New Bounds for the Generalized Marcum Q-Function

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***Abstract*—In this paper, we study the generalized Marcum Q-function $Q_\nu(a,b)$, where $a,\nu > 0$ and $b \geq 0$. Our aim is to extend the results of Corazza and Ferrari (*IEEE Trans. Inf. Theory*, vol. 48, pp. 3003–3008, 2002) to the generalized Marcum Q-function in order to deduce some new tight lower and upper bounds. The key tools in our proofs are some monotonicity properties of certain functions involving the modified Bessel function of the first kind and some classical inequalities, i.e., the Cauchy–Buniakowski–Schwarz and Chebyshev integral inequalities. These bounds are shown to be very tight for large b, i.e., the relative errors of our bounds converge to zero as b increases. Both theoretical analysis and numerical results are provided to show the tightness of our bounds.**

***Index Terms*—Cauchy–Buniakowski–Schwarz and Chebyshev integral inequalities, generalized Marcum Q-function, lower and upper bounds, modified Bessel functions of the first kind.**

I. Introduction

FOR ν unrestricted real (or complex) number, let I_ν be the modified Bessel function of the first kind of order ν, defined by the relation [1, pp. 77]

$$I_\nu(x) = \sum_{k\geq 0} \frac{(x/2)^{2k+\nu}}{k!\Gamma(\nu+k+1)}$$

which is of frequent occurrence in problems of electrical engineering, finite elasticity, quantum billiard, wave mechanics, mathematical physics, and chemistry, to cite a few domains of applications. Here, as usual

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \mathrm{dt}$$

denotes the Euler gamma function, where $x > 0$. Further, let $Q_\nu(a,b)$ be the generalized Marcum Q-function, defined by

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\mathrm{dt} \quad (1)$$

where $b \geq 0$ and $a,\nu > 0$. When $\nu = 1$, the function

$$Q_1(a,b) = \int_b^\infty t e^{-\frac{t^2+a^2}{2}} I_0(at)\mathrm{dt}$$

is known in the literature as the Marcum Q-function. The Marcum Q-function and the generalized Marcum Q-function, defined above, are widely used in radar communications and have important applications in error performance analysis of digital communication problems dealing with partially coherent, differentially coherent, and noncoherent detections [2], [3]. Since, the precise computations of the Marcum Q-function and generalized Marcum Q-function are quite difficult, in the last few decades several authors worked on precise and stable numerical calculation algorithms for the functions. Moreover, many tight lower and upper bounds for the Marcum Q-function and generalized Marcum Q-function were proposed as simpler alternative evaluating methods or intermediate results for further integrations. See, for example, the papers [4]–[12] and the references therein. In this field, the order ν is the number of independent samples of the output of a square-law detector, and hence in most of the papers the authors deduce lower and upper bounds for the generalized Marcum Q-function with order ν integer. However, in our analysis of this paper, ν is not necessarily an integer number. During the revision of this paper, we discovered the work [13], in which similarly to our work, it is also suggested to study lower and upper bounds for the generalized Marcum Q-function of real order $\nu > 0$ and for all ranges of the parameters a, b.

An important contribution to the subject is the publication of Corazza and Ferrari [7], which is the starting point of our present paper. Namely, our main motivation to write this paper is the work [7], which we wish to complement.

This paper is organized as follows.

In Section II, we present some preliminary results which will be useful to deduce lower and upper bounds for the generalized Marcum Q-function. Our main results of this section are some monotonicity properties of some functions which involve the modified Bessel functions of the first kind and the key tools are some classical results of Boyd [14], Gronwall [15], and Nasell [16].

In Section III, we show that all results of Corazza and Ferrari from [7] can be extended to the generalized Marcum Q-function with ν real order. However, the integrals which appear in our study are also quite difficult to compute, and consequently, we apply some classical integral inequalities (Cauchy–Buniakowski–Schwarz and Chebyshev) to have computable lower and upper bounds for $Q_\nu(a,b)$ when ν is an arbitrary real number greater than or equal to $1/2$, and in some cases than 1, respectively. As far as we know, these are the first bounds for the generalized Marcum Q-function with noninteger order. In the case of $\nu = n$ positive integer, we deduce computable lower and upper bounds derived from the general case, and these bounds can be applied without any difficulty to approximate the generalized Marcum Q-function of integer order. Our notation is standard and the basic ideas

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are taken from Corazza and Ferrari's paper [7]. However, we found that there is an incompleteness in the above mentioned paper [7], and in the next section we also clarify this issue in order to have complete proof. More precisely, equation (8) in [7] is stated without proof, even if some computer-generated pictures appear [7, p. 3005], which suggest the validity of that equation. But, computer generated pictures can be misleading and rigorous mathematical proof is required. In part (a) of Lemma 1 below, we point out that in fact [7, eq. (8)] can be deduced easily from a result of Gronwall [15], which was used in wave mechanics. For comments see also Remark 1.

In Section IV, we study the tightness of our bounds and we prove that in the case $b \geq a > 0$ the relative errors of the lower and upper bounds tend to zero as b approaches infinity. This completes also the results from [7] where numerical experiments were given for the case $\nu = 1$. Moreover, we compare our results with the bounds given in [4], [5], [8]. It is shown numerically that our bounds are tighter than the bounds from [4], [5] in most of the cases, and for large values of b are tighter than the bounds given in [8]. Here it is important to note that the relative errors of the bounds given in [8] do not tend to zero as b tends to infinity. For the case $a > b > 0$,, the numerical results show that our bounds are tighter than the bounds from [4], [5] in most of the cases. Moreover, for b there exists a subinterval of $(0, a)$ on which our bounds are tighter than the bounds of Li and Kam [8].

Finally, in Section V, the conclusions of this paper are given.

