\beta-1)n}.$$

Hence, if $(1 - p - \beta)n - 2 > r \ge 0$, [24, Appendix A.2] (or the discussion on [6, pp. 66–67]) implies that ϕ is in continuity class C^r. □

Since the mask P in (9) has symmetry and B^2 is the dyadic dilation, according to Theorem 2.3 (see also Example 2.4) of [14], a much simpler proof can be given for the estimation of the regularity in Proposition 2 by using the dilation matrix B instead of R.

Once we have the trigonometric polynomial P defined by (9) and a refinable function ϕ as in Proposition 2, we show two constructions of functions $\{\psi_1, \dots, \psi_5\}$ that are tight framelets associated to the dilation matrix R.

2.1. **First construction.** In this subsection we apply the algorithm described in [21, Theorem 1] or [22, Theorem B] to construct a family of tight framelets $\Psi = \{\psi_1, \dots, \psi_5\}$ associated to the dilation matrix R. All that remains is to find trigonometric polynomials $\widetilde{P}(A^T\mathbf{t})$ such that the identity (12) in [21] (or (4) in [22]) holds when $A = R$. For $n \in \mathbb{N}$, let u_n, v_n and w_n be trigonometric polynomials on $\mathbb{R}$ such that

$$|u_n(t)|^2 = 1 - |g_n(t)|^2, \qquad |v_n(t)|^2 = 1 - |g_n(t + \tfrac{1}{2})|^2 \tag{16}$$

and

$$|w_n(t)|^2 = 1 - |g_n(t)|^2 - |g_n(t + \tfrac{1}{2})|^2. \tag{17}$$

To see that these polynomials exist note that, for example,

$$1 - |g_n(t)|^2 - |g_n(t + \tfrac{1}{2})|^2 \ge 0,$$

and the assertion follows applying a lemma of Riesz (cf., e.g., [7, Lemma 6.1.3] or [17, Lemma 10, p. 102]). The coefficients of the polynomials u_n, v_n and w_n may be obtained by *spectral factorization* ([9]) and they can be real numbers. Note that u_n, v_n and w_n are nontrivial polynomials because [24, Lemma 4.8] implies that

$$|g_n(t)|^2 + |g_n(t + \tfrac{1}{2})|^2 < 1$$

except for a countable set of points where the equality holds.

We need:

Lemma 2. *Let $n \in \mathbb{N}$, let $P(\mathbf{t})$ be defined by* (9)*, let $u_n(t), v_n(t)$ and $w_n(t)$ be trigonometric polynomials that satisfy* (16) *and* (17) *respectively, and let the trigonometric polynomials $L_0(\mathbf{t})$ and $L_1(\mathbf{t})$ be defined by*

$$L_0(R^T\mathbf{t}) := \frac{1}{\sqrt{2}} \left(P(\mathbf{t}) + P(\mathbf{t} + \mathbf{r}_1)\right) \quad \text{and} \quad L_1(R^T\mathbf{t}) := \frac{e^{-i2\pi t_1}}{\sqrt{2}} \left(P(\mathbf{t}) - P(\mathbf{t} + \mathbf{r}_1)\right), \tag{18}$$

where $\mathbf{r}_1$ is defined in (7)*. If*

$$\widetilde{P}_1(R^T\mathbf{t}) := u_n(t_1) g_n(t_2),$$
$$\widetilde{P}_2(R^T\mathbf{t}) := v_n(t_1) g_n(t_2 + \tfrac{1}{2}),$$
$$\widetilde{P}_3(R^T\mathbf{t}) := w_n(t_2)$$

then

$$\sum_{k=0}^{1}|L_k(R^T\mathbf{t})|^2+\sum_{j=1}^{3}|\widetilde{P}_j(R^T\mathbf{t})|^2=1.$$

Proof. We have:

$$\begin{aligned}
&\sum_{k=0}^{1}|L_k(R^T\mathbf{t})|^2+\sum_{j=1}^{3}|\widetilde{P}_j(R^T\mathbf{t})|^2\\
=\;&\frac{1}{2}\left[g_n(t_1)g_n(t_2)+g_n(t_1+\tfrac{1}{2})g_n(t_2+\tfrac{1}{2})\right]^2\\
&+\frac{1}{2}\left[g_n(t_1)g_n(t_2)-g_n(t_1+\tfrac{1}{2})g_n(t_2+\tfrac{1}{2})\right]^2\\
&\quad+(1-|g_n(t_1)|^2)|g_n(t_2)|^2+(1-|g_n(t_1+\tfrac{1}{2})|^2)|g_n(t_2+\tfrac{1}{2})|^2\\
&\qquad+1-|g_n(t_2)|^2-|g_n(t_2+\tfrac{1}{2})|^2=1.
\end{aligned}$$

□

We are now ready to describe our first construction.

