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# Inequalities for the generalized Marcum Q-function

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ABSTRACT

In this paper, we consider the generalized Marcum Q-function of order $\nu > 0$ real, defined by

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\,dt,$$

where $a, b \geqslant 0$, $I_\nu$ stands for the modified Bessel function of the first kind and the right hand side of the above equation is replaced by its limiting value when $a = 0$. Our aim is to prove that the function $\nu \mapsto Q_\nu(a,b)$ is strictly increasing on $(0,\infty)$ for each $a \geqslant 0$, $b > 0$, and to deduce some interesting inequalities for the function $Q_\nu$. Moreover, we present a somewhat new viewpoint of the generalized Marcum Q-function, by showing that satisfies the new-is-better-than-used (nbu) property, which arises in economic theory.



## 1. Introduction and preliminaries

For $\nu$ unrestricted real (or complex) number let $I_\nu$ be the modified Bessel function of the first kind of order $\nu$, defined by the relation [19, p. 77]

$$I_\nu(x) = \sum_{k \geqslant 0} \frac{(x/2)^{2k+\nu}}{k!\Gamma(\nu+k+1)},$$

which is of frequent occurrence in problems of mathematical physics and chemistry. Further, let $Q_\nu(a,b)$ be the generalized Marcum Q-function, defined by

$$Q_\nu(a,b) = \begin{cases} \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\,dt & \text{if } a > 0, \\ \frac{1}{2^{\nu-1}\Gamma(\nu)} \int_b^\infty t^{2\nu-1} e^{-\frac{t^2}{2}}\,dt & \text{if } a = 0, \end{cases} \tag{1.1}$$

where $b \geqslant 0$ and $\nu > 0$. Clearly the function $a \mapsto Q_\nu(a,b)$ is continuous, because for each $t \geqslant b$ fixed we have

$$\lim_{a\to 0} \left[2^{\nu-1}\Gamma(\nu)(at)^{1-\nu} I_{\nu-1}(at)\right] = 1,$$

which implies that

$$\lim_{a\to 0} Q_\nu(a,b) = Q_\nu(0,b)$$

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for all $b \geqslant 0$ and $\nu > 0$. The generalized Marcum $Q$-function defined above is widely used in radar signal processing and has important applications in error performance analysis of multichannel dealing with partially coherent, differentially coherent, and non-coherent detections in digital communications. For further details the interested reader is referred to the book [17] and to the references therein. Since, the precise computation of the generalized Marcum $Q$-function is quite difficult, in the last few decades several authors established approximation formulas and bounds for the function $Q_\nu(a, b)$.

This paper is a further contribution to the subject and is organized as follows: in Section 2 we prove that the function $b \mapsto Q_\nu(a, b)$ is strictly log-concave on $(0, \infty)$, which implies that the generalized Marcum $Q$-function satisfies the nbu property (see Section 2), which is of importance in economic theory. In Section 3 we prove that the function $\nu \mapsto Q_\nu(a, b)$ is strictly increasing on $(0, \infty)$, and we deduce some new inequalities for the function $Q_\nu(a, b)$.

It is worth mentioning that the generalized Marcum $Q$-function has an important interpretation in probability theory, namely that is the complement (with respect to unity) to the cumulative distribution function (cdf) of the non-central chi distribution with $2\nu$ degrees of freedom. We note here that in probability theory and in economic theory the complement (with respect to unity) of a cdf is called a survival (or a reliability) function. For these we refer the reader to the papers [1,2,4]. To be more precise for the reader's convenience we recall some basic facts. First note that when $a > 0$ the integrand in (1.1) is a probability density function (pdf). For this, consider the Sonine formula [19, p. 394]

$$\int_0^\infty J_\nu(at)\mathrm{e}^{-pt^2}t^{\nu+1}\mathrm{d}t = \frac{a^\nu \mathrm{e}^{-\frac{a^2}{4p}}}{(2p)^{\nu+1}},$$

which holds for all $a, p, \nu$ complex numbers such that $\mathrm{Re}(p) > 0$, $\mathrm{Re}(\nu) > -1$ and where $J_\nu$ stands for the Bessel function of the first kind. Taking into account the relation $I_\nu(x) = \mathrm{i}^{-\nu}J_\nu(\mathrm{i}x)$ and changing in the above Sonine formula $a$ with $\mathrm{i}a$ we easily get that

$$\int_0^\infty I_\nu(at)\mathrm{e}^{-pt^2}t^{\nu+1}\mathrm{d}t = \frac{a^\nu \mathrm{e}^{\frac{a^2}{4p}}}{(2p)^{\nu+1}},$$

which implies that for each $\nu, a > 0$ we have

$$Q_\nu(a, 0) = \frac{1}{a^{\nu-1}}\int_0^\infty t^\nu \mathrm{e}^{-\frac{t^2+a^2}{2}}I_{\nu-1}(at)\mathrm{d}t = 1$$

as we required. When $a = 0$ clearly we have for each $\nu > 0$ that

$$Q_\nu(0, 0) = \frac{1}{2^{\nu-1}\Gamma(\nu)}\int_0^\infty t^{2\nu-1}\mathrm{e}^{-\frac{t^2}{2}}\mathrm{d}t = \frac{1}{\Gamma(\nu)}\int_0^\infty \mathrm{e}^{-u}u^{\nu-1}\mathrm{d}u = 1.$$