II. Preliminary Results

It is worth mentioning that the generalized Marcum Q-function has an important interpretation in probability theory, namely, that (as a function of b) is the complement (with respect to unity) of the cumulative distribution function of the noncentral chi-distribution with 2ν degrees of freedom. We note here that in probability theory and in economic theory the complement (with respect to unity) of a cumulative distribution function is called a survival (or a reliability) function. More precisely, the generalized Marcum Q-function, i.e., $b \mapsto Q_\nu(a,b)$ is exactly the reliability function of the noncentral chi-distribution with 2ν degrees of freedom (where ν is not necessarily an integer) and noncentrality parameter a. For more details, we refer to the recent paper [17] where a somewhat new viewpoint of the generalized Marcum Q-function is presented, by showing that it satisfies the new-is-better-than-used property, which arises in economic theory, and to the paper [18] where it is shown that the generalized Marcum function with real order is in fact the tail probability function of a noncentral χ^2-distribution. However, for readers's convenience, we recall here some basic facts. First note that when $a, \nu > 0$ the integrand in (1) is a probability density function. This is well known when $\nu = n$ is an integer. For the sake of completeness, we note that this is also true when ν is not necessarily an integer. For this, consider the Sonine formula [19, eq. 6.631.4]

$$\int_0^\infty J_\nu(at)e^{-pt^2}t^{\nu+1}\mathrm{dt} = \frac{a^\nu e^{-\frac{a^2}{4p}}}{(2p)^{\nu+1}}$$

which holds for all a, p, ν complex numbers such that $\mathrm{Re}(p) > 0, \mathrm{Re}(\nu) > -1$ and where J_ν stands for the Bessel function of the first kind [1, p. 40]. Taking into account the relation $I_\nu(x) = \mathrm{i}^{-\nu}J_\nu(\mathrm{i}x)$ and changing in the above Sonine formula a with $\mathrm{i}a$, we easily get that

$$\int_0^\infty I_\nu(at)e^{-pt^2}t^{\nu+1}\mathrm{dt} = \frac{a^\nu e^{\frac{a^2}{4p}}}{(2p)^{\nu+1}}$$

which implies that for each $\nu, a > 0$ we have

$$Q_\nu(a,0) = \frac{1}{a^{\nu-1}}\int_0^\infty t^\nu e^{-\frac{t^2+a^2}{2}}I_{\nu-1}(at)\mathrm{dt} = 1$$

as required. Thus, for all $b \geq 0$ and $a, \nu > 0$, the function $Q_\nu(a,b)$ can be rewritten as

$$Q_\nu(a,b) = 1 - \frac{1}{a^{\nu-1}}\int_0^b t^\nu e^{-\frac{t^2+a^2}{2}}I_{\nu-1}(at)\mathrm{dt} \qquad (2)$$

and clearly $b \mapsto Q_\nu(a,b)$ maps $[0,\infty)$ into $(0,1]$,; moreover, it is decreasing on $[0,\infty)$.

The following result is one of the crucial facts in the proof of our main results of the following section. In what follows, we denote with $\log x$ the natural logarithm of the real number $x > 0$.

Lemma 1: The following assertions are true:

(a) the function $x \mapsto x^{\nu+1}I_\nu(x)e^{-x}$ is increasing on $(0,\infty)$ for all $\nu \geq 0$;

(b) the function $x \mapsto x^{-\nu}I_\nu(x)e^{-x}$ is strictly decreasing on $(0,\infty)$ for all $\nu \geq -1/2$;

(c) the function $x \mapsto x^{-1}\log I_\nu(x)$ is increasing $(0,\infty)$ for all $\nu \geq 0$.

Proof: (a) In order to prove the asserted result, we show that for all $x > 0$ and $\nu \geq 0$ we have $xI'_\nu(x)/I_\nu(x) > x - 1$. We note that Gronwall [15, eq. 5] has proved a sharper version of this, namely, that $xI'_\nu(x)/I_\nu(x) > x - 1/2$, but just for $\nu \geq 1/2$. As we will see, in our discussion it will be enough that $xI'_\nu(x)/I_\nu(x) > x - 1$ holds for all $\nu \geq 0$. However, it can be shown easily that for $\nu \leq 1/2$, the inequality $xI'_\nu(x)/I_\nu(x) > x - 1/2$ does not hold for all $x > 0$.

Due to Gronwall [15, eq. 17] it is known that the function $\nu \mapsto xI'_\nu(x)/I_\nu(x)$ is increasing on $[0,\infty)$ for each fixed $x > 0$. Hence, it is enough to show that $xI'_0(x)/I_0(x) > x - 1$ holds for all $x > 0$. On the other hand, Nasell [20, eq. 11] has shown that

$$I_\nu(x) < \frac{1 + 2(\nu+1)/x + 2(\nu+1)(\nu+3/2)/x^2}{1 + (\nu+3/2)/x}I_{\nu+1}(x)$$

where $x > 0$ and $\nu > -3/2$. Choosing in the above inequality $\nu = 0$ and taking into account that $I'_0(x) = I_1(x)$, we have

$$\frac{xI'_0(x)}{I_0(x)} = \frac{xI_1(x)}{I_0(x)} > \frac{x + 3/2}{1 + 2/x + 3/x^2} > x - 1$$

for all $x > 0$. Now, using the above result we conclude that

$$\frac{[x^{\nu+1}I_\nu(x)e^{-x}]'}{x^\nu I_\nu(x)e^{-x}} = \frac{xI'_\nu(x)}{I_\nu(x)} + (\nu+1) - x$$
$$> x - 1 + (\nu+1) - x = \nu \geq 0$$

i.e., the function $x \mapsto x^{\nu+1} I_\nu(x) e^{-x}$ is increasing on $(0,\infty)$ for all $\nu \geq 0$, as we required.

