TWO FAMILIES OF COMPACTLY SUPPORTED PARSEVAL FRAMELETS IN $L^2(\mathbb{R}^d)$

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ABSTRACT. For any dilation matrix with integral entries $A \in \mathbb{R}^{d\times d}$, $d \geq 1$, we construct two families of Parseval wavelet frames in $L^2(\mathbb{R}^d)$. Both families have compact support and any desired number of vanishing moments. The first family has $|\det A|$ generators. The second family has, in addition, any desired degree of regularity. In this case, the number of generators depends on the dilation matrix A and the dimension d, but never exceeds $|\det A| + d$. Our construction involves trigonometric polynomials developed by Heller to obtain refinable functions, the Oblique Extension Principle, and a slight generalization of a theorem of Lai and Stöckler.

1. INTRODUCTION

We are interested in methods for constructing compactly supported wavelet frames with good properties of approximation. Tight wavelet frames and orthonormal wavelets are used, for instance, in image processing and data compression. Indeed, tight framelets have recently become the focus of increased interest because they can be computed and applied just as easily as orthonormal wavelets, but they are easier to construct.

In dimension 1 and with dyadic dilations, compactly supported orthonormal wavelets with any number of vanishing moments and any degree of regularity were constructed in [5] (see also [6]).

For the multivariate case with a dilation matrix $A \in \mathbb{R}^{d\times d}$, $d \geq 1$, with integer entries, the problem of constructing compactly supported wavelet frames where the number of generators does not depend on degree of regularity nor on the number of vanishing moments in $L^2(\mathbb{R}^d)$ is of a different nature.

Although there is a rich literature that discusses different constructions of compactly supported wavelet frames, we only mention a few results that are closely related to our constructions. Ron and Shen [25] (see also [26]) focus on constructions of compactly supported tight famelets in dimension 2, and they also describe an algorithm for constructing compactly supported tight affine frames in any dimension d and with any dilation matrix. Gröchenig and Ron [11] construct compactly supported framelets with any desired degree of smoothness in $L^2(\mathbb{R}^d)$ and associated to any dilation matrix. In these constructions, the number of generators increase with the degree of smoothness. As far as we know, the paper by Han [16] is the only one that provides a constructive proof of the existence of tight wavelet frames associated to a general dilation matrix and such that their generators are

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compactly supported, with any given degree of regularity, any fixed number of vanishing moments, and a fixed number of generators. In particular this number of generators is bounded by $(3/2)^d|\det A|$. The proposed method by Skopina [32] (see also [33] and [34]) for the construction of compactly supported Parseval wavelet frames with high approximation order is given by a simple and explicit algorithm. The number of generators does not exceed $(d+1)|\det A|-d$. Moreover, the number of generators can be reduced to $|\det A|$ for a large class of dilation matrices A. In Krivoshein [19], wavelet frame systems providing any desired approximation order are constructed for any matrix dilation. For some particular dilation matrices there are other types of constructions, see e.g. [2, 4, 20, 28, 29, 30, 31].

An extensive study of multivariate wavelet frames can be found in the books by Han [12] and Krivoshein, Protasov and Skopina [18].

In this paper, for any dilation matrix A we construct two families of Parseval wavelet frames in $L^2(\mathbb{R}^d)$ associated to A, with compact support and any desired number of vanishing moments. The first family has $|\det A|$ generators. For the second family the number of generators depends on the dilation matrix and on the dimension d, but never exceeds $|\det A|+d$; it has the additional property of having any given degree of regularity. Our starting point is the paper by Han [16]. In order to construct a refinable function we will need the trigonometric polynomials developed by Heller [17] associated to a dilation factor and we will use the Smith normal form of A. After constructing the refinable function we will use a slight generalization of a theorem of Lai and Stöckler [21] and the Unitary Extension Principle to obtain our familes of Parseval wavelet frames. Our construction is made on the Fourier transform side.

We now introduce the notation and definitions that we shall use in what follows. The sets of strictly positive integers, integers, rational numbers and real numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively. For $0 \le k \le n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the usual binomial coefficient. Given a set $S \subset \mathbb{R}^d$, χ_S will denote its characteristic function.

Given a real–valued matrix A, its transpose will be denoted by A^*. The identity matrix will be denoted by $\mathbf{I}$.

We say that $A \in \mathbb{R}^{d\times d}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$ if all eigenvalues of A have modulus greater than 1 and $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$. The set of all $d \times d$ dilation matrices preserving the lattice $\mathbb{Z}^d$ will be denoted by $\mathbf{E}_d(\mathbb{Z})$. In what follows, A will denote a fixed element of $\mathbf{E}_d(\mathbb{Z})$,and $D := |\det A|$. With some abuse in the notation, we will use the same letter A for a linear map $\mathbb{R}^d \to \mathbb{R}^d$ and for its associated matrix with respect to the canonical basis. Note that if $A \in \mathbf{E}_d(\mathbb{Z})$ then $|\det A|$ is an integer greater than 1, and the quotient groups $\mathbb{Z}^d/A\mathbb{Z}^d$ and $A^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ are well defined. By $\mathbf{\Delta}_A \subset \mathbb{Z}^d$ and $\mathbf{\Gamma}_A$ we will denote a full collection of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$ and $(A^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ respectively. From [10, Lemma 2] we know that $\mathbb{Z}^d/A\mathbb{Z}^d$ has exactly $|\det A|$ cosets, which readily implies that also $A^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ has exactly $|\det A|$ cosets. For a real number r, $r\mathbb{Z}^d := \{(rk_1, \dots, rk_d)\,;\, k_1, \dots, k_d \in \mathbb{Z}\}$.

A sequence $\{\phi_n\}_{n=1}^\infty$ of elements in a separable Hilbert space $\mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|h\|^2 \le \sum_{n=1}^{\infty} |\langle h, \phi_n\rangle|^2 \le C_2\|h\|^2, \qquad \forall h \in \mathbb{H},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{H}$. The constants C_1 and C_2 are called *frame bounds*. The definition implies that a frame is a complete sequence of elements of $\mathbb{H}$. A frame $\{\phi_n\}_{n=1}^{\infty}$ is *tight* if we may choose $C_1 = C_2$.