Thus in fact for all $b \geqslant 0$ and $\nu > 0$ we have

$$Q_\nu(a, b) = \begin{cases} 1 - \frac{1}{a^{\nu-1}}\int_0^b t^\nu \mathrm{e}^{-\frac{t^2+a^2}{2}}I_{\nu-1}(at)\mathrm{d}t, & \text{if } a > 0, \\ 1 - \frac{1}{2^{\nu-1}\Gamma(\nu)}\int_0^b t^{2\nu-1}\mathrm{e}^{-\frac{t^2}{2}}\mathrm{d}t, & \text{if } a = 0. \end{cases} \tag{1.2}$$

On the other hand, it is known that if $X_1, X_2, \ldots, X_n$ are random variables that are normally distributed with unit variance and nonzero mean $\mu_1, \mu_2, \ldots, \mu_n$, then the random variable $[X_1^2 + X_2^2 + \cdots + X_n^2]^{1/2}$ has the non-central chi distribution with $n = 1, 2, 3, \ldots$ degrees of freedom and non-centrality parameter $\tau = [\mu_1^2 + \mu_2^2 + \cdots + \mu_n^2]^{1/2}$. The pdf $\chi_{n,\tau} : (0, \infty) \to (0, \infty)$ of the non-central chi distribution [13] is defined as

$$\chi_{n,\tau}(x) = 2^{-\frac{n}{2}+1}\mathrm{e}^{-\frac{x^2+\tau^2}{2}}\sum_{k\geqslant 0}\frac{x^{n+2k-1}(\tau/2)^{2k}}{\Gamma(n/2+k)k!} = \tau \mathrm{e}^{-\frac{x^2+\tau^2}{2}}\left(\frac{x}{\tau}\right)^{n/2}I_{\frac{n}{2}-1}(\tau x).$$

Observe that when $\mu_1 = \mu_2 = \cdots = \mu_n = 0$, i.e. $\tau = 0$, the above distribution reduces to the classical chi distribution with pdf $\chi_{n,0} : (0, \infty) \to (0, \infty)$ given by

$$\chi_n(x) = \chi_{n,0}(x) = \frac{x^{n-1}\mathrm{e}^{-x^2/2}}{2^{n/2-1}\Gamma(n/2)}.$$

Thus taking into account the above definitions and (1.2), in particular, when $n = 2\nu$ is an integer the generalized Marcum $Q$-function is exactly the reliability function of the non-central chi distribution with $2\nu$ degrees of freedom and non-centrality parameter $\tau = a$. In fact there is another probabilistic interpretation of the generalized Marcum $Q$-function, i.e. a transformation of this function is connected with the non-central chi-squared distribution. For this let $Y_1, Y_2, \ldots, Y_m$ be random variables that are normally distributed with unit variance and nonzero mean $\gamma_i$, where $i = 1, 2, \ldots, m$. It is known that $Y_1^2 + Y_2^2 + \cdots + Y_m^2$ has the non-central chi-squared distribution with $m = 1, 2, 3, \ldots$ degrees of freedom and non-centrality parameter $\lambda = \gamma_1^2 + \gamma_2^2 + \cdots + \gamma_m^2$. The pdf $\chi^2_{m,\lambda} : (0, \infty) \to (0, \infty)$ of the non-central chi-squared distribution [13] is defined as

$$\chi^2_{m,\lambda}(x) = 2^{-m/2}\mathrm{e}^{-(x+\lambda)/2}\sum_{k\geqslant 0}\frac{x^{m/2+k-1}(\lambda/4)^k}{\Gamma(m/2+k)k!} = \frac{\mathrm{e}^{-(x+\lambda)/2}}{2}\left(\frac{x}{\lambda}\right)^{m/4-1/2}I_{m/2-1}(\sqrt{\lambda x}).$$

Recall that when $\gamma_1 = \gamma_2 = \cdots = \gamma_m = 0$, i.e. $\lambda = 0$, the above distribution reduces to the classical chi-squared distribution. The pdf $\chi^2_{m,0} : (0,\infty) \to (0,\infty)$ of this distribution is given by

$$\chi^2_m(x) = \chi^2_{m,0}(x) = \frac{x^{m/2-1}\mathrm{e}^{-x/2}}{2^{m/2}\Gamma(m/2)}.$$