(b) In the case $\nu > -1/2$, for a proof see Nasell's work [16, p. 2]. Nasell actually has proved a much stronger statement. More precisely, he proved that for all $x > 0, \nu > -1/2$, and $k \in \{0,1,2,\ldots\}$ the inequality $(-1)^k [x^{-\nu} I_\nu(x) e^{-x}]^{(k)} > 0$ holds [16, eq. 2], i.e., the function $x \mapsto x^{-\nu} I_\nu(x) e^{-x}$ is strictly completely monotonic on $(0,\infty)$. Moreover, the required monotonicity property of the function $x \mapsto x^{-\nu} I_\nu(x) e^{-x}$ for $\nu > -1/2$ has been also verified by Laforgia [21], who used only the Soni inequality $I_\nu(x) > I_{\nu+1}(x)$. However, for the sake of completeness, we give here an alternative proof for the complete monotonicity which does not require Laplace transform and Bernstein theorem. It is known that the function $f_\nu : (0,\infty) \to (1,\infty)$, defined by

$$f_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} I_\nu(x)$$

for $\nu > -1/2$ has the integral representation [22, eq. 3]

$$f_\nu(x) = \int_0^1 e^{x(2u-1)} \varphi(u) \mathrm{du}$$

where

$$\varphi(u) = \frac{\Gamma(2\nu+1)}{\Gamma^2(\nu+1/2)} (u-u^2)^{\nu-1/2}$$

is a Dirichlet density function on the interval J. Here $J=[0,1]$ if $\nu \geq 1/2$ and $J=(0,1)$ otherwise. First suppose that $\nu > -1/2$. Then from the above integral representation, we easily get

$$[f_\nu(x) e^{-x}]^{(k)} = \int_0^1 [2(u-1)]^k e^{2x(u-1)} \varphi(u) \mathrm{du}$$

which implies that

$$(-1)^k [f_\nu(x) e^{-x}]^{(k)} = 2^k \int_0^1 (1-u)^k e^{2x(u-1)} \varphi(u) \mathrm{du} > 0$$

for all $\nu > -1/2$, $k \in \{0,1,2,\ldots\}$, and $x > 0$. Now, consider the case when $\nu = -1/2$. Since $I_{-1/2}(x) = \sqrt{2/\pi} \cdot x^{-1/2} \cosh x$, it is easy to verify that for all $k \in \{0,1,2,\ldots\}$ and $x > 0$ we have

$$\begin{aligned}(-1)^k [f_{-1/2}(x) e^{-x}]^{(k)} &= (-1)^k [e^{-x} \cosh x]^{(k)} \\ &= \frac{1}{2} (-1)^k [e^{-2x}+1]^{(k)} = 2^{k-1} e^{-2x} > 0.\end{aligned}$$

Thus, choosing $k=1$ we obtain $[x^{-\nu} I_\nu(x) e^{-x}] < 0$ for all $\nu \geq -1/2$ and $x > 0$, i.e., the function $x \mapsto x^{-\nu} I_\nu(x) e^{-x}$ is strictly decreasing on $(0,\infty)$ for all $\nu \geq -1/2$.

(c) We note that in fact it is enough to show the assertion for $\nu = 0$. Namely, to prove that the function $x \mapsto x^{-1} \log I_\nu(x)$ is increasing $(0,\infty)$ for all $\nu \geq 0$, we just need to show that for all $x > 0$ and $\nu \geq 0$ we have

$$\frac{d}{dx}\left[\frac{\log I_\nu(x)}{x}\right] = \frac{1}{x^2}\left[\frac{x I_\nu'(x)}{I_\nu(x)} - \log I_\nu(x)\right] \geq 0.$$

But, due to Cochran [23] we know that the function $\nu \mapsto I_\nu(x)$ is strictly decreasing on $[0,\infty)$ for all fixed $x > 0$, and, consequently, $\log I_0(x) \geq \log I_\nu(x)$ holds true for all $\nu \geq 0$ and $x > 0$. On the other hand, appealing again on Gronwall's result [15, eq. 17], which states that the function $\nu \mapsto x I_\nu'(x)/I_\nu(x)$ is increasing on $[0,\infty)$ for each fixed $x > 0$, we have $x I_\nu'(x)/I_\nu(x) \geq x I_0'(x)/I_0(x)$ for all $\nu \geq 0$ and $x > 0$. Summarizing the above facts, we obtain that

$$\nu \mapsto \frac{d}{dx}\left[\frac{\log I_\nu(x)}{x}\right]$$

is increasing on $[0,\infty)$ for all fixed $x > 0$ and consequently in particular we have

$$\frac{1}{x^2}\left[\frac{x I_\nu'(x)}{I_\nu(x)} - \log I_\nu(x)\right] \geq \frac{1}{x^2}\left[\frac{x I_0'(x)}{I_0(x)} - \log I_0(x)\right]. \quad (3)$$

Thus, indeed we just need to show that $x \mapsto x^{-1} \log I_0(x)$ is increasing on $(0,\infty)$. Note that this was proved implicitly by Corazza and Ferrari [7]. However, we present here two somewhat different analytical proofs for this. Since $I_0(0)=1$ and

$$\frac{\log I_0(x)}{x} = \frac{\log I_0(x) - \log I_0(0)}{x-0}$$

in view of the monotone form of l'Hospital's rule [24, Lemma 2.2], to prove that $x \mapsto [\log I_0(x)]/x$ is increasing on $(0,\infty)$ it is enough to show that $x \mapsto [\log I_0(x)]'/x' = I_1(x)/I_0(x)$ is increasing on $(0,\infty)$. But, this is true due to Boyd [14]. Moreover, the function $x \mapsto I_1(x)/I_0(x)$ is in fact strictly increasing. More precisely, since $I_0'(x) = I_1(x)$, the above result is in fact equivalent to the strict log-convexity of the function I_0. This result appears in Neuman's paper [22, Corollary 2], where it is stated that the function I_ν is strictly log-convex on $(0,\infty)$ for all $\nu \in (-1/2, 0]$. From this, $x \mapsto [\log I_0(x)]' = I_1(x)/I_0(x)$ is strictly increasing, and indeed the right-hand side of (3) is positive. With this the proof is complete.