Theorem 1. *Let $n\in\mathbb{N}$, let $P(\mathbf{t})$ be defined by* (9)*, let*

$$\mathcal{P}(\mathbf{t}):=\left(P(\mathbf{t}),P(\mathbf{t}+\mathbf{r}_1)\right)^T,$$

and let

$$\mathcal{M}(\mathbf{t}):=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & e^{i2\pi t_1}\\ 1 & -e^{i2\pi t_1}\end{pmatrix}.$$

Let the trigonometric polynomials $L_0(\mathbf{t})$ and $L_1(\mathbf{t})$ be defined by

$$L_0(R^T\mathbf{t}):=\frac{1}{\sqrt{2}}\left(P(\mathbf{t})+P(\mathbf{t}+\mathbf{r}_1)\right)\quad\text{and}\quad L_1(R^T\mathbf{t}):=\frac{e^{-i2\pi t_1}}{\sqrt{2}}\left(P(\mathbf{t})-P(\mathbf{t}+\mathbf{r}_1)\right),$$

and let the trigonometric polynomials $\widetilde{P}_j(R^T\mathbf{t})$, $j=1,2,3$, be defined as in Lemma 2.

Let matrix function $\mathcal{G}(\mathbf{t})$ be defined by

$$\mathcal{G}(\mathbf{t}):=\Big(L_k(R^T\mathbf{t});k=0,1,\widetilde{P}_j(R^T\mathbf{t});1\le j\le 3\Big)^T,$$

and

$$\widetilde{\mathcal{Q}}(\mathbf{t}):=I_{5\times 5}-\mathcal{G}(\mathbf{t})\mathcal{G}^*(\mathbf{t}).$$

Let $K(\mathbf{t})$ denote the first 2×5 block matrix of $\widetilde{\mathcal{Q}}(\mathbf{t})$,

$$\mathcal{Q}(\mathbf{t}):=\mathcal{M}(\mathbf{t})K(\mathbf{t}),$$

and let $[Q_1(\mathbf{t}),\ldots,Q_5(\mathbf{t})]$ denote the first row of $\mathcal{Q}(\mathbf{t})$.

Let ϕ be the function defined in Proposition 2,

$$\widehat{\psi_\ell}(R^T\mathbf{t}):=Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}),\quad \ell=1,\ldots,5,\tag{19}$$

and let

$$\Psi=\{\psi_\ell(\mathbf{t})\ ;\ \ell=1,\ldots,5\}$$

be the set of inverse Fourier transforms of the functions $\widehat{\psi_\ell}(\mathbf{t})$ defined by (19)*. Then Ψ is a tight framelet in $L^2(\mathbb{R}^2)$ with dilation R and frame constant equal to 1, and the functions ψ_ℓ have compact support. In addition, Ψ has vanishing moments of order n. Moreover, let $p:=\log_2(\frac{8}{3\sqrt{3}})\approx 0.6226$. Then for any $\beta>0$ there are a* *constant C_n and an integer N, such that if $n \geq N$ then* (12) *holds. In particular, if*

$$(1 - p - \beta)n - 2 > r \geq 0$$

and $n \geq N$, the functions ψ_ℓ, $\ell = 1, \dots, 5$ are in continuity class C^r.

Proof. That Ψ is a tight framelet follows from Lemma 2, [22, Theorem B] and the Oblique Extension Principle.

Since ϕ has compact support and the functions Q_ℓ are trigonometric polynomials, it follows that the functions in Ψ are compactly supported.

The smoothness follows from Proposition 2.