Now from (1.1) and (1.2) it is easy to verify that

$$Q_\nu\left(\sqrt{a},\sqrt{b}\right) = \begin{cases} 1 - \frac{1}{2}\int_0^b \left(\frac{t}{a}\right)^{\frac{\nu}{2}-\frac{1}{2}}\mathrm{e}^{-\frac{t+a}{2}} I_{\nu-1}\left(\sqrt{at}\right)\mathrm{d}t & \text{if } a > 0,\\ 1 - \frac{1}{2^\nu\Gamma(\nu)}\int_0^b t^{\nu-1}\mathrm{e}^{-\frac{t}{2}}\mathrm{d}t & \text{if } a = 0, \end{cases} \tag{1.3}$$

i.e. the function $Q_\nu(\sqrt{a},\sqrt{b})$ in particular is the survival function of the non-central chi-squared distribution with $m = 2\nu$ degrees of freedom and non-centrality parameter $\lambda = a$.

## 2. The nbu property for the generalized Marcum Q-function

Solving a problem which arises in random flights, Findling [9, Theorem 8], using an interesting method, proved that the function $x \mapsto xI_1(x)$ is strictly log-concave on $\mathbb{R}\setminus\{0\}$. The following result – which is of independent interest – improves Findling's result when $x > 0$ and is useful in establishing the nbu property for the generalized Marcum Q-function.

**Proposition 2.1.** *Let $\nu$ be a real number and let $x > 0$. The following assertions are true:*

(a) *the function $x \mapsto xI_\nu(x)$ is log-concave for each $\nu \geqslant 1/2$;*
(b) *the function $x \mapsto x^\nu I_\nu(x)$ is strictly log-concave for each $\nu \geqslant 1$.*

**Proof**

(a) Let us consider the modified Bessel function of the second kind (which is called sometimes as the MacDonald function) $K_\nu$, defined by [19, p. 78]

$$K_\nu(x) := \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi},$$

where the right hand side of this equation is replaced by its limiting value if $\nu$ is an integer or zero. Due to Hartman [12] it is known that the function $x \mapsto xI_\nu(x)K_\nu(x)$ is concave on $(0,\infty)$ for all $\nu > 1/2$. Since $x \mapsto 2xI_{1/2}(x)K_{1/2}(x) = 1 - \mathrm{e}^{-2x}$ is concave on $(0,\infty)$, we conclude that in fact the function $x \mapsto xI_\nu(x)K_\nu(x)$ is concave on $(0,\infty)$ for all $\nu \geqslant 1/2$.

On the other hand it is known that the function $x \mapsto K_\nu(x)$ is log-convex on $(0,\infty)$, which result was stated in [11, Remark 3.2] without proof. For the sake of completeness we include here the proof. For this recall the following integral representation [19, p. 181] of the modified Bessel function of the second kind

$$K_\nu(x) = \int_0^\infty \mathrm{e}^{-x\cosh t}\cosh(\nu t)\mathrm{d}t, \tag{2.2}$$

which holds for each $x > 0$ and $\nu \in \mathbb{R}$. Further consider the well-known Hölder-Rogers inequality [16, p. 54], that is

$$\int_a^b |f(t)g(t)|\mathrm{d}t \leqslant \left[\int_a^b |f(t)|^p\mathrm{d}t\right]^{1/p}\left[\int_a^b |g(t)|^q\mathrm{d}t\right]^{1/q}, \tag{2.3}$$

where $p > 1$, $1/p + 1/q = 1$, $f$ and $g$ are real functions defined on $[a,b]$ and $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$. Using (2.2) and (2.3) we conclude that

$$\begin{aligned} K_\nu(\alpha x_1 + (1-\alpha)x_2) &= \int_0^\infty \mathrm{e}^{-(\alpha x_1+(1-\alpha)x_2)\cosh t}\cosh(\nu t)\mathrm{d}t = \int_0^\infty \left[\mathrm{e}^{-x_1\cosh t}\cosh(\nu t)\right]^\alpha\left[\mathrm{e}^{-x_2\cosh t}\cosh(\nu t)\right]^{1-\alpha}\mathrm{d}t\\ &\leqslant \left[\int_0^\infty \mathrm{e}^{-x_1\cosh t}\cosh(\nu t)\mathrm{d}t\right]^\alpha\left[\int_0^\infty \mathrm{e}^{-x_2\cosh t}\cosh(\nu t)\mathrm{d}t\right]^{1-\alpha} = [K_\nu(x_1)]^\alpha[K_\nu(x_2)]^{1-\alpha} \end{aligned}$$

holds for all $\alpha \in [0,1]$, $x_1, x_2 > 0$ and $\nu \in \mathbb{R}$, i.e. $x \mapsto K_\nu(x)$ is log-convex on $(0,\infty)$ for all $\nu \in \mathbb{R}$. Thus, we have that the function $x \mapsto 1/K_\nu(x)$ is log-concave on $(0,\infty)$ for each $\nu \in \mathbb{R}$.