We note that there in another proof for the monotonicity of the function $x \mapsto x^{-1} \log I_0(x)$, which does not require explicitly the monotone form of l'Hospital's rule. Using the definition of convexity, it is easy to verify [25, p. 12] that if $f : [0,\infty) \to [0,\infty)$ is convex and $f(0)=0$, then the function $x \mapsto x^{-1} f(x)$ is increasing on $(0,\infty)$. Now, Boyd's result stated above implies that $x \mapsto \log I_0(x)$ is convex on $(0,\infty)$, i.e., I_0 is log-convex. On the other hand, $\log I_0(0) = 0$, thus using the above result for $f = \log I_0$, the function $x \mapsto x^{-1} \log I_0(x)$ is indeed increasing on $(0,\infty)$.

Finally, we would like to note that in fact from the monotone form of l'Hospital rule the above result [25, p. 12] can be deduced. Namely, since $f(0)=0$, to prove that $x \mapsto f(x)/x = [f(x)-f(0)]/[x-0]$ is increasing, it is enough to show that $x \mapsto f'(x)/x' = f'(x)$ is increasing, which is clearly true, because f is convex. □

III. Lower and Upper Bounds for the Generalized Marcum Q-Function

In this section, we establish some new tight lower and upper bounds for the generalized Marcum Q-function by using the results of Lemma 1. These bounds are natural extensions of the bounds stated in [7] and the proofs of Theorem 1 and 2 are similar to those given in [7]. In the followings, erfc stands for the complementary error function, which is defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\mathrm{dt}.$$

Moreover, for all $\lambda > 0$ and x real, let

$$\Gamma(\lambda, x) = \int_x^\infty t^{\lambda-1} e^{-t}\mathrm{dt}$$

denotes, as usual, the upper incomplete gamma function. Notice that for all x real $\Gamma(1/2, x^2) = \sqrt{\pi}\cdot \mathrm{erfc}(x)$.

Our first main result reads as follows.

Theorem 1: If $\nu \geq 1$ and $b \geq a > 0$, then the following inequalities hold:

$$Q_\nu(a,b) \geq \sqrt{\frac{\pi}{2}} \frac{b^\nu I_{\nu-1}(ab)}{a^{\nu-1}e^{ab}} \mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \quad (4)$$

$$Q_\nu(a,b) \leq \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}e^{ab}} \int_{b-a}^\infty (u+a)^{2\nu-1} e^{-\frac{u^2}{2}}\mathrm{du}. \quad (5)$$

Moreover, the inequality (5) holds true for all $\nu \geq 1/2$.

Proof: First let us focus on the lower bound in (4). Since from part (a) of Lemma 1 the function $x \mapsto x^{\nu+1} I_\nu(x) e^{-x}$ is increasing on $(0,\infty)$, for all $t \geq b$ and $\nu \geq 0$ we have

$$I_\nu(t) \geq \frac{e^t b^{\nu+1}}{e^b t^{\nu+1}} I_\nu(b). \quad (6)$$

Replacing in (6) t with at, b with ab, and ν with $\nu-1$, from (1) we obtain

$$\begin{aligned} Q_\nu(a,b) &= \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\mathrm{dt} \\ &\geq \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} \frac{e^{at}(ab)^\nu}{e^{ab}(at)^\nu} I_{\nu-1}(ab)\mathrm{dt} \\ &= \frac{b^\nu I_{\nu-1}(ab)}{a^{\nu-1}e^{ab}} \int_b^\infty e^{-\frac{(t-a)^2}{2}}\mathrm{dt} \\ &= \frac{b^\nu I_{\nu-1}(ab)}{a^{\nu-1}e^{ab}} \int_{b-a}^\infty e^{-\frac{u^2}{2}}\mathrm{du} \\ &= \sqrt{\frac{\pi}{2}} \frac{b^\nu I_{\nu-1}(ab)}{a^{\nu-1}e^{ab}} \mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right). \end{aligned}$$

Now, let us concentrate on the upper bound in (5). From part (b) of Lemma 1, the function $x \mapsto x^{-\nu} e^{-x} I_\nu(x)$ is decreasing on $(0,\infty)$. Thus, for all $t \geq b$ and $\nu \geq -1/2$, we have

$$I_\nu(t) \leq \frac{e^t t^\nu}{e^b b^\nu} I_\nu(b). \quad (7)$$

Replacing in (7) t with at, b with ab, and ν with $\nu-1$, and using again (1), we get that

$$\begin{aligned} Q_\nu(a,b) &= \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\mathrm{dt} \\ &\leq \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} \frac{e^{at}(at)^{\nu-1}}{e^{ab}(ab)^{\nu-1}} I_{\nu-1}(ab)\mathrm{dt} \\ &= \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}e^{ab}} \int_b^\infty t^{2\nu-1} e^{-\frac{(t-a)^2}{2}}\mathrm{dt} \\ &= \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}e^{ab}} \int_{b-a}^\infty (u+a)^{2\nu-1} e^{-\frac{u^2}{2}}\mathrm{du}. \end{aligned}$$

□

It is worth mentioning that the upper bound of (5), even if it seems to be very tight, is not very useful, because the computation of the integral, which is on the right-hand side of (5), appears to be difficult. In other words, the upper bound derived in Theorem 1 cannot be used in its current form, since it requires numerical integration. This stems from the fact that numerical approximations can lie below or above the exact values of the integral. However, in particular when $\nu = n$ is an integer, easy computations yield the following result, which is an immediate consequence of Theorem 1. In the followings, as usual, for $k \in \{0,1,2,\ldots,n\}$

$$C_n^k = \frac{n!}{(n-k)!k!}$$

is the binomial coefficient and we use the familiar notations $(2k)!! = 2\cdot 4\cdot\cdots\cdot(2k)$ and $(2k-1)!! = 1\cdot 3\cdot\cdots\cdot(2k-1)$.