We now verify that Ψ has vanishing moments of order n. For $\ell \in \{1, \cdots, 5\}$, using the identity (6) in [22] with $S(\mathbf{t}) = 1$, and the fact that $0 \leq \widehat{\phi}(\mathbf{t}) \leq 1$, we have

$$\begin{aligned}
&\lim_{(t_1,t_2)\to(0,0)} \frac{|\widehat{\psi}_\ell(t_1,t_2)|^2}{\|(t_1,t_2)\|^{2n}} \leq \lim_{(t_1,t_2)\to(0,0)} \frac{|Q_\ell(t_1,t_2)|^2}{\|(t_1,t_2)\|^{2n}} \\
\leq & \lim_{(t_1,t_2)\to(0,0)} \frac{1 - |P(t_1,t_2)|^2}{\|(t_1,t_2)\|^{2n}} \\
= & \lim_{(t_1,t_2)\to(0,0)} \frac{1 - (1 - c_n \int_0^{t_1} (\sin 2\pi\xi)^{2n+1} d\xi)^2 (1 - c_n \int_0^{t_2} (\sin 2\pi\xi)^{2n+1} d\xi)^2}{(t_1^2 + t_2^2)^n} \\
\leq & \lim_{(t_1,t_2)\to(0,0)} \frac{|c_n \int_0^{t_1} (\sin 2\pi\xi)^{2n+1} d\xi| + |c_n \int_0^{t_2} (\sin 2\pi\xi)^{2n+1} d\xi|}{(t_1^2 + t_2^2)^n} \\
& + \frac{1}{2} \lim_{(t_1,t_2)\to(0,0)} \frac{|c_n \int_0^{t_1} (\sin 2\pi\xi)^{2n+1} d\xi|^2 + |c_n \int_0^{t_2} (\sin 2\pi\xi)^{2n+1} d\xi|^2}{(t_1^2 + t_2^2)^n} = 0.
\end{aligned}$$

Hence, $\widehat{\psi}_\ell$ has a zero of order n at the origin. □

2.2. **Second construction.** In this subsection we apply the Unitary Extension Principle (OEP with $S(\mathbf{t}) = 1$) and adapt some well known constructions of univariate tight framelets with two generators associated to the dyadic dilation, to construct a family of tight framelets Ψ with five generators and associated to the dilation matrix R.

We need the following well known result. Versions can be found in Theorem 2 of [4] and Theorem 4.1 of [18]. Although it is a result on the existence of tight framelets with two generators, the proof is constructive. We state it here in a form that is more suitable to our needs.

Theorem B. *Let P be a trigonometric polynomial on $\mathbb{R}$ such that $P(0) = 1$ and $|P(t)|^2 + |P(t + \frac{1}{2})|^2 \leq 1$. Then there exists a pair of trigonometric polynomials Q_1 and Q_2 on $\mathbb{R}$ such that* (3) *holds with $S(\mathbf{t}) = 1$.*

An immediate consequence of Theorem B is the following:

Corollary 1. *There exists a pair of trigonometric polynomials q_1 and q_2 such that*

$$|g_n(t)|^2 + |q_1(t)|^2 + |q_2(t)|^2 = 1$$

and

$$g_n(t)\overline{g_n(t + \tfrac{1}{2})} + q_1(t)\overline{q_1(t + \tfrac{1}{2})} + q_2(t)\overline{q_2(t + \tfrac{1}{2})} = 0,$$

where $g_n(t)$ is defined by (5).

Our second construction is the following:

Theorem 2. *Let $n \in \mathbb{N}$, let $P(\mathbf{t})$ be defined by* (9)*, let q_1 and q_2 be trigonometric polynomials as in Corollary 1, let ϕ be the inverse Fourier transform of the function $\widehat{\phi}$ whose existence is established in Proposition 1, and let u_n be a trigonometric polynomial on $\mathbb{R}$ that satisfies* (16)*. Let*

$$\widehat{\psi_\ell}(R^T\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad \ell = 1, \dots, 5, \tag{20}$$

where

$$Q_1(\mathbf{t}) := q_1(t_1)g_n(t_2), \qquad Q_2(\mathbf{t}) := q_2(t_1)g_n(t_2), \qquad Q_3(\mathbf{t}) := g_n(t_1)u_n(t_2),$$
$$Q_4(\mathbf{t}) := q_1(t_1)u_n(t_2), \qquad Q_5(\mathbf{t}) := q_2(t_1)u_n(t_2),$$