Now, since the function $x \mapsto xI_\nu(x)K_\nu(x)$ is concave on $(0,\infty)$ for all $\nu \geqslant 1/2$, it follows that it is log-concave on $(0,\infty)$ for all $\nu \geqslant 1/2$. Consequently we have that the function $x \mapsto xI_\nu(x)$ is log-concave on $(0,\infty)$ for all $\nu \geqslant 1/2$, as a product of two log-concave functions.

(b) First suppose that $\nu = 1$. Recall that due to Findling [9] it is known that the function $x \mapsto xI_1(x)$ is strictly log-concave on $(0,\infty)$. Now assume that $\nu > 1$. Since the function $x \mapsto x^{\nu-1}$ is strictly log-concave on $(0,\infty)$, using part (a) of this proposition, we deduce that $x \mapsto x^\nu I_\nu(x)$ is strictly log-concave as a product of a strictly log-concave and log-concave functions, as we required. □

In probability theory usually the cumulative distribution functions (cdf-s) does not have closed-form, and consequently is quite difficult to study their properties directly. In statistics, economics and industrial engineering frequently appears some problems which are related to the study of log-concavity (log-convexity) of some univariate distributions. An interesting unified exposition of related results on the log-concavity and log-convexity of many distributions – including applications – were communicated by Bagnoli and Bergstrom [4]. The next results are widely used in economic theory and for proofs the interested reader is referred to the following papers [1,3,4,6]. We note that in economics, the inequality (2.5) is called the new-is-better-than-used (nbu) property [1, p. 21], because if $X$ is the time of death of a physical object, then the probability $P(X \geqslant x) = S(x)$ that a new unit will survive to age $x$, is greater than the probability

$$\frac{P(X \geqslant x+y)}{P(X \geqslant y)} = \frac{S(x+y)}{S(y)}$$

that a survived unit of age $y$ will survive for an additional time $x$.

**Lemma 2.4.** *Let $f : [u, v] \to [0, \infty)$ be a continuously differentiable pdf and consider the survival function $S : [u, v] \to [0, 1]$, defined by*

$$S(x) = \int_x^v f(t)\mathrm{d}t.$$

*If $f$ is (strictly) log-concave, then the reliability function $S$ is (strictly) log-concave too. Moreover, if the random variable $X$ has positive support and its survival function $S$ is log-concave, then for all $x, y \geqslant 0$ the following inequality*

$$S(x+y) \leqslant S(x)S(y) \tag{2.5}$$

*holds true. If the survival function $S$ is strictly log-concave, then the inequality (2.5) is strict.*

For the shape parameter $\nu > 0$ and $\lambda, \tau \geqslant 0$ consider the pdf-s of the non-central chi-squared and non-central chi distributions $\chi^2_{\nu,\lambda}, \chi_{\nu,\tau} : [0, \infty) \to [0, \infty)$, defined by the relations

$$\chi^2_{\nu,\lambda}(x) = \mathrm{e}^{-(x+\lambda)/2} \sum_{k \geqslant 0} \frac{(x/2)^{\nu/2}(\lambda/4)^k}{\Gamma(\nu/2+k)k!} x^{k-1},$$

$$\chi_{\nu,\tau}(x) = \mathrm{e}^{-(x^2+\tau^2)/2} \sum_{k \geqslant 0} \frac{x^\nu(\tau/2)^{2k}}{2^{\nu/2-1}\Gamma(\nu/2+k)k!} x^{2k-1}.$$

Further, let us denote simply $\chi^2_{\nu,0}(x) = \chi^2_\nu(x)$ and $\chi_{\nu,0}(x) = \chi_\nu(x)$. Recently, the second author in [6], among other things, proved that the survival function of the (central) chi and chi-squared distributions satisfies the nbu property (2.5), i.e. the functions

$$Q_{\nu/2}(0, \sqrt{b}) = S_{\chi^2_\nu}(b) = 1 - \int_0^b \frac{t^{\nu/2-1}\mathrm{e}^{-t/2}}{2^{\nu/2}\Gamma(\nu/2)}\mathrm{d}t, \quad \nu \geqslant 2,$$