Let A be any dilation matrix in $\mathbf{E}_d(\mathbb{Z})$. A set of functions $\Psi = \{\psi_1, \ldots, \psi_N\} \subset L^2(\mathbb{R}^d)$ is called a *wavelet frame* or *framelet* associated to the dilation A, if the system

$$\{\psi_{\ell,j,\mathbf{k}}(\mathbf{x}); j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le N\},$$

where $\psi_{\ell,j,\mathbf{k}}(\mathbf{x}) := |\det A|^{j/2}\psi_\ell(A^j\mathbf{x} + \mathbf{k})$, is a frame for $L^2(\mathbb{R}^d)$. If this system is a tight frame then it is called a tight wavelet frame or tight framelet. If the functions ψ_ℓ, $\ell = 1, \ldots N$ are linearly independent they are called the *generators* of the frame. If the frame constant is equal to 1 it will be called a Parseval wavelet frame or a Parseval framelet in $L^2(\mathbb{R}^d)$. Thus we have:

$$\sum_{\ell=1}^{N}\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^d} |\langle f, \psi_{\ell,j,\mathbf{k}}\rangle|^2 = ||f||^2 \qquad \forall f \in L^2(\mathbb{R}^d).$$

A wavelet frame $\Psi = \{\psi_1, \ldots, \psi_N\} \subset L^2(\mathbb{R}^d)$ has vanishing moments of order $m \in \{0, 1, \cdots\}$, if $\widehat{\psi}_\ell$, $\ell = 1, \cdots, N$ has a zero of order m at the origin.

Let $\widehat{f}$ denote the Fourier transform of the function f. Thus, if $f \in L^1(\mathbb{R}^d)$ and $\mathbf{x}, \mathbf{t} \in \mathbb{R}^d$, then

$$\widehat{f}(\mathbf{t}) := \int_{\mathbb{R}^d} f(\mathbf{x})e^{-2\pi i\mathbf{x}\cdot\mathbf{t}}d\mathbf{x},$$

where $\mathbf{x}\cdot\mathbf{t}$ denotes the dot product of vectors $\mathbf{x}$ and $\mathbf{t}$. The Fourier transform is extended to $L^2(\mathbb{R}^d)$ in the usual way.

The remainder of this paper is organized as follows: in Section 2, we will construct our first family of Parseval wavelet frames while the second family will be described in Section 3.

2. First Construction

Let $A \in \mathbb{R}^{d\times d}$, $d \ge 1$, be a dilation matrix with integer entries. In this section we construct a family of compactly supported Parseval framelets in $L^2(\mathbb{R}^d)$, associated to A, with any desired number of vanishing moments and $|\det A|$ generators.

Since A is invertible and all the entries of A are integers, there exist two $d \times d$ integer matrices U and V with $|\det U| = |\det V| = 1$, and a $d \times d$ diagonal matrix $S = \operatorname{diag}(s_1, s_2, s_3, \ldots, s_r, \ldots, s_d)$, $s_i \in \mathbb{N}$, such that

$$A = USV. \tag{1}$$

Moreover, there is an integer $r \le d$ such that $s_i > 1$ when $1 \le i \le r$ and $s_i = 1$ when $r < i$. In the case $r = d$, all s_i will be larger than 1, and if $i \le r$, then $s_{i-1}|s_i$ (i.e. s_i is divisible by s_{i-1}), This is the Smith normal form of A (see e.g. [23]). This number r will play an important role in this paper because it will appear in the construction of refinable functions and in the number of generators of the family of Parseval wavelet frame we will construct in the next section.

Now we focus on the structure of the quotient group $(A^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d$. We consider $\mathbf{\Gamma}$, the set of d-tuples defined by

$$\mathbf{\Gamma} = \{\frac{0}{s_1}, \frac{1}{s_1}, \ldots, \frac{s_1 - 1}{s_1}\} \times \cdots \times \{\frac{0}{s_r}, \frac{1}{s_r}, \ldots, \frac{s_r - 1}{s_r}\} \times \{0\} \times \cdots \times \{0\}.$$

Since $(A^*)^{-1}\mathbb{Z}^d = (U^*)^{-1}S^{-1}(V^*)^{-1}\mathbb{Z}^d = (U^*)^{-1}S^{-1}\mathbb{Z}^d$, it is not hard to see that

$$\{(U^*)^{-1}\gamma : \gamma \in \Gamma\}$$

is a full collection of representatives of the cosets of $(A^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d$. We define $\mathbf{\Gamma}_A$ by

$$\mathbf{\Gamma}_A := \{(U^*)^{-1}\gamma : \gamma \in \Gamma\},$$

and, when appropriate, we will also write

$$\mathbf{\Gamma}_A := \{\mathbf{r}_a\}_{a=0}^{|\det A|-1}$$

with $\mathbf{r}_0 = \mathbf{0}$. Let $\mathbf{\Delta}_A := \{\mathbf{q}_a\}_{a=0}^{|\det A|-1}$ be a full collection of representatives of the cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$ with $\mathbf{q}_0 = \mathbf{0}$.

To construct a refinable function we need the following trigonometric polynomials on $\mathbb{R}$ introduced by Heller [17]. Let s and m be two integers larger than one and let

$$p_{s,m}(t) := \left(\frac{1}{m^2}\left|1 + e^{2\pi it} + \cdots + e^{2\pi i(m-1)t}\right|^2\right)\left(\sum_{j=0}^{m-1} q_j(1-\cos 2\pi t)^j\right) \tag{2}$$

where q_j are defined as follows:

If s is even, take $b = s/2$ and

$$\begin{aligned} q_j :&= \sum_{k_1+k_2+\cdots+k_b=j} \{\prod_{c=1}^{b-1}(\binom{2m+k_c-1}{2m-1})(1-\cos(\frac{2\pi c}{s}))^{-k_c}\} \\ &\times \binom{m+k_b-1}{m-1}(1-\cos\pi)^{-k_b}, \end{aligned} \tag{3}$$

where k_i, $i = 1, \dots, b$, are non negative integers.

If s is odd, take $b = (s-1)/2$ and

$$q_j := \sum_{k_1+k_2+\cdots+k_b=j} \{\prod_{c=1}^{b}(\binom{2m+k_c-1}{2m-1})(1-\cos(\frac{2\pi c}{s}))^{-k_c}\}, \tag{4}$$

where k_i, $i = 1, \dots, b$, are non negative integers.