and let

$$\Psi = \{\psi_\ell(\mathbf{t}) \ ; \ \ell = 1, \dots, 5\}$$

be the set of inverse Fourier transforms of the functions $\widehat{\psi_\ell}(\mathbf{t})$ defined by (20)*. Then Ψ is a tight framelet in $L^2(\mathbb{R}^2)$ with dilation R and frame constant equal to 1, and the functions ψ_ℓ have compact support. In addition, Ψ has vanishing moments of order n. Moreover, if $p := \log_2(\frac{8}{3\sqrt{3}})$ then for any $\beta > 0$ there are a constant C_n and an integer N such that if $n \geq N$ then* (12) *holds. In particular, if*

$$(1 - p - \beta)n - 2 > r \geq 0$$

and $n \geq N$, the functions ψ_ℓ, $\ell = 1, \dots, 5$ are in continuity class C^r.

Proof. By direct computation, it is easy to prove that P and Q_ℓ, $\ell = 1, \dots, 5$, satisfy the Oblique Extension Principle with $S(\mathbf{t}) = 1$. The remaining assertions follow as in the proof of Theorem 1. □

3. Tight framelets with three generators

In this section we will construct tight wavelet frames associated to the dilation matrix R defined in (4) with three smooth compactly supported generators and an even refinable functions. Since the arguments in this section are similar to those used in the previous section, some details will be skipped. We note that the tight wavelet frames associated to R constructed in this section are also tight wavelet frames associated to B. This fact holds because the two lattices $R\mathbb{Z}^d$ and $B\mathbb{Z}^d$ coincide and by the observation in pag. 215 of [14]. However, the regularity of tight wavelet frames associated to B will be different to the regularity for using R, due to the lack of symmetry of the considered mask P.

The following proposition is similar to Proposition 1. The proof is obtained proceeding as in the proof [3, Theorem 1] (see also [13, Lemma 2.1]), and will be omitted.

Proposition 3. *Let $n \in \mathbb{N}$, let $g_n(t)$ be defined by (5), and let P be the trigonometric polynomial in $\mathbb{R}^2$ defined by*

$$P(\mathbf{t}) := g_n(2t_1)g_n(t_1). \tag{21}$$

Then the infinite product

$$\prod_{j=1}^{\infty} P((R^T)^{-j}\mathbf{t})$$

converges to a nonnegative continuous function $\widehat{\phi}$ *in* $L^2(\mathbb{R}^2)$ *such that* $\|\widehat{\phi}\|_{L^2(\mathbb{R}^2)} \leq 1$, $\widehat{\phi}(\mathbf{0}) = 1$, *and satisfies the refinement equation*

$$\widehat{\phi}(R^T\mathbf{t}) = P(\mathbf{t})\widehat{\phi}(\mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}^2.$$

We also have

Proposition 4. *Let* $n \in \mathbb{N}$ *and let the function* $\widehat{\phi} \in L^2(\mathbb{R}^2)$ *be defined as in Proposition 3. Then the inverse Fourier transform* ϕ *of* $\widehat{\phi}$ *is not identically zero and is even, real valued and compactly supported, and* $\|\phi\|_{L_2(\mathbb{R}^2)} \leq 1$. *Moreover, if* $p := \log_2(\frac{8}{3\sqrt{3}})$ *then for any* $\beta > 0$ *there are a constant* C_n *and an integer* N *such that if* $n \geq N$ *then* (12) *holds. In particular, if*

$$(1 - p - \beta)n - 2 > r \geq 0$$

and $n \geq N$, *then* ϕ *is in continuity class* C^r.