$$Q_{\nu/2}(0, b) = S_{\chi_\nu}(b) = 1 - \int_0^b \frac{t^{\nu-1}\mathrm{e}^{-t^2/2}}{2^{\nu/2-1}\Gamma(\nu/2)}\mathrm{d}t, \quad \nu \geqslant 1,$$

satisfies the inequality (2.5). We note that since $t \mapsto t^{\nu-1}$ is log-concave on $(0,\infty)$ for all $\nu \geqslant 1$, using part (a) of Proposition 2.1 and the formula (1.2), we conclude that the pdf $b \mapsto \chi_{2\nu,a}(b)$ of the survival function $b \mapsto Q_\nu(a, b)$ is log-concave on $(0,\infty)$ for all $a > 0$ and $\nu \geqslant 3/2$. Therefore, in view of Lemma 2.4 the function $b \mapsto Q_\nu(a, b)$ is log-concave too on $(0,\infty)$, and consequently satisfies the nbu property, that is we have

$$Q_\nu(a, b_1 + b_2) \leqslant Q_\nu(a, b_1)Q_\nu(a, b_2) \leqslant Q^2_\nu\left(a, \frac{b_1+b_2}{2}\right) \tag{2.6}$$

for all $a > 0$, $\nu \geqslant 3/2$ and $b_1, b_2 > 0$. However, using a slightly different approach, in the followings we prove that in fact the strict version of (2.6) holds for each $\nu > 1$ and $a \geqslant 0$.

The main result of this section improves the above mentioned results.

**Theorem 2.7.** *Let $a \geqslant 0$ and $\nu > 1$. Then the following assertions are true:*

(a) *the function $b \mapsto Q_\nu(a, \sqrt{b})$ is strictly log-concave on $(0, \infty)$;*
(b) *the function $b \mapsto Q_\nu(a, b)$ is strictly log-concave on $(0, \infty)$;*
(c) *the strict version of inequality (2.6) and*

$$Q_\nu\left(a, \sqrt{b_1+b_2}\right) < Q_\nu(a, \sqrt{b_1})Q_\nu(a, \sqrt{b_2}) < Q^2_\nu\left(a, \sqrt{\frac{b_1+b_2}{2}}\right), \tag{2.8}$$

*hold true for all $b_1, b_2 > 0$ and $b_1 \neq b_2$. Moreover, the inequality (2.6) is weaker than the inequality (2.8) in the sense that for all $b_1, b_2 > 0$ and $b_1 \neq b_2$ we have*

$$Q_\nu(a, b_1+b_2) < Q_\nu\left(a, \sqrt{b_1^2+b_2^2}\right) < Q_\nu(a, b_1)Q_\nu(a, b_2) < Q^2_\nu\left(a, \sqrt{\frac{b_1^2+b_2^2}{2}}\right) < Q^2_\nu\left(a, \frac{b_1+b_2}{2}\right). \tag{2.9}$$

**Proof**

(a) It is known [7, Theorem 1.5] that the function $b \mapsto \chi^2_{\nu,a}(b)$ is strictly log-concave on $(0,\infty)$ for all $a \geqslant 0$ and $\nu > 2$. Hence, in view of (1.3), the pdf of the survival function $b \mapsto Q_\nu(\sqrt{a},\sqrt{b})$ is strictly log-concave, i.e. the function $b \mapsto \chi^2_{2\nu,a}(b)$ is strictly log-concave on $(0,\infty)$ for all $a \geqslant 0$ and $\nu > 1$. Consequently from Lemma 2.4 we have that the function $b \mapsto Q_\nu(\sqrt{a},\sqrt{b})$, as well as the function $b \mapsto Q_\nu(a,\sqrt{b})$ are strictly log-concave, as we required.

(b) Since $b \mapsto Q_\nu(a,\sqrt{b})$ is strictly log-concave and the function $b \mapsto Q_\nu(a,b)$ is decreasing, by definition we have that for all $a \geqslant 0$, $b_1, b_2 > 0$, $b_1 \neq b_2$, $\alpha \in (0,1)$ and $\nu > 1$ the inequality

$$[Q_\nu(a,b_1)]^\alpha [Q_\nu(a,b_2)]^{1-\alpha} < Q_\nu\left(a, \sqrt{\alpha b_1^2 + (1-\alpha) b_2^2}\right) < Q_\nu(a, \alpha b_1 + (1-\alpha) b_2)$$

holds, where we used the inequality $\alpha b_1^2 + (1-\alpha)b_2^2 > [\alpha b_1 + (1-\alpha)b_2]^2$. With other words, in the proof of strict log-concavity of $b \mapsto Q_\nu(a,b)$ we have used the following property: if a positive function $f$ is strictly log-concave and decreasing, and $g$ is convex, then the composite function $f \circ g$ is strictly log-concave too. Here $f(b) = Q_\nu(a,\sqrt{b})$, which is decreasing and strictly log-concave, and $g(b) = b^2$, which is clearly convex.