Corollary 1: If $n \in \{1,2,3,\ldots\}$ and $b \geq a > 0$, then the inequalities

$$Q_n(a,b) \geq \frac{b^n I_{n-1}(ab)}{a^{n-1}e^{ab}} A_0(b-a) \quad (8)$$

$$Q_n(a,b) \leq \frac{I_{n-1}(ab)}{(ab)^{n-1}e^{ab}} \sum_{j=0}^{2n-1} C_{2n-1}^j a^j A_{2n-j-1}(b-a) \quad (9)$$

hold, where for all $m \in \{0,1,2,\ldots\}$

$$A_m(\alpha) = \int_\alpha^\infty u^m e^{-\frac{u^2}{2}}\mathrm{du} = 2^{\frac{m-1}{2}} \Gamma\left(\frac{m+1}{2}, \frac{\alpha^2}{2}\right)$$

which can be rewritten as follows:

$$A_0(\alpha) = \sqrt{\frac{\pi}{2}} \mathrm{erfc}\left(\frac{\alpha}{\sqrt{2}}\right), \quad A_1(\alpha) = e^{-\frac{\alpha^2}{2}}$$

and for all $k \in \{1,2,3,\ldots\}$

$$A_{2k}(\alpha) = e^{-\frac{\alpha^2}{2}} \sum_{i=1}^k \alpha^{2i-1} \frac{(2k-1)!!}{(2i-1)!!} + (2k-1)!! A_0(\alpha) \quad (10)$$

$$A_{2k+1}(\alpha) = e^{-\frac{\alpha^2}{2}} \sum_{i=1}^k \alpha^{2i} \frac{(2k)!!}{(2i)!!} + (2k)!! A_1(\alpha). \quad (11)$$

Proof: Choosing $\nu = n$ in (4) and (5), the lower bound in (8) is clear, while for the upper bound in (9) we need to evaluate the integral

$$\int_{b-a}^\infty (u+a)^{2n-1} e^{-\frac{u^2}{2}}\mathrm{du}.$$

Using the Newton binomial formula, we conclude that

$$\begin{aligned} &\int_{b-a}^\infty (u+a)^{2n-1} e^{-u^2/2}\mathrm{du} \\ &\quad= \int_{b-a}^\infty \left[\sum_{j=0}^{2n-1} C_{2n-1}^j u^{2n-j-1} a^j\right] e^{-\frac{u^2}{2}}\mathrm{du} \\ &\quad= \sum_{j=0}^{2n-1} C_{2n-1}^j a^j \left[\int_{b-a}^\infty u^{2n-j-1} e^{-\frac{u^2}{2}}\mathrm{du}\right] \\ &\quad= \sum_{j=0}^{2n-1} C_{2n-1}^j a^j A_{2n-j-1}(b-a) \end{aligned}$$

as we requested. It remains just to compute the coefficients $A_m(\alpha)$. Clearly, we have

$$A_0(\alpha) = \sqrt{\frac{\pi}{2}} \mathrm{erfc}\left(\frac{\alpha}{\sqrt{2}}\right) \quad \text{and} \quad A_1(\alpha) = e^{-\frac{\alpha^2}{2}}.$$

On the other hand, partial integration yields

$$A_m(\alpha) = e^{-\frac{\alpha^2}{2}}\alpha^{m-1} + (m-1)A_{m-2}(\alpha)$$

and consequently for all $k \in \{1, 2, 3, \ldots\}$, one has

$$A_{2k}(\alpha) = e^{-\frac{\alpha^2}{2}}\alpha^{2k-1} + (2k-1)A_{2k-2}(\alpha)$$

and

$$A_{2k+1}(\alpha) = e^{-\frac{\alpha^2}{2}}\alpha^{2k} + (2k)A_{2k-1}(\alpha).$$

From these relations, it is just straightforward to verify that the relations (10) and (11) hold. □

Remark 1: We note that from (8) and (9) for $n = 1$, we re-obtain the results of Corazza and Ferrari [7]

$$\frac{bI_0(ab)}{e^{ab}}A_0(b-a) \le Q_1(a,b) \le \frac{I_0(ab)}{e^{ab}}\sum_{j=0}^{1} C_1^j a^j A_{1-j}(b-a)$$

i.e.,

$$Q_1(a,b) \ge \sqrt{\frac{\pi}{2}}\frac{bI_0(ab)}{e^{ab}}\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right)$$

and

$$Q_1(a,b) \le \frac{I_0(ab)}{e^{ab}}\left[e^{-\frac{(b-a)^2}{2}} + a\sqrt{\frac{\pi}{2}}\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right)\right].$$

In other words, the lower bound in (8) is a natural extension of (9) from [7], while the upper bound in (9) is a natural generalization of (7) from [7]. It is important to note that Corazza and Ferrari, in order to deduce the above lower bound [7, eq. 9], have used, taking into account some numerical experiments, the fact that the function $x \mapsto xI_0(x)e^{-x}$ is increasing on $(0,\infty)$, i.e., for all $t \ge b > 0$ the inequality $tI_0(t)e^{-t} \ge bI_0(b)e^{-b}$ holds. But the above inequality is stated in [7, eq. 8] without proof. However, part (a) of Lemma 1 guarantees that indeed the function $x \mapsto xI_0(x)e^{-x}$ is increasing on $(0,\infty)$.

The following theorem completes the results from Theorem 1. In this case, since the integration domains are finite intervals, in addition we deduce computable lower and upper bounds for the generalized Marcum Q-function for ν real. The key tools in our proofs are two classical integral inequalities.