Proof. As in Proposition 2, we only need to prove the estimates for the degree of smoothness of ϕ. By analogous computations as in Lemma 1 and bearing in mind that g_n is even and $0 \leq g_n(t) \leq 1$, we have

$$\begin{aligned}
\widehat{\phi}(\mathbf{t}) &= \prod_{j=1}^{\infty} P((R^T)^{-j}\mathbf{t}) \\
&= g_n(t_1 - t_2)g_n(t_2) \prod_{j=1}^{\infty} g_n(2^{-j}(t_1 - t_2))g_n(2^{-j}(t_1 + t_2))g_n(2^{-j}t_1)g_n(2^{-j}t_2) \\
&\leq \prod_{j=1}^{\infty} g_n(2^{-j}t_1)g_n(2^{-j}t_2),
\end{aligned}$$

and the assertion follows by proceeding as in the proof of Proposition 2. $\square$

For $n \in \mathbb{N}$, let h_n be trigonometric polynomials on $\mathbb{R}$ such that

$$|h_n(t)|^2 = 1 - g_n^2(2t)(g_n^2(t) + g_n^2(t + \tfrac{1}{2})). \tag{22}$$

We are now ready to state the main result of this section.

Theorem 3. *Let* $n \in \mathbb{N}$, *let* $P(\mathbf{t})$ *and* $h(t)$ *be defined by* (21) *and* (22) *respectively, and let the trigonometric polynomial* $\widetilde{P}_1(\mathbf{t})$ *in* $\mathbb{R}^2$ *be defined by* $\widetilde{P}_1(R^T\mathbf{t}) := h_n(t_1)$. *Let the trigonometric polynomials* $Q_1(\mathbf{t})$, $Q_2(\mathbf{t})$, *and* $Q_3(\mathbf{t})$ *be defined by*

$$Q_1(\mathbf{t}) := \frac{1}{\sqrt{2}}\left[1 - g_n^2(2t_1)g_n(t_1)\right],$$

$$Q_2(\mathbf{t}) := \frac{e^{i2\pi t_1}}{\sqrt{2}}\left[1 - g_n^2(2t_1)g_n(t_1)\left(g_n(t_1) - g_n(t_1 + \tfrac{1}{2})\right)\right],$$

and

$$Q_3(\mathbf{t}) := -\overline{\widetilde{P}_1(R^T\mathbf{t})}g_n(2t_1)g_n(t_1).$$

Let ϕ *be the function defined in Proposition 4,*

$$\widehat{\psi_\ell}(R^T\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad \ell = 1, 2, 3, \tag{23}$$

and let

$$\Psi = \{\psi_\ell(\mathbf{t})\ ;\ \ell = 1, 2, 3\}$$

be the set of inverse Fourier transforms of the functions $\widehat{\psi_\ell}(\mathbf{t})$ *defined by* (23). *Then* Ψ *is a tight framelet in* $L^2(\mathbb{R}^2)$ *with dilation* R *and frame constant equal to 1, and the functions* ψ_ℓ *have compact support. In addition,* Ψ *has vanishing moments of order* n. *Moreover, if* $p := \log_2(\frac{8}{3\sqrt{3}})$ *then for any* $\beta > 0$ *there are a constant* C_n *and an integer* N, *such that if* $n \geq N$ *then* (12) *holds. In particular, if*

$$(1 - p - \beta)n - 2 > r \geq 0$$

and $n \geq N$, *the functions* ψ_ℓ, $\ell = 1, 2, 3$, *are in continuity class* C^r.

Proof. Referring to (7) and noting that, since

$$L_0(R^T\mathbf{t}) = \frac{1}{\sqrt{2}}\left(P(\mathbf{t}) + P(\mathbf{t} + \mathbf{r}_1)\right) \quad \text{and} \quad L_1(R^T\mathbf{t}) = \frac{e^{-i2\pi t_1}}{\sqrt{2}}\left(P(\mathbf{t}) - P(\mathbf{t} + \mathbf{r}_1)\right)$$

are the corresponding trigonometric polynomials in [21, Theorem 1] or [22, Theorem B], we see that

$$\begin{aligned} & |L_0(R^T\mathbf{t})|^2 + |L_1(R^T\mathbf{t})|^2 + |\widetilde{P}_1(R^T\mathbf{t})|^2 \\ = & \frac{1}{2}\left(g_n(2t_1)g_n(t_1) + g_n(2t_1)g_n(t_1 + \frac{1}{2}))\right)^2 \\ & + \frac{1}{2}\left(g_n(2t_1)g_n(t_1) - g_n(2t_1)g_n(t_1 + \frac{1}{2}))\right)^2 \\ + & \quad 1 - g_n^2(2t_1)(g_n^2(t_1) + g_n^2(t_1 + \tfrac{1}{2})) = 1. \end{aligned}$$