The following is a version of Lemma 3.1 in [17].

Lemma A. *Let s and m be two integers larger than one. Then the univariate trigonometric polynomials $p_{s,m}$ defined by (2) are non negative and satisfy the following conditions:*

(i)
$$\sum_{j=0}^{s-1} p_{s,m}(t + \frac{j}{s}) = 1;$$

(ii)
$$\lim_{t\to 0} \frac{1 - p_{s,m}(t)}{|t|^{2m}} = 0.$$

Since these are nonnegative trigonometric polynomials of a single variable, from a lemma of Riesz (cf., e.g., [6, Lemma 6.1.3] or [35, Lemma 4.6]) we know that there are non null trigonometric polynomials $h_{s,m}$ on $\mathbb{R}$ such that

$$|h_{s,m}(t)|^2 = p_{s,m}(t) \tag{5}$$

The coefficients of the polynomials $h_{s,m}(t)$ may be obtained by *spectral factorization* ([9]).

Now, let $s_1, \dots, s_r$ be the terms larger than 1 in the main diagonal of the matrix S defined by (1) and let $m_1, \dots, m_r$ be integers larger than one. Let

$$H(\mathbf{t}) := Q(U^*\mathbf{t}), \tag{6}$$

where

$$Q(\mathbf{t}) = Q(t_1, \dots, t_d) := \prod_{j=1}^{r} h_{s_j,m_j}(t_j).$$

We have the following.

Lemma 1. *The trigonometric polynomial H defined in* (6) *satisfies*

(*i*)

$$\sum_{\mathbf{r}\in\mathbf{\Gamma}_A} |H(\mathbf{t}+\mathbf{r})|^2 = 1$$

and

(*ii*)

$$\lim_{\|\mathbf{t}\|\to 0} \frac{1-|H(\mathbf{t})|^2}{\|\mathbf{t}\|^{2m_0}} = 0,$$

where $m_0 = \min\{m_1, \dots, m_r\}$.

Proof. We first verify (*i*). We have

$$\begin{aligned}
\sum_{\mathbf{r}\in\mathbf{\Gamma}_A} |H((U^*)^{-1}\mathbf{t}+\mathbf{r})|^2 &= \sum_{\gamma\in\mathbf{\Gamma}} |H((U^*)^{-1}\mathbf{t}+(U^*)^{-1}\gamma)|^2 = \sum_{\gamma\in\mathbf{\Gamma}} |Q(\mathbf{t}+\gamma)|^2 \\
&= \sum_{\ell_1=0}^{s_1-1}\cdots\sum_{\ell_r=0}^{s_r-1}\left(\prod_{j=1}^{r} |h_{s_j,m_j}(t_j+\frac{\ell_j}{s_j})|^2\right) \\
&= \prod_{j=1}^{r}\left(\sum_{\ell=0}^{s_j-1} |h_{s_j,m_j}(t_j+\frac{\ell}{s_j})|^2\right) = 1,
\end{aligned}$$

where the last equality follows from (*i*) in Lemma A.

Let us now verify (*ii*). Using (*i*), we have

$$\begin{aligned}
&\lim_{\|\mathbf{t}\|\to 0} \frac{1-|H((U^*)^{-1}\mathbf{t})|^2}{\|(U^*)^{-1}\mathbf{t}\|^{2m_0}} = \lim_{\|\mathbf{t}\|\to 0} \frac{\sum_{\mathbf{r}\in\mathbf{\Gamma}_A\setminus\{\mathbf{0}\}} |H((U^*)^{-1}\mathbf{t}+\mathbf{r})|^2}{\|(U^*)^{-1}\mathbf{t}\|^{2m_0}} \\
&= \sum_{\gamma\in\mathbf{\Gamma}\setminus\{\mathbf{0}\}} \lim_{\|\mathbf{t}\|\to 0} \frac{|Q(\mathbf{t}+\gamma)|^2}{\|(U^*)^{-1}\mathbf{t}\|^{2m_0}} \\
&= \sum_{(\ell_1,\cdots,\ell_r)\in\mathbf{\Gamma}\setminus\{\mathbf{0}\}} \lim_{\|\mathbf{t}\|\to 0} \frac{\left(\prod_{j=1}^{r} |h_{s_j,m_j}(t_j+\frac{\ell_j}{s_j})|^2\right)}{\|\mathbf{t}\|^{2m_0}} \frac{\|\mathbf{t}\|^{2m_0}}{\|(U^*)^{-1}\mathbf{t}\|^{2m_0}}.
\end{aligned}$$

Since $(U^*)^{-1}$ is a continuous linear operator such that $\det(U^*)^{-1} \neq 0$ and the unit sphere in $\mathbb{R}^d$ is a compact set, there is a positive constant C such that

$$\max_{\mathbf{t}\in\mathbb{R}^d\setminus\{\mathbf{0}\}} \frac{\|\mathbf{t}\|^{2m_0}}{\|(U^*)^{-1}\mathbf{t}\|^{2m_0}} = \max_{\mathbf{y}\in\{\mathbb{R}^d:\|\mathbf{y}\|=1\}} \frac{1}{\|(U^*)^{-1}\mathbf{y}\|^{2m_0}} \leq C^{2m_0}.$$

On the other hand, given $(\ell_1, \cdots, \ell_r) \in \mathbf{\Gamma} \setminus \{\mathbf{0}\}$, there is $j_0 \in \{1, 2, \ldots, r\}$ such that $\ell_{j_0} \neq 0$. Then, from (i) in Lemma A

$$\begin{aligned} & \lim_{\|\mathbf{t}\| \to 0} \frac{\prod_{j=1}^r |h_{s_j,m_j}(t_j + \frac{\ell_j}{s_j})|^2}{\|\mathbf{t}\|^{2m_0}} \\ \leq & \lim_{\|\mathbf{t}\| \to 0} \frac{|h_{s_{j_0},m_{j_0}}(t_{j_0} + \frac{\ell_{j_0}}{s_{j_0}})|^2}{\|\mathbf{t}\|^{2m_0}} \leq \lim_{\|\mathbf{t}\| \to 0} \frac{1 - |h_{s_{j_0},m_{j_0}}(t_{j_0})|^2}{\|\mathbf{t}\|^{2m_{j_0}}} \frac{\|\mathbf{t}\|^{2m_{j_0}}}{\|\mathbf{t}\|^{2m_0}} \\ \leq & \lim_{\|\mathbf{t}\| \to 0} \frac{1 - |h_{s_{j_0},m_{j_0}}(t_{j_0})|^2}{|t_{j_0}|^{2m_{j_0}}} \frac{\|\mathbf{t}\|^{2m_{j_0}}}{\|\mathbf{t}\|^{2m_0}} = 0, \end{aligned}$$

where the last equality follows from (ii) in Lemma A. Combining these estimates we have

$$\lim_{\|\mathbf{t}\| \to 0} \frac{1 - |H((U^*)^{-1}\mathbf{t})|^2}{\|(U^*)^{-1}\mathbf{t}\|^{2m_0}} = 0$$

and the assertion follows. □

We need the following refinable function.