Theorem 2: If $\nu \ge 1$, then the following inequalities hold:

$$Q_\nu(a,b) \ge 1 - \frac{e^{\frac{\zeta_\nu^2 - a^2}{2}}}{a^{\nu-1}}\int_{-\zeta_\nu}^{b-\zeta_\nu}(u+\zeta_\nu)^\nu e^{-\frac{u^2}{2}}\mathrm{du} \tag{12}$$

$$Q_\nu(a,b) \le 1 - \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}e^{ab}}\int_{-a}^{b-a}(u+a)^{2\nu-1}e^{-\frac{u^2}{2}}\mathrm{du} \tag{13}$$

where $a > b > 0$ and $\zeta_\nu = b^{-1}\log I_{\nu-1}(ab)$. Moreover, (13) holds true for all $\nu \ge 1/2$. Consequently, for all $\nu \ge 1$ and $a > b > 0$ we have

$$Q_\nu(a,b) \ge 1 - \frac{e^{\frac{\zeta_\nu^2 - a^2}{2}}}{a^{\nu-1}}\sqrt{\frac{b^{2\nu+1}}{2\nu+1}\frac{\sqrt{\pi}}{2}[\mathrm{erfc}(-\zeta_\nu) - \mathrm{erfc}(b-\zeta_\nu)]} \tag{14}$$

and in addition the inequality

$$Q_\nu(a,b) \le 1 - \frac{b^\nu I_{\nu-1}(ab)}{2\nu a^{\nu-1}e^{ab}}\sqrt{\frac{\pi}{2}}\left[\mathrm{erfc}\left(\frac{-a}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right)\right] \tag{15}$$

holds for all $\nu \ge 1/2$.

Proof: In order to prove (12), we use part (c) of Lemma 1 and conclude that for all $t \in (0,b]$ and $\nu \ge 0$ we have $b\log I_\nu(t) \le t\log I_\nu(b)$. Replacing in this inequality t with at, b with ab, and ν with $\nu - 1$, one has $b\log I_{\nu-1}(at) \le t\log I_{\nu-1}(ab)$, which holds for all $t \in (0,b]$ and $\nu \ge 1$. This implies that

$$I_{\nu-1}(at) \le e^{\zeta_\nu t} \tag{16}$$

holds for all $t \in [0,b]$ and $\nu \ge 1$. Using (2) and (16) yields

$$\begin{aligned} Q_\nu(a,b) &= 1 - \frac{1}{a^{\nu-1}}\int_0^b t^\nu e^{-\frac{t^2+a^2}{2}}I_{\nu-1}(at)\mathrm{dt} \\ &\ge 1 - \frac{1}{a^{\nu-1}}\int_0^b t^\nu e^{-\frac{t^2+a^2}{2}}e^{\zeta_\nu t}\mathrm{dt} \\ &= 1 - \frac{e^{\frac{\zeta_\nu^2 - a^2}{2}}}{a^{\nu-1}}\int_0^b t^\nu e^{-\frac{(t-\zeta_\nu)^2}{2}}\mathrm{dt} \\ &= 1 - \frac{e^{\frac{\zeta_\nu^2 - a^2}{2}}}{a^{\nu-1}}\int_{-\zeta_\nu}^{b-\zeta_\nu}(u+\zeta_\nu)^\nu e^{-\frac{u^2}{2}}\mathrm{du}. \end{aligned}$$

To prove the inequality (13), we proceed exactly as in the proof of Theorem 1. Using again the fact that the function $x \mapsto x^{-\nu}e^{-x}I_\nu(x)$ is decreasing on $(0,\infty)$, it follows that for all $t \in [0,b]$ and $\nu \ge -1/2$ we have

$$I_\nu(t) \ge \frac{e^t t^\nu}{e^b b^\nu}I_\nu(b). \tag{17}$$

Replacing in (17) t with at, b with ab, and ν with $\nu - 1$ and in view of (2), we get that

$$\begin{aligned} Q_\nu(a,b) &= 1 - \frac{1}{a^{\nu-1}}\int_0^b t^\nu e^{-\frac{t^2+a^2}{2}}I_{\nu-1}(at)\mathrm{dt} \\ &\le 1 - \frac{1}{a^{\nu-1}}\int_0^b t^\nu e^{-\frac{t^2+a^2}{2}}\frac{e^{at}(at)^{\nu-1}}{e^{ab}(ab)^{\nu-1}}I_{\nu-1}(ab)\mathrm{dt} \\ &= 1 - \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}e^{ab}}\int_0^b t^{2\nu-1}e^{-\frac{(t-a)^2}{2}}\mathrm{dt} \\ &= 1 - \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}e^{ab}}\int_{-a}^{b-a}(u+a)^{2\nu-1}e^{-\frac{u^2}{2}}\mathrm{du}. \end{aligned}$$

Now, we are going to prove (14). For this, let us recall the Cauchy–Buniakowski–Schwarz inequality [26, p. 43]

$$\left[\int_c^d f(u)g(u)\mathrm{du}\right]^2 \le \left[\int_c^d f^2(u)\mathrm{du}\right]\left[\int_c^d g^2(u)\mathrm{du}\right]$$

where f and g are real and integrable functions on $[c,d]$. Thus, using the Cauchy–Buniakowski–Schwarz inequality, we easily obtain that

$$\begin{aligned} &\int_{-\zeta_\nu}^{b-\zeta_\nu}(u+\zeta_\nu)^\nu e^{-\frac{u^2}{2}}\mathrm{du} \\ &\quad\le \left[\int_{-\zeta_\nu}^{b-\zeta_\nu}(u+\zeta_\nu)^{2\nu}\mathrm{du}\right]^{1/2}\left[\int_{-\zeta_\nu}^{b-\zeta_\nu}e^{-u^2}\mathrm{du}\right]^{1/2} \end{aligned}$$

$$= \left\{ \frac{b^{2\nu+1}}{2\nu+1} \frac{\sqrt{\pi}}{2} \left[\mathrm{erfc}(-\zeta_\nu) - \mathrm{erfc}(b-\zeta_\nu) \right] \right\}^{1/2}$$

and in view of (12) this in turn implies that (14) holds.