This allows us to apply the algorithm described in [21, Theorem 1] or [22, Theorem B] to obtain

$$\begin{aligned} Q_1(\mathbf{t}) &= \frac{1}{\sqrt{2}}[1 - |L_0(R_k^T\mathbf{t})|^2 - e^{i2\pi(2b-1)t_1}L_1(R_k^T\mathbf{t})L_0(R_k^T\mathbf{t})] \\ &= \frac{1}{\sqrt{2}}\left[1 - g_n^2(2t_1)g_n(t_1)\right], \end{aligned}$$

$$\begin{aligned} Q_2(\mathbf{t}) &= \frac{1}{\sqrt{2}}[e^{i2\pi t_1} - L_0(R_k^T\mathbf{t})\overline{L_1(R_k^T\mathbf{t})} - e^{i2\pi t_1}|L_1(R_k^T\mathbf{t})|^2] \\ &= \frac{e^{i2\pi t_1}}{\sqrt{2}}\left[1 - g_n^2(2t_1)g_n(t_1)\left(g_n(t_1) - g_n(t_1 + \tfrac{1}{2})\right)\right], \end{aligned}$$

and

$$Q_3(\mathbf{t}) = -\frac{1}{\sqrt{2}}[L_0(R_k^T\mathbf{t}) + e^{i2\pi t_1}L_1(R_k^T\mathbf{t})]\ \overline{\widetilde{P}_1(R^T\mathbf{t})} = -\overline{\widetilde{P}_1(R^T\mathbf{t})}g_n(2t_1)g_n(t_1).$$

Thus, that Ψ is a tight framelet with frame constant 1 follows from [22, Theorem B] and the Oblique Extension Principle.

Since ϕ has compactly support and the functions Q_ℓ are trigonometric polynomials, it follows that the functions in Ψ are compactly supported.

The smoothness follows from Proposition 4.

Finally, that Ψ has vanishing moments of order n can be verified as in the proof of Theorem 1.

□

4. Tight wavelet frames with two generators

With the same refinable function ϕ as in Section 3, we now construct tight framelets associated to the dilation matrix R, having only two smooth compactly supported generators.

4.1. First construction.

Our first construction in this section is the following.

Theorem 4. *Let $n \in \mathbb{N}$, let $P(\mathbf{t})$ and $h_n(t)$ be defined by (21) and (22) respectively, and let the trigonometric polynomial on $\mathbb{R}^2$ be defined by $H(\mathbf{t}) := h_n(t_1)$. Let the trigonometric polynomials $Q_1(\mathbf{t})$ and $Q_2(\mathbf{t})$ be defined by*

$$Q_1(\mathbf{t}) := e^{2\pi i t_1} g_n(2t_1) g_n(t_1 + \tfrac{1}{2})$$

and

$$Q_2(\mathbf{t}) := g_n(2t_1) g_n(t_1) H(R^T \mathbf{t})$$

Let ϕ be the function defined in Proposition 4,

$$\widehat{\psi_\ell}(A^T \mathbf{t}) := Q_\ell(\mathbf{t}) \widehat{\phi}(\mathbf{t}), \quad \ell = 1, 2, \tag{24}$$

and let

$$\Psi = \{\psi_\ell(\mathbf{t}) \ ; \ \ell = 1, 2\}$$

be the set of inverse Fourier transforms of the functions $\widehat{\psi_\ell}(\mathbf{t})$ defined by (24). Then Ψ is a tight framelet in $L^2(\mathbb{R}^2)$ with dilation R and frame constant equal to 1, and the functions ψ_ℓ have compact support. In addition, Ψ has vanishing moments of order n. Moreover, if $p := \log_2(\frac{8}{3\sqrt{3}})$ then for any $\beta > 0$ there are a constant C_n and an integer N, such that if $n \geq N$ then (12) holds. In particular, if

$$(1 - p - \beta)n - 2 > r \geq 0$$

and $n \geq N$, the functions ψ_1 and ψ_2 are in continuity class C^r.