Proposition 1. *Let H be the trigonometric polynomial in $\mathbb{R}^d$ defined by* (6). *Then the infinite product*

$$\prod_{j=1}^{\infty} H((A^*)^{-j}\mathbf{t}) \tag{7}$$

converges to a non null continuous function $\widehat{\phi}$ in $L^2(\mathbb{R}^d)$ that satisfies the refinement equation

$$\widehat{\phi}(A^*\mathbf{t}) = H(\mathbf{t})\widehat{\phi}(\mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}^d \tag{8}$$

and such that $\|\widehat{\phi}\|_{L^2(\mathbb{R}^d)} \leq 1$ and $\widehat{\phi}(\mathbf{0}) = 1$. Moreover, there exist two positive constants C and ν such that

$$|\widehat{\phi}(\mathbf{t})| \leq C(1 + \|\mathbf{t}\|)^{-\nu}. \tag{9}$$

In addition, the function $\phi \in L^2(\mathbb{R}^d)$, whose Fourier transform is $\widehat{\phi}$, is non null, compactly supported and $\|\phi\|_{L_2(\mathbb{R}^d)} \leq 1$.

Proof. By Lemma 1 and [16, Lemma 2.1], the infinite product (7) converges to a non null continuous function $\widehat{\phi}$ in $L^2(\mathbb{R}^d)$ that satisfies the refinement equation (8) and such that $\|\widehat{\phi}\|_{L^2(\mathbb{R}^d)} \leq 1$ and $\widehat{\phi}(\mathbf{0}) = 1$. .

Since $\widehat{\phi}$ is in $L^2(\mathbb{R}^d)$ and is non null, it follows that also ϕ is in $L^2(\mathbb{R}^d)$ and is non null. Moreover, $\|\phi\|_{L^2(\mathbb{R}^d)} = \|\widehat{\phi}\|_{L^2(\mathbb{R}^d)} \leq 1$.

Replicating an argument of Wojtaszczyk [35, p. 79] it is easy to see that ϕ is compactly supported on $\mathbb{R}^d$.

The existence of the constants C and ν in (9) follow by Theorem 2.2 of [16]. □

Using ideas developed in the proof of [21, Theorem 3] and using the refinable functions of Proposition 1, we construct the following compactly supported Parseval wavelet frames.

Theorem 1. *Let $D := |\det A|$, let H be the trigonometric polynomial in $\mathbb{R}^d$ defined by (6), and let ϕ be the associated function defined in Proposition 1. For $\ell \in \{0, \ldots, D-1\}$ define*

$$\widehat{\psi_\ell}(A^*\mathbf{t}) := D^{-1/2}\left[e^{2\pi i \mathbf{q}_\ell\cdot\mathbf{t}} - H(\mathbf{t})\left(\sum_{a=0}^{D-1} e^{2\pi i \mathbf{q}_\ell\cdot(\mathbf{t}+\mathbf{r}_a)}\overline{H(\mathbf{t}+\mathbf{r}_a)}\right)\right]\widehat{\phi}(\mathbf{t})$$

and let

$$\Psi = \{\psi_\ell(\mathbf{t})\ ;\ \ell = 0, \ldots, D-1\} \tag{10}$$

be the set of inverse Fourier transforms of the functions defined in the preceding displayed identities. Then Ψ is a compactly supported Parseval wavelet frame in $L^2(\mathbb{R}^d)$ associated to the dilation matrix A with vanishing moments of order m_0.

To prove Theorem 1 we need the Unitary Extension Principle (UEP) or the Oblique Extension Principle (OEP), a more flexible method based on the UEP. These principles are very useful tools for constructing wavelet frames. References on different versions of these principles are e.g. Ron and Shen [24], Chui, He and Stöckler [3], Daubechies, Han, Ron, and Shen [7], Han [15, 13], Atreas, Melas and Stavropoulos [1], Li and Zhang [36] and [27].

The version of the OEP we will use here may be formulated as follows:

Theorem B. *Let $A \in \mathbf{E}_d(\mathbb{Z})$. Let $\phi \in L^2(\mathbb{R}^d)$ be compactly supported and refinable, i.e.*

$$\widehat{\phi}(A^*\mathbf{t}) = P(\mathbf{t})\widehat{\phi}(\mathbf{t}),$$

where $P(\mathbf{x})$ is a trigonometric polynomial. Assume moreover that $|\widehat{\phi}(\mathbf{0})| = 1$. Let $S(\mathbf{t})$ be another trigonometric polynomial such that $S(\mathbf{t}) \geq 0$ and $S(\mathbf{0}) = 1$. Assume there are trigonometric polynomials or rational functions Q_ℓ, $\ell = 1, \cdots, N$, that satisfy the OEP condition

$$\begin{aligned} & S(A^*\mathbf{t})P(\mathbf{t})\overline{P(\mathbf{t}+\mathbf{j})} + \sum_{\ell=1}^{N} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} \\ &= \begin{cases} S(\mathbf{t}) & if \quad \mathbf{j} \in \mathbb{Z}^d, \\ 0 & if \quad \mathbf{j} \in \Big((A^*)^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d\Big) \setminus \mathbb{Z}^d \end{cases} . \end{aligned} \tag{11}$$

If

$$\widehat{\psi_\ell}(A^*\mathbf{t}) := Q_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad \ell = 1, \ldots, N,$$

then $\Psi = \{\psi_1, \ldots, \psi_N\}$ is a tight framelet in $L^2(\mathbb{R}^d)$ with dilation A and frame constant 1.