In order to prove (15), we use the well-known Chebyshev integral inequality [26, p. 40] which reads as follows: if f and g are integrable functions defined on $[c,d]$, both increasing or both decreasing and p is a positive integrable function defined on $[c,d]$, then

$$\int_c^d p(u)f(u)\mathrm{d}u \int_c^d p(u)g(u)\mathrm{d}u \leq \int_c^d p(u)\mathrm{d}u \int_c^d p(u)f(u)g(u)\mathrm{d}u.$$

Note that if one of the functions f or g is decreasing and the other is increasing, then the above inequality is reversed. Since the function $u \mapsto f(u) = (u+a)^{2\nu-1}$ is increasing on $[-a, b-a]$ for all $\nu \geq 1/2$ and the function $u \mapsto g(u) = e^{-u^2/2}$ is increasing too on $[-a, b-a]$, from Chebyshev's inequality we deduce that

$$\begin{aligned} &\int_{-a}^{b-a} (u+a)^{2\nu-1} e^{-\frac{u^2}{2}} \mathrm{d}u \\ &\quad \geq \frac{1}{b} \int_{-a}^{b-a} (u+a)^{2\nu-1} \mathrm{d}u \int_{-a}^{b-a} e^{-\frac{u^2}{2}} \mathrm{d}u \\ &\quad = \frac{1}{b} \frac{b^{2\nu}}{2\nu} \sqrt{\frac{\pi}{2}} \left[\mathrm{erfc}\left(-\frac{a}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \right] \end{aligned}$$

where we used $p(u) \equiv 1$ for all $u \in [-a, b-a]$. Finally, using (13), the proof is complete. □

It is important to note here that since the bounds in (14) and (15) were deduced from (12) and (13), clearly they are weaker than the bounds given in (12) and (13). However, the bounds (14) and (15) can be computed without any difficulty and can be used to approximate the generalized Marcum Q-function $Q_\nu(a,b)$ for each $\nu \geq 1$, and $\nu \geq 1/2$, respectively. As far as we know, no previous lower and upper bounds have been introduced for ν noninteger or nonrational.

The following result is an immediate consequence of Theorem 2 and provides a generalization of the results of Corazza and Ferrari [7, eqs. 12 and 14].

Corollary 2: If $n \in \{1,2,3,\ldots\}$ and $a > b > 0$, then the inequalities

$$Q_n(a,b) \geq 1 - \frac{e^{\frac{\zeta_n^2 - a^2}{2}}}{a^{n-1}} \sum_{j=0}^{n} C_n^j \zeta_n^j B_{n-j}(\zeta_n), \quad (18)$$

$$Q_n(a,b) \leq 1 - \frac{I_{n-1}(ab)}{(ab)^{n-1} e^{ab}} \sum_{j=0}^{2n-1} C_{2n-1}^j a^j B_{2n-j-1}(a) \quad (19)$$

hold, where $\zeta_n = b^{-1} \log I_{n-1}(ab)$ and for all $m \in \{0,1,2,\ldots\}$

$$B_m(\alpha) = \int_{-\alpha}^{b-\alpha} u^m e^{-\frac{u^2}{2}} \mathrm{d}u$$

which can be rewritten as

$$B_0(\alpha) = \sqrt{\frac{\pi}{2}} \left[\mathrm{erfc}\left(-\frac{\alpha}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{b-\alpha}{\sqrt{2}}\right) \right]$$
$$B_1(\alpha) = e^{-\frac{\alpha^2}{2}} - e^{-\frac{(b-\alpha)^2}{2}}$$

and for all $k \in \{1,2,3,\ldots\}$

$$B_{2k}(\alpha) = -\sum_{i=1}^{k} \Delta_{2i-1}(b,\alpha) \frac{(2k-1)!!}{(2i-1)!!} + (2k-1)!! B_0(\alpha) \quad (20)$$

$$B_{2k+1}(\alpha) = \sum_{i=1}^{k} \Delta_{2i}(b,\alpha) \frac{(2k)!!}{(2i)!!} + (2k)!! B_1(\alpha) \quad (21)$$

where

$$\Delta_i(b,\alpha) = e^{-\frac{\alpha^2}{2}} \alpha^i - (-1)^i e^{-\frac{(b-\alpha)^2}{2}} (b-\alpha)^i$$

for all $i \in \{1,2,3,\ldots,k\}$.

Proof: Choosing $\nu = n$ in (12) and (13), we just need to compute the integrals

$$\int_{-\zeta_n}^{b-\zeta_n} (u+\zeta_n)^n e^{-\frac{u^2}{2}} \mathrm{d}u \quad \text{and} \quad \int_{-a}^{b-a} (u+a)^{2n-1} e^{-\frac{u^2}{2}} \mathrm{d}u.$$

Thus, using again the Newton binomial formula we conclude that

$$\begin{aligned} &\int_{-\zeta_n}^{b-\zeta_n} (u+\zeta_n)^n e^{-\frac{u^2}{2}} \mathrm{d}u \\ &\quad = \int_{-\zeta_n}^{b-\zeta_n} \left[\sum_{j=0}^{n} C_n^j u^{n-j} \zeta_n^j \right] e^{-\frac{u^2}{2}} \mathrm{d}u \\ &\quad = \sum_{j=0}^{n} C_n^j \zeta_n^j \left[\int_{-\zeta_n}^{b-\zeta_n} u^{n-j} e^{-\frac{u^2}{2}} \mathrm{d}u \right] \\ &\quad = \sum_{j=0}^{n} C_n^j \zeta_n^j B_{n-j}(\zeta_n) \end{aligned}$$

and

$$\begin{aligned} &\int_{-a}^{b-a} (u+a)^{2n-1} e^{-\frac{u^2}{2}} \mathrm{d}u \\ &\quad = \int_{-a}^{b-a} \left[\sum_{j=0}^{2n-1} C_{2n-1}^j u^{2n-j-1} a^j \right] e^{-\frac{u^2}{2}} \mathrm{d}u \\ &\quad = \sum_{j=0}^{2n-1} C_{2n-1}^j a^j \left[\int_{-a}^{b-a} u^{2n-j-1} e^{-\frac{u^2}{2}} \mathrm{d}u \right] \\ &\quad = \sum_{j=0}^{2n-1} C_{2n-1}^j a^j B_{2n-j-1}(a). \end{aligned}$$