(c) Using Lemma 2.4, we conclude that the inequalities (2.6) and (2.8) follows easily from parts (a) and (b) of this theorem. On the other hand, since the function $b \mapsto Q_\nu(a,b)$ is a survival function, clearly it is decreasing. Therefore changing in (2.8) $b_1$ with $b_1^2$ and $b_2$ with $b_2^2$, we immediately get (2.9). □

**Remark 2.10.** We note that, since $b \mapsto Q_\nu(a,b)$ is a survival function, clearly it is decreasing on $(0,\infty)$ for all $\nu > 0$ and $a \geqslant 0$. Hence the function $b \mapsto Q_\nu(a,b)/b$ is strictly decreasing on $(0,\infty)$ for all $a \geqslant 0$ and $\nu > 0$. Thus we have that the survival function $b \mapsto Q_\nu(a,b)$ is strictly sub-additive on $(0,\infty)$, that is for all $b_1, b_2, \nu > 0$ and $a \geqslant 0$ the inequality

$$Q_\nu(a, b_1 + b_2) < Q_\nu(a,b_1) + Q_\nu(a,b_2) \tag{2.11}$$

holds true. In fact, the same argument can be applied to the survival function $b \mapsto Q_\nu(\sqrt{a},\sqrt{b})$. Namely, the function $b \mapsto Q_\nu(\sqrt{a},\sqrt{b})$ is strictly sub-additive on $(0,\infty)$, that is for all $b_1, b_2, \nu > 0$ and $a \geqslant 0$ the inequality

$$Q_\nu\left(a, \sqrt{b_1 + b_2}\right) < Q_\nu(a, \sqrt{b_1}) + Q_\nu(a, \sqrt{b_2}) \tag{2.12}$$

holds true. Moreover, since $b \mapsto Q_\nu(a,b)$ is decreasing, we have that (2.11) is weaker than (2.12), that is we have the inequality

$$Q_\nu(a, b_1 + b_2) < Q_\nu\left(a, \sqrt{b_1^2 + b_2^2}\right) < Q_\nu(a,b_1) + Q_\nu(a,b_2),$$

which holds for all for all $b_1, b_2, \nu > 0$ and $a \geqslant 0$.

## 3. New inequalities for the generalized Marcum *Q*-function

Recently Li and Kam [14], using an interesting geometric interpretation of the function $Q_\nu(a,b)$, proved that for all $\nu = m$ natural number and $a, b > 0$ the inequalities

$$Q_\nu(a,b) < Q_{\nu+1/2}(a,b) < Q_{\nu+1}(a,b)$$

hold. The following result improves the above inequalities.

**Theorem 3.1.** *Let $b > 0$ and $a \geqslant 0$. Then the following assertions are true:*

(a) *the function $\nu \mapsto Q_\nu(a,b)$ is strictly increasing on $(0,\infty)$;*
(b) *the function $\nu \mapsto Q_{\nu+1}(a,b) - Q_\nu(a,b)$ is strictly decreasing on $(0,\infty)$ provided $a \geqslant b$;*
(c) *the function $\nu \mapsto Q_{\nu+1}(a,b) - Q_\nu(a,b)$ is log-concave on $(0,\infty)$;*
(d) *the inequality*

$$Q_{\nu+1}(a,b) > \frac{Q_\nu(a,b) + Q_{\nu+2}(a,b)}{2} > \sqrt{Q_\nu(a,b) Q_{\nu+2}(a,b)} \tag{3.2}$$

*holds for all $a \geqslant b > 0$ and $\nu > 0$, while the inequalities*

$$[Q_{\nu+2}(a,b) - Q_{\nu+1}(a,b)]^2 > [Q_{\nu+1}(a,b) - Q_\nu(a,b)][Q_{\nu+3}(a,b) - Q_{\nu+2}(a,b)], \tag{3.3}$$

$$b^2 Q^2_{\nu+1}(a,b) + Q^2_{\nu+2}(a,b) > Q_{\nu+2}(a,b) Q_{\nu+1}(a,b) + b^2 Q_{\nu+2}(a,b) Q_\nu(a,b) \tag{3.4}$$