We also need the following technical result, see e.g. [18, Lemma 2.1.5].

Lemma C. Let $D = |\det A|$. The following equalities hold:

$$\sum_{a=0}^{D-1} e^{2\pi i \mathbf{q}_\ell\cdot\mathbf{r}_a} = \begin{cases} D & if\ \ell = 0; \\ 0 & if\ \ell \in \{1, 2, \ldots, D-1\} \end{cases}$$

and

$$\sum_{\ell=0}^{D-1} e^{2\pi i \mathbf{q}_\ell\cdot\mathbf{r}_a} = \begin{cases} D & if\ a = 0; \\ 0 & if\ a \in \{1, 2, \ldots, D-1\}. \end{cases}$$

Proof of Theorem 1. Let $\ell \in \{0, 1, \dots, D-1\}$, we denote

$$Q_\ell(\mathbf{t}) := D^{-1/2}\left[e^{2\pi i \mathbf{q}_\ell\cdot\mathbf{t}} - H(\mathbf{t})\sum_{a=0}^{D-1} e^{2\pi i \mathbf{q}_\ell\cdot(\mathbf{t}+\mathbf{r}_a)}\overline{H(\mathbf{t}+\mathbf{r}_a)}\right].$$

For $\mathbf{j} \in \mathbf{\Gamma}_A$ we have

$$\begin{aligned}\sum_{\ell=0}^{D-1} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} = D^{-1}\sum_{\ell=0}^{D-1} e^{-2\pi i \mathbf{q}_\ell\cdot\mathbf{j}} \\ - D^{-1}\overline{H(\mathbf{t}+\mathbf{j})}\sum_{a=0}^{D-1} H(\mathbf{t}+\mathbf{j}+\mathbf{r}_a)\sum_{\ell=0}^{D-1} e^{-2\pi i \mathbf{q}_\ell\cdot(\mathbf{j}+\mathbf{r}_a)} \\ - D^{-1}H(\mathbf{t})\sum_{a=0}^{D-1}\overline{H(\mathbf{t}+\mathbf{r}_a)}\sum_{\ell=0}^{D-1} e^{-2\pi i \mathbf{q}_\ell\cdot(\mathbf{j}-\mathbf{r}_a)} \\ + D^{-1}H(\mathbf{t})\overline{H(\mathbf{t}+\mathbf{j})}\sum_{a,\mu=0}^{D-1}\overline{H(\mathbf{t}+\mathbf{r}_a)}H(\mathbf{t}+\mathbf{j}+\mathbf{r}_\mu)\sum_{\ell=0}^{D-1} e^{-2\pi i \mathbf{q}_\ell\cdot(\mathbf{j}+\mathbf{r}_\mu-\mathbf{r}_a)}\end{aligned}$$

Since $\{\mathbf{j}+\mathbf{r}_a; a = 0, \cdots, D-1\}$ and $\{\mathbf{j}-\mathbf{r}_a; a = 0, \cdots, D-1\}$ are full collections of representatives of $(A^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d$, using Lemma C we readily see that

$$\begin{aligned}\sum_{\ell=0}^{D-1} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} = D^{-1}\sum_{\ell=0}^{D-1} e^{-2\pi i \mathbf{q}_\ell\cdot\mathbf{j}} - \overline{H(\mathbf{t}+\mathbf{j})}H(\mathbf{t}) - H(\mathbf{t})\overline{H(\mathbf{t}+\mathbf{j})} \\ + H(\mathbf{t})\overline{H(\mathbf{t}+\mathbf{j})}\sum_{\mu=0}^{D-1}\overline{H(\mathbf{t}+\mathbf{j}+\mathbf{r}_\mu)}H(\mathbf{t}+\mathbf{j}+\mathbf{r}_\mu).\end{aligned}$$

By (i) in Lemma 1, we have

$$\sum_{\ell=0}^{D-1} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} = D^{-1}\sum_{\ell=0}^{D-1} e^{-2\pi i \mathbf{q}_\ell\cdot\mathbf{j}} - \overline{H(\mathbf{t}+\mathbf{j})}H(\mathbf{t}).$$

We now apply Lemma C again and distinguish two cases: if $\mathbf{j} = \mathbf{0}$ we get

$$\sum_{\ell=0}^{D-1} |Q_\ell(\mathbf{t})|^2 = 1 - |H(\mathbf{t})|^2, \tag{12}$$

whereas if $\mathbf{j} \in \mathbf{\Gamma}_A \setminus \{\mathbf{0}\}$, we obtain

$$\sum_{\ell=0}^{D-1} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} = -H(\mathbf{t})\overline{H(\mathbf{t}+\mathbf{j})}. \tag{13}$$

From (i) in Lemma 1, (12) and (13), we deduce that the conditions of Theorem B are satisfied and therefore that Ψ defined as in (10) is a Parseval wavelet frame in $L^2(\mathbb{R}^d)$ associated to the dilation matrix A.

Since Q_ℓ, $\ell \in \{0, 1, \cdots, D-1\}$, is a trigonometric polynomial and ϕ has compact support, we conclude that also ψ_ℓ is compactly supported.

We now verify that ψ_ℓ, $\ell \in \{0, 1, \cdots, D-1\}$, has vanishing moments of order m_0. Since A^* is an expansive linear map, there exists $C > 0$ such that $\|A^*\mathbf{t}\| \geq C\|\mathbf{t}\|$.

Therefore, bearing in mind that $0 \leq |\widehat{\phi}(\mathbf{t})| \leq 1$ and using (12), we have:

$$\lim_{\mathbf{t}\to\mathbf{0}} \frac{|\widehat{\psi}_\ell(A^*\mathbf{t})|^2}{\|A^*\mathbf{t}\|^{2m_0}} \leq \lim_{\mathbf{t}\to\mathbf{0}} \frac{|Q_\ell(\mathbf{t})|^2}{C^{2m_0}\|\mathbf{t}\|^{2m_0}} \leq \lim_{\mathbf{t}\to\mathbf{0}} \frac{1-|H(\mathbf{t})|^2}{C^{2m_0}\|\mathbf{t}\|^{2m_0}} = 0. \tag{14}$$

where the last equality holds by (ii) in Lemma 1. This finishes the proof. □

We have the following direct consequence of Theorem 1.