Proof. Let $S(\mathbf{t}) := 1 - |H(\mathbf{t})|^2$. Then

$$\begin{aligned}
& S(R^T \mathbf{t}) |P(\mathbf{t})|^2 + \sum_{\ell=1}^{2} |Q_\ell(\mathbf{t})|^2 \\
&= S(R^T \mathbf{t}) g_n^2(2t_1) g_n^2(t_1) + g_n^2(2t_1) g_n^2(t_1 + \tfrac{1}{2}) + g_n^2(2t_1) g_n^2(t_1)(1 - S(R^T \mathbf{t})) = S(\mathbf{t}).
\end{aligned}$$

Moreover, bearing in mind that $R^T \mathbf{r}_1 \in \mathbb{Z}^2$, we have

$$\begin{aligned}
& S(R^T \mathbf{t}) P(\mathbf{t}) \overline{P(\mathbf{t} + \mathbf{r}_1)} + \sum_{\ell=1}^{2} Q_\ell(\mathbf{t}) \overline{Q_\ell(\mathbf{t} + \mathbf{r}_1)} \\
= \ & S(R^T \mathbf{t}) g_n^2(2t_1) g_n(t_1) g_n(t_1 + \tfrac{1}{2}) - g_n^2(2t_1) g_n(t_1 + \tfrac{1}{2}) g_n(t_1) \\
& + g_n^2(2t_1) g_n(t_1) g_n(t_1 + \tfrac{1}{2}) H(R^T \mathbf{t}) \overline{H(R^T(\mathbf{t} + \mathbf{r}_1))} = 0
\end{aligned}$$

and applying the Oblique Extension Principle we conclude that $\Psi_{n,m}$ is a tight framelet.

The remaining assertions follow as in the proof of Theorem 2. $\square$

4.2. **Second construction.** We need the following corollary of Theorem B.

Corollary 2. *For $n \in \mathbb{N}$, there exists a pair of trigonometric polynomials q_1 and q_2 on $\mathbb{R}$ such that*

$$|g_n(2t)g_n(t)|^2 + |q_1(t)|^2 + |q_2(t)|^2 = 1$$

and

$$g_n(2t)g_n(t)\overline{g_n(2t)g_n(t+\tfrac{1}{2})} + q_1(t)\overline{q_1(t+\tfrac{1}{2})} + q_2(t)\overline{q_2(t+\tfrac{1}{2})} = 0,$$

where $g_n(t)$ is defined by (5).

We have:

Theorem 5. *Let $n \in \mathbb{N}$, let $P(\mathbf{t})$ and $h_n(t)$ be defined by* (21) *and* (22) *respectively. Let q_1 and q_2 be trigonometric polynomials on $\mathbb{R}$ as in Corollary 2. Let the trigonometric polynomials $Q_1(\mathbf{t})$ and $Q_2(\mathbf{t})$ be defined by*

$$Q_1(\mathbf{t}) := q_1(t_1) \qquad \textit{and} \qquad Q_2(\mathbf{t}) := q_2(t_1)$$

Let ϕ be the function defined in Proposition 4,

$$\widehat{\psi_\ell}(R^T\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad \ell = 1, 2, \tag{25}$$

and let

$$\Psi = \{\psi_\ell(\mathbf{t}) \ ; \ \ell = 1, 2\}$$

be the set of inverse Fourier transforms of the functions $\widehat{\psi_\ell}(\mathbf{t})$ defined by (24). *Then Ψ is a tight framelet in $L^2(\mathbb{R}^2)$ with dilation R and frame constant equal to 1, and the functions ψ_ℓ have compact support. In addition, Ψ has vanishing moments of order n. Moreover, if $p := \log_2(\frac{8}{3\sqrt{3}})$ then for any $\beta > 0$ there are a constant C_n and an integer N, such that if $n \geq N$ then* (12) *holds. In particular, if*

$$(1 - p - \beta)n - 2 > r \geq 0$$

and $n \geq N$, the functions ψ_1 and ψ_2 are in continuity class C^r.

Proof. By Corollary 2, we readily see that the polynomials P, Q_1 and Q_2 satisfy the Oblique Extension Principle with $S(\mathbf{t}) = 1$. The remaining assertions follow as in the proof of Theorem 1. □

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