*hold true for all $a \geqslant 0$ and $b, \nu > 0$.*

**Proof**

(a) To show that $\nu \mapsto Q_\nu(a,b)$ is strictly increasing we prove that for all $\nu_1, \nu_2 > 0$ we have that

$$Q_{\nu_1+\nu_2}(a,b) > Q_{\nu_1}(a,b), \tag{3.5}$$

where $a \geqslant 0$ and $b > 0$. For this, let $X$ be a random variable which has non-central chi-squared distribution with shape parameter (degree of freedom) $2v_1$ and non-centrality parameter $a$. Further let $Y$ be a random variable which has chi-squared distribution with the shape parameter $2v_2$. Using the characteristic functions of the non-central chi-squared and (central) chi-squared distributions, it is easy to verify that the random variable $X + Y$ has non-central chi-squared distribution with shape parameter $2(v_1 + v_2)$ and non-centrality parameter $a$. Namely, the characteristic functions of the independent random variables $X$ and $Y$ are defined as follows

$$\varphi_X(t) = e^{\frac{iat}{1-2it}}(1-2it)^{-v_1}, \quad \varphi_Y(t) = (1-2it)^{-v_2}$$

and then we have

$$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t) = e^{\frac{iat}{1-2it}}(1-2it)^{-(v_1+v_2)},$$

which is the characteristic function of the non-central chi-squared distribution with shape parameter $2(v_1 + v_2)$ and non-centrality parameter $a$. Therefore we conclude that the random variable $X + Y$ indeed has non-central chi-squared distribution with shape parameter $2(v_1 + v_2)$ and non-centrality parameter $a$. Thus, in view of (1.3), we have

$$\begin{aligned}
Q_{v_1+v_2}(\sqrt{a},\sqrt{b}) &= 1 - \frac{1}{2}\int_0^b \left(\frac{t}{a}\right)^{\frac{v_1+v_2}{2}-\frac{1}{2}} e^{-\frac{t+a}{2}} I_{v_1+v_2-1}(\sqrt{at})\mathrm{d}t \\
&= 1 - P(X+Y<b) = P(X+Y \geqslant b) \\
&= P(X \geqslant b, X+Y \geqslant b) + P(X<b, X+Y \geqslant b) \\
&\geqslant P(X \geqslant b) + P(X<b, Y \geqslant b) \\
&= P(X \geqslant b) + P(X<b)P(Y \geqslant b) \\
&> P(X \geqslant b) = 1 - P(X<b) \\
&= 1 - \frac{1}{2}\int_0^b \left(\frac{t}{a}\right)^{\frac{v_1}{2}-\frac{1}{2}} e^{-\frac{t+a}{2}} I_{v_1-1}(\sqrt{at})\mathrm{d}t = Q_{v_1}(\sqrt{a},\sqrt{b}),
\end{aligned}$$

for all $v_1, v_2, b, a > 0$. When $a = 0$, using the characteristic function of the (central) chi-squared distribution, the same argument can be applied to show that for all $v_1, v_2, b > 0$ we have

$$Q_{v_1+v_2}(0,\sqrt{b}) > Q_{v_1}(0,\sqrt{b}).$$

Thus we have proved that for all $v_1, v_2, b > 0$ and $a \geqslant 0$ we have

$$Q_{v_1+v_2}(\sqrt{a},\sqrt{b}) > Q_{v_1}(\sqrt{a},\sqrt{b})$$

consequently changing $a$ with $a^2$ and $b$ with $b^2$, the required inequality (3.5) follows.

(b) It is known that the generalized Marcum $Q$-function satisfies the recurrence formula [17, p. 82]

$$Q_{v+1}(a,b) = \left(\frac{b}{a}\right)^v e^{-\frac{a^2+b^2}{2}} I_v(ab) + Q_v(a,b). \tag{3.6}$$

On the other hand due to Cochran [8] we know that $\mathrm{d}I_v(ab)/\mathrm{d}v < 0$ for all $a, b, v > 0$. Thus from (3.6) we have that for all $a \geqslant b > 0$ and $v > 0$

$$\frac{\mathrm{d}}{\mathrm{d}v}[Q_{v+1}(a,b) - Q_v(a,b)] = \left(\frac{b}{a}\right)^v e^{-\frac{a^2+b^2}{2}}\left[I_v(ab)\log\left(\frac{b}{a}\right) + \frac{\mathrm{d}}{\mathrm{d}v}I_v(ab)\right] < 0,$$

i.e. the function $v \mapsto Q_{v+1}(a,b) - Q_v(a,b)$ is strictly decreasing on $(0,\infty)$, as we required.

(c) Observe that with our notations the relation (3.6) can be written as

$$b[Q_{v+1}(a,b) - Q_v(a,b)] = \chi_{2v+2,a}(b). \tag{3.7}$$

Recently, the second author, by showing that $v \mapsto I_v(x)$ is log-concave on $(-1,\infty)$, deduced that [7, Theorem 1.5] the function $v \mapsto \chi_{v,a}(b)$ is log-concave on $(0,\infty)$ for each $a \geqslant 0$ and $b > 0$. From this we clearly have that the function $v \mapsto \chi_{2v+2,a}(b)$ is log-concave too on $(0,\infty)$. Application of (3.7) yields the asserted result.