Corollary 1. *For every dilation matrix A and any integer $m > 0$, there exists a compactly supported Parseval wavelet frame in $L^2(\mathbb{R}^d)$ associated to A with D generators and m vanishing moments.*

3. Second construction

In this section we construct a family of Parseval framelets $\Psi = \{\psi_1, \psi_2, \ldots, \psi_N\}$ in $L^2(\mathbb{R}^d)$ associated to a dilation matrix A where the generators ψ_ℓ are compactly supported, with any desired number of vanishing moments and having any given degree of regularity. Here, the number of generators is $N = D + r$. Therefore, the number of generators will not depend on the number of vanishing moments nor on the degree of regularity, and it will always smaller or equal to $D + d$.

We first construct a suitable refinable function from the refinable function in Section 2.

Let $n \in \mathbb{N}$ and

$$\widehat{\theta}(\mathbf{t}) := |\widehat{\phi}(\mathbf{t})|^{2n} \tag{15}$$

where $\widehat{\phi}$ is the function defined in Proposition 1. Thus

$$\widehat{\theta}(A^*\mathbf{t}) = |H(\mathbf{t})|^{2n}\widehat{\theta}(\mathbf{t}) = P(\mathbf{t})\widehat{\theta}(\mathbf{t}), \tag{16}$$

where $P(\mathbf{t}) = |H(\mathbf{t})|^{2n}$ and H is defined in (6).

It is easy to see that P is a trigonometric polynomial on $\mathbb{R}^d$ with $P(\mathbf{0}) = 1$. Moreover, from Lemma 1,

$$\sum_{s=0}^{D-1} |P(\mathbf{t}+\mathbf{r}_s)|^2 \leq \sum_{s=0}^{D-1} |H(\mathbf{t}+\mathbf{r}_s)|^2 = 1. \tag{17}$$

We have:

Proposition 2. *The function $\widehat{\theta}$ defined by (15) is in $L^2(\mathbb{R}^d)$. In addition, the function θ whose Fourier transform is $\widehat{\theta}$ is non null, refinable, compactly supported and in $L^2(\mathbb{R}^d)$. Moreover, $\|\theta\|_{L_2(\mathbb{R}^d)} \leq 1$ and, if $2n\nu - d > \alpha > 1$, θ is in continuity class C^α.*

Proof. From Proposition 1 we know that $|\widehat{\phi}(\mathbf{t})| \leq 1$, that $\widehat{\phi}$ is nonnull, and that $\widehat{\phi} \in L^2(\mathbb{R}^d)$ with $\|\phi\|_{L_2(\mathbb{R}^d)} \leq 1$. Its definition therefore implies that $0 \leq \widehat{\theta}(\mathbf{t}) \leq |\widehat{\phi}(\mathbf{t})|$, that $\widehat{\theta}$ is nonnull, and that $\widehat{\theta} \in L^2(\mathbb{R}^d)$ with $\|\widehat{\theta}\|_{L_2(\mathbb{R}^d)} \leq 1$. From (8) we also know that ϕ is refinable; thus also θ is refinable.

Since θ is the convolution of ϕ with itself and ϕ has compact support, we conclude that also θ has compact support.

We now study the regularity of θ. By definition of $\widehat{\theta}$ and (9), we have

$$\widehat{\theta}(\mathbf{t}) = |\widehat{\phi}(\mathbf{t})|^{2n} \leq C^{2n}(1 + \|\mathbf{t}\|)^{-2n\nu}. \tag{18}$$

By the well known Sobolev embedding theorems, see e.g. [8, Theorem 9.17], we conclude that θ is in continuity class C^α if $2n\nu - d > \alpha > 1$. This completes the proof. □

At this point we need to define some auxiliary trigonometric polynomials. We will use the integers $s_1, \dots, s_r$ of the matrix S defined by (1) and the integers $m_1, \dots, m_r$ used in (6). all these numbers s_j and m_j, $j = 1, \dots r$ are larger than 1. For $e \in \{1, \dots, r\}$, let h_{m_e,s_e} be the trigonometric polynomial defined as in (5) and let q_e be a trigonometric polynomial on $\mathbb{R}$ such that

$$|q_e(t)|^2 = 1 - \sum_{\ell_e=0}^{s_e-1} |h_{m_e,s_e}(t + \frac{\ell_e}{s_e})|^{4n}. \tag{19}$$

Since Lemma A (i) implies that

$$\sum_{\ell_e=0}^{s_e-1} |h_{m_e,s_e}(t + \frac{\ell_e}{s_e})|^{4n} \le \sum_{\ell_e=0}^{s_e-1} |h_{m_e,s_e}(t + \frac{\ell_e}{s_e}) = 1,$$

the existence of q_e follows applying a lemma of Riesz (cf., e.g., [6, Lemma 6.1.3]) or [22, Lemma 10, p. 102]). Since

$$h_{m_e,s_e}(t + \frac{1}{s_e} + \frac{\ell_e}{s_e}) = h_{m_e,s_e}(t),$$

we readily see that

$$1 - \sum_{\ell_e=0}^{s_e-1} |h_{m_e,s_e}(t + \frac{\ell_e}{s_e})|^{4n}$$

is $1/(s_e)$-periodic. Thus q_e can be chosen to be $1/(s_e)$-periodic.

We now define the following trigonometric polynomials on $\mathbb{R}^d$.

$$G_e(\mathbf{t}) := q_e(t_e) \prod_{b=e+1}^{r} \left(\sum_{\ell_b=0}^{s_b-1} |h_{m_b,s_b}(t + \frac{\ell_b}{s_b})|^{2n} \right),\ e \in \{1, \dots, r-1\},\ G_r(\mathbf{t}) = q_r(t_r),$$

and

$$F_e(\mathbf{t}) := G_e(U^*\mathbf{t}), \quad e \in \{1, \dots, r\}. \tag{20}$$

We need the following technical lemma.