(d) The first inequality in (3.2) follows from part (b), while the second inequality in (3.2) follows from the well-known arithmetic–geometric mean value inequality. Now, application of part (c) yields inequality (3.3). Finally, recall that from part (b) of Theorem 2.7, the reliability function $b \mapsto Q_v(a,b)$ is strictly log-concave on $(0,\infty)$ for all $a \geqslant 0$ and $v > 1$. From this we conclude that the function $b \mapsto Q_{v+1}(a,b)$ is strictly log-concave too on $(0,\infty)$ for all $a \geqslant 0$ and $v > 0$. Thus, in view of (3.7), we have that the function

$$b \mapsto \frac{\mathrm{d}}{\mathrm{db}}[\log Q_{v+1}(a,b)] = -\frac{\chi_{2v+2,a}(b)}{Q_{v+1}(a,b)} = b\frac{Q_v(a,b)}{Q_{v+1}(a,b)} - b$$

is strictly decreasing on $(0,\infty)$. Thus, applying again (3.7), it is just straightforward to verify that the inequality (3.4) holds. □

## 4. Concluding remarks

1. We note that the first part of Theorem 3.1, namely the fact that the function $\nu \mapsto Q_\nu(a,b)$ is strictly increasing on $(0,\infty)$ for each fixed $b>0$ and $a \geqslant 0$, can be proved also by an analytical argument. More precisely, the anonymous referee of this paper has communicated to us the following simple proof: since due to Tricomi [18] the incomplete gamma function ratio:

$$\nu \mapsto Q(\nu,x) = \frac{1}{\Gamma(\nu)}\int_x^\infty t^{\nu-1}\mathrm{e}^{-t}\mathrm{d}t$$

is strictly increasing on $(0,\infty)$ for each fixed $x>0$, it follows that the function:

$$\nu \mapsto Q_\nu(a,b) = \mathrm{e}^{-a^2/2}\sum_{n\geqslant 0}\frac{(a^2/2)^n}{n!}Q(\nu+n,b^2/2)$$

is strictly increasing too on $(0,\infty)$ for each $a,b>0$ fixed. The later expansion follows from substituting the power series of the modified Bessel function into the integral in the first line of (1.1). We are grateful to the referee for this important information.
Notice that the above mentioned result of Tricomi on the incomplete gamma function ratio in fact can be deduced from the first part of Theorem 3.1. Namely, since there is a close connection between the gamma and the chi-squared distributions, it is easy to see that $Q(\nu,x) = Q_\nu(0,\sqrt{2x})$ for each $\nu,x>0$ and thus applying part (a) of Theorem 3.1 we obtain that the function $\nu \mapsto Q(\nu,x)$ is indeed strictly increasing on $(0,\infty)$ for each fixed $x>0$. The above relation in turn implies that Tricomi's result implies in fact that the function $\nu \mapsto Q_\nu(a,b)$ is strictly increasing on $(0,\infty)$ for each $a \geqslant 0$ and $b>0$ fixed. With other words the first part of Theorem 3.1 is in fact equivalent with Tricomi's result.
We note also that after we have completed the first draft of this manuscript we have found that a similar analytical proof of the monotonicity of $\nu \mapsto Q_\nu(a,b)$ has been given recently by Mihos et al. [15]. Moreover, a slightly different analytical proof can be found in Ghosh's paper [10, Theorem 1].
2. It is worth mentioning here that the inequality

$$Q^2_{\nu+1}(a,b) \geqslant Q_\nu(a,b)Q_{\nu+2}(a,b) \tag{3.8}$$

it is a little surprising. As we mentioned in the proof of Theorem 3.1 the integrand of $Q_\nu(a,b)$ as a function of $\nu$, i.e. $\nu \mapsto \chi_{2\nu,a}(b)$ is log-concave on $(0,\infty)$, and surprisingly part (d) of Theorem 3.1 states that this log-concavity property remains true after integration, of course with some assumptions on parameters. Moreover, we note that the inequality (3.8) is interesting in its own right, because similar inequalities appears in literature as Turán type inequalities. Turán type inequalities have an extensive literature, and in the last six decades it was proved by several researchers that the most important special functions, orthogonal polynomials satisfies a Turán type inequality. For further details and for a large list of references on this topic, the interested reader is referred to the recent papers [5,7].

Our numerical experiments suggest the following conjecture.

**Conjecture 3.9.** *The function $\nu \mapsto Q_\nu(a,b)$ is strictly log-concave on $(0,\infty)$ for all $a \geqslant 0$ and $b>0$.*


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