Lemma 2. *Let P be the trigonometric polynomial defined in* (16) *and F_e, $e \in \{1, \dots, r\}$, be defined in* (20)*. Then F_e is $(A^*)^{-1}\mathbb{Z}^d$-periodic and*

$$\sum_{s=0}^{|detA|-1} |P(\mathbf{t} + \mathbf{r}_s)|^2 + \sum_{e=1}^{r} |F_e(\mathbf{t})|^2 = 1. \tag{21}$$

Proof. Assume that $r > 2$. Then

$$\begin{aligned}
&\sum_{\mathbf{r}\in\mathbf{\Gamma}_A} |P((U^*)^{-1}\mathbf{t}+\mathbf{r})|^2 + \sum_{e=1}^{r} |F_e((U^*)^{-1}\mathbf{t})|^2 \\
= \;& \prod_{j=1}^{r}\left(\sum_{\ell_j=0}^{s_j-1} |h_{s_j,m_j}(t_j+\frac{\ell_j}{s_j})|^{4n}\right) \\
&+|q_1(t_1)|^2 \prod_{b=2}^{r}\left(\sum_{\ell_b=0}^{s_b-1} |h_{s_b,m_b}(t+\frac{\ell_b}{s_b})|^{4n}\right) \\
&+\sum_{e=2}^{r-1} |q_e(t_e)|^2 \prod_{b=e+1}^{r}\left(\sum_{\ell_e=0}^{s_b-1} |h_{s_b,m_b}(t+\frac{\ell_b}{s_b})|^{4n}\right) + |q_r(t_r)|^2.
\end{aligned}$$

Since

$$\left(\sum_{\ell_1=0}^{s_1-1} |h_{s_1,m_1}(t+\frac{\ell_1}{s_1})|^{4n}\right) + |q_1(t_1)|^2 = 1,$$

we have

$$\begin{aligned}
&\sum_{\mathbf{r}\in\mathbf{\Gamma}_A} |P((U^*)^{-1}\mathbf{t}+\mathbf{r})|^2 + \sum_{e=1}^{r} |F_e(((U^*)^{-1}\mathbf{t})|^2 \\
= \;& \prod_{b=2}^{r}\left(\sum_{\ell_b=0}^{s_b-1} |h_{s_b,m_b}(t+\frac{\ell_b}{s_b})|^{4n}\right) \\
&+\sum_{e=2}^{r-1} |q_e(t_e)|^2 \prod_{b=e+1}^{r}\left(\sum_{\ell_b=0}^{s_b-1} |h_{h_{s_b,m_b}}(t+\frac{\ell_b}{s_b})|^{4n}\right) + |q_r(t_r)|^2.
\end{aligned}$$

Repeating this procedure a finite number of times, we finally obtain

$$\begin{aligned}
&\sum_{\mathbf{r}\in\mathbf{\Gamma}_A} |P((U^*)^{-1}\mathbf{t}+\mathbf{r})|^2 + \sum_{e=1}^{r} |F_e((U^*)^{-1}\mathbf{t})|^2 \\
= \;& \left(\sum_{\ell_r=0}^{s_r-1} |h_{s_r,m_r}(t+\frac{\ell_r}{s_r})|^{4n}\right) + |q_r(t_r)|^2 = 1.
\end{aligned}$$

We now show that F_e, $e \in \{1,\dots,r\}$, is $(A^*)^{-1}$-periodic: let $\mathbf{k}\in\mathbb{Z}^d$, then there exists $s_{\mathbf{k}} \in \{0,\dots,D-1\}$, $\mathbf{a_k}\in\mathbb{Z}^d$ and $\gamma_{\mathbf{k}}\in\Gamma$ such that $(A^*)^{-1}\mathbf{k} = \mathbf{r}_{s_{\mathbf{k}}} + \mathbf{a_k} = (U^*)^{-1}\gamma_{\mathbf{k}} + \mathbf{a_k}$. Since G_e is $1/s_b$–periodic in each variable, $b\in\{e,\dots,r\}$. Therefore

$$\begin{aligned}
F_e(\mathbf{t}+(A^*)^{-1}\mathbf{k}) &= G_e(U^*(\mathbf{t}+(A^*)^{-1}\mathbf{k})) = G_e(U^*((\mathbf{t}+(U^*)^{-1}\gamma_{\mathbf{k}}+\mathbf{a_k})) \\
&= G_e(U^*\mathbf{t}+\gamma_{\mathbf{k}}) = G_e(U^*\mathbf{t}) = F_e(\mathbf{t})
\end{aligned}$$

and the conclusion follows. For $r = 1, 2$ the proof is similar but simpler and will be omitted. □

Using the refinable function of Proposition 2 and adapting the method used in the proof of [21, Theorem 3.4] we obtain our second family of compactly supported Parseval wavelet frames:

Theorem 2. *Let $D := |\det A|$, let P be the real valued trigonometric polynomial in $\mathbb{R}^d$ defined by* (16)*, and let θ be the associated function defined in Proposition 2. Define*

$$\widehat{\psi_\ell}(A^*\mathbf{t}) := D^{-1/2}\left[e^{2\pi i\mathbf{q}_\ell\cdot\mathbf{t}} - P(\mathbf{t})\left(\sum_{a=0}^{D-1} e^{2\pi i\mathbf{q}_\ell\cdot(\mathbf{t}+\mathbf{r}_a)}P(\mathbf{t}+\mathbf{r}_a)\right)\right]\widehat{\theta}(\mathbf{t})$$

for $\ell \in \{0,\dots,D-1\}$, and

$$\widehat{\psi_\ell}(A^*\mathbf{t}) := P(\mathbf{t})F_{\ell-D+1}(\mathbf{t})\widehat{\theta}(\mathbf{t})$$

for $\ell \in \{D,\dots,D+r-1\}$, where the functions F_e are defined in (20)*. Moreover, let*

$$\Psi = \{\psi_\ell(\mathbf{t})\ ;\ \ell = 0,\dots,D+r-1,\} \tag{22}$$

be the set of inverse Fourier transforms of the functions defined in the preceding displayed identities. Then Ψ is a compactly supported Parseval wavelet frame in $L^2(\mathbb{R}^d)$ associated to the dilation matrix A with vanishing moments of order m_0. In addition, if $2n\nu - d > \alpha > 1$, then ψ_ℓ is in continuity class C^α.

Proof. We first prove that Ψ is a compactly supported Parseval wavelet frame in $L^2(\mathbb{R}^d)$ associated to the dilation matrix A.

For $\ell \in \{0,1,\dots,D-1\}$ define $Q_\ell(\mathbf{t})$ by

$$Q_\ell(\mathbf{t}) := D^{-1/2}\left[e^{2\pi i\mathbf{q}_\ell\cdot\mathbf{t}} - P(\mathbf{t})\left(\sum_{a=0}^{D-1} e^{2\pi i\mathbf{q}_\ell\cdot(\mathbf{t}+\mathbf{r}_a)}P(\mathbf{t}+\mathbf{r}_a)\right)\right]$$

and for $\ell \in \{D,\dots,D+r-1,\}$,

$$Q_\ell(\mathbf{t}) := P(\mathbf{t})F_{\ell-D+1}(\mathbf{t}).$$

Proceeding as in the proof of Theorem 1, we see that for $\mathbf{j} \in \mathbf{\Gamma}_A$,

$$\begin{aligned}
&\sum_{\ell=0}^{D-1} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} + \sum_{\ell=D}^{D+r-1} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})}\\
= \quad & D^{-1}\left(\sum_{\ell=0}^{D-1} e^{-2\pi i\mathbf{q}_\ell\cdot\mathbf{j}}\right) - P(\mathbf{t}+\mathbf{j})P(\mathbf{t}) - P(\mathbf{t})P(\mathbf{t}+\mathbf{j})\\
&+P(\mathbf{t})P(\mathbf{t}+\mathbf{j})\left(\sum_{\mu=0}^{D-1}|P(\mathbf{t}+\mathbf{j}+\mathbf{r}_\mu)|^2\right)\\
&+P(\mathbf{t})P(\mathbf{t}+\mathbf{j})\left(\sum_{\ell=D}^{D+r-1} F_{\ell-D+1}(\mathbf{t})\overline{F_{\ell-D+1}(\mathbf{t}+\mathbf{j})}\right).
\end{aligned}$$

From Lemma 2, we know that F_e is $(A^*)^{-1}\mathbb{Z}^d$–periodic and that (21) is satisfied; therefore

$$\sum_{\mu=0}^{D-1}|P(\mathbf{t}+\mathbf{j}+\mathbf{r}_\mu)|^2 + \sum_{\ell=D}^{D+r-1} F_{\ell-D+1}(\mathbf{t})\overline{F_{\ell-D+1}(\mathbf{t}+\mathbf{j})} = 1.$$

Thus,

$$\sum_{\ell=0}^{D-1} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} + \sum_{\ell=D}^{D+r-1} Q_\ell(\mathbf{t})\overline{Q_\ell(\mathbf{t}+\mathbf{j})} = D^{-1}\left(\sum_{\ell=0}^{D-1} e^{-2\pi i\mathbf{q}_\ell\cdot\mathbf{j}}\right) - P(\mathbf{t})P(\mathbf{t}+\mathbf{j}).$$

Proceeding now as in the proof of Theorem 1 we conclude that Ψ is a compactly supported Parseval wavelet frame.

We now show that ψ_ℓ, $\ell \in \{0, 1, \cdots, D + r - 1\}$, has vanishing moments of order m_0. Since $0 \leq |\widehat{\theta}(\mathbf{t})| \leq 1$ and the precding displayed identity implies that $|Q_\ell(\mathbf{t})|^2 + |P(\mathbf{t})|^2 \leq 1$, we obtain

$$\lim_{\mathbf{t}\to\mathbf{0}} \frac{|\widehat{\psi}_\ell(A^*\mathbf{t})|^2}{\|A^*\mathbf{t}\|^{2m_0}} \leq \lim_{\mathbf{t}\to\mathbf{0}} \frac{|Q_\ell(\mathbf{t})|^2}{\|A^*\mathbf{t}\|^{2m_0}} \leq \lim_{\mathbf{t}\to\mathbf{0}} \frac{1-|P(\mathbf{t})|^2}{\|A^*\mathbf{t}\|^{2m_0}} = \lim_{\mathbf{t}\to\mathbf{0}} \frac{1-|H(\mathbf{t})|^{4n}}{\|A^*\mathbf{t}\|^{2m_0}}.$$

Since, if $|H(\mathbf{t})|^2 \neq 1$,

$$\sum_{\mu=1}^{2n-1} \left(|H(\mathbf{t})|^2\right)^\mu = \frac{|H(\mathbf{t})|^{4n}-1}{|H(\mathbf{t})|^2-1},$$

we readily see that

$$1 - |H(\mathbf{t})|^{4n} = (1 - |H(\mathbf{t})|^2) \sum_{\mu=1}^{2n-1} \left(|H(\mathbf{t})|^2\right)^\mu .$$

Therefore

$$\lim_{\mathbf{t}\to\mathbf{0}} \frac{1-|H(\mathbf{t})|^{4n}}{\|A^*\mathbf{t}\|^{2m_0}} = \lim_{\mathbf{t}\to\mathbf{0}} \left(\sum_{\mu=1}^{2n-1} |H(\mathbf{t})|^{2\mu} \right) \frac{1-|H(\mathbf{t})|^2}{\|A^*\|^{2m_0}}$$

As remarked in the proof of Theorem 1, since A^* is an expansive linear map, there exists $C > 0$ such that $\|A^*\mathbf{t}\| \geq C\|\mathbf{t}\|$. Bearing in mind that $|H(\mathbf{0})|^2 = 1$ and proceeding as in (14) we see that

$$\lim_{\mathbf{t}\to\mathbf{0}} \frac{|\widehat{\psi}_\ell(A^*\mathbf{t})|^2}{\|A^*\mathbf{t}\|^{2m_0}} = 0.$$

We have therefore shown that ψ_ℓ has vanishing moments of order m_0.

It remains to find the degree of regularity of ψ_ℓ, $\ell \in \{0, 1, \cdots, D+r-1\}$. Since $|Q_\ell|$ is bounded, from (18) we have

$$|\widehat{\psi_\ell}(\mathbf{t})| \leq K(1 + \|\mathbf{t}\|)^{-2n\nu}.$$

Applying again the Sobolev embedding theorems we conclude that ψ_ℓ is in continuity class C^α if $2n\nu - d > \alpha > 1$. □

The following is a direct consequence of Theorem 2.

Corollary 2. *Let $m > 0$ be an integer. There exists a compactly supported Parseval wavelet frame in $L^2(\mathbb{R}^d)$ associated to a dilation matrix A with $D + r$ generators, m vanishing moments and any desired degree of regularity.*

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