Now, it remains to evaluate the coefficients $B_m(\alpha)$. Clearly, we have

$$B_0(\alpha) = \sqrt{\frac{\pi}{2}} \left[\mathrm{erfc}\left(-\frac{\alpha}{\sqrt{2}}\right) - \mathrm{erfc}\left(\frac{b-\alpha}{\sqrt{2}}\right) \right]$$

and

$$B_1(\alpha) = e^{-\frac{\alpha^2}{2}} - e^{-\frac{(b-\alpha)^2}{2}}.$$

On the other hand, partial integration yields

$$B_m(\alpha) = (-1)^{m-1}\Delta_{m-1}(b,\alpha) + (m-1)B_{m-2}(\alpha),$$

and, consequently for all $k \in \{1,2,3,\ldots\}$, one has

$$B_{2k}(\alpha) = -\Delta_{2k-1}(b,\alpha) + (2k-1)B_{2k-2}(\alpha)$$

and

$$B_{2k+1}(\alpha) = \Delta_{2k}(b,\alpha) + (2k)B_{2k-1}(\alpha).$$

From these relations it is just straightforward to verify that the relations (20) and (21) holds. □

Remark 2: It is worth mentioning that, from (19) in particular for $n = 1$,, we get that

$$\begin{aligned} Q_1(a,b) &\le 1 - \frac{I_0(ab)}{e^{ab}} \sum_{j=0}^{1} C_1^j a^j B_{1-j}(a) \\ &= 1 - \frac{I_0(ab)}{e^{ab}} \left[C_1^0 a^0 B_1(a) + C_1^1 a^1 B_0(a)\right] \\ &= 1 - \frac{I_0(ab)}{e^{ab}} \left\{ e^{-\frac{a^2}{2}} - e^{-\frac{(b-a)^2}{2}} \right. \\ &\quad \left. + a\sqrt{\frac{\pi}{2}}\left[\operatorname{erfc}\left(-\frac{a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right)\right]\right\} \end{aligned}$$

which was proved recently by Corazza and Ferrari [7, eq. 12].

On the other hand, from (18) in particular when $n = 1$ for all $a > b > 0$,, we have the following inequality [7, eq. 14]

$$\begin{aligned} Q_1(a,b) &\ge 1 - e^{-\frac{a^2-\zeta^2}{2}} \sum_{j=0}^{1} C_1^j \zeta^j B_{1-j}(\zeta) \\ &= 1 - e^{-\frac{a^2-\zeta^2}{2}} \left[C_1^0 \zeta^0 B_1(\zeta) + C_1^1 \zeta^1 B_0(\zeta)\right] \\ &= 1 - e^{-\frac{a^2-\zeta^2}{2}} \left\{ e^{-\frac{\zeta^2}{2}} - e^{-\frac{(b-\zeta)^2}{2}} \right. \\ &\quad \left. + \zeta\sqrt{\frac{\pi}{2}}\left[\operatorname{erfc}\left(-\frac{\zeta}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b-\zeta}{\sqrt{2}}\right)\right]\right\} \end{aligned}$$

where $\zeta = \zeta_1 = b^{-1}\log I_0(ab)$.

IV. Tightness of the Bounds and Numerical Results

In this section, we will study the tightness of our proposed bounds for the generalized Marcum Q-function, and compare our bounds with the bounds previously introduced in the literature.

A. Tightness of the Bounds

First, let us discuss the tightness of the bounds stated in Theorem 1 and Corollary 1. Let $a > 0$ and $\nu \ge 1$ be fixed. Recall that the function $b \mapsto Q_\nu(a,b)$ is a survival function and then we have $Q_\nu(a,b) \to 0$ as $b \to \infty$. Moreover, it is easy to see that the right-hand side of (4) tends also to zero as b approaches infinity. On the other hand, it is known that the asymptotic formula [27, p. 377]

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left[1 - \frac{4\nu^2-1}{1!(8x)} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8x)^2} - \cdots\right]$$

holds for large values of x and for fixed $\nu > -1$. Using this, it follows that

$$\lim_{b\to\infty} \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}e^{ab}} = 0$$

and then the right-hand side of (5) also tends to zero as b tends to infinity. These show that the lower and upper bounds (4) and (5) as well as (8) and (9) are tight as b tends to infinity.

Now, let us study the relative errors of the lower and upper bounds (4) and (5), and in particular (8) and (9). For this, let us consider the expressions

$$L_\nu(a,b) = \sqrt{\frac{\pi}{2}}\frac{b^\nu I_{\nu-1}(ab)}{a^{\nu-1}e^{ab}}\operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right)$$

and

$$U_\nu(a,b) = \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}e^{ab}}\int_{b-a}^{\infty}(u+a)^{2\nu-1}e^{-\frac{u^2}{2}}\mathrm{du}.$$

Due to Theorem 1 we know that if $b \ge a > 0$ and $\nu \ge 1$, we have $L_\nu(a,b) \le Q_\nu(a,b) \le U_\nu(a,b)$. In what follows, we show that if $\nu \ge 1$ and $a > 0$ are fixed, the relative errors

$$\frac{L_\nu(a,b) - Q_\nu(a,b)}{Q_\nu(a,b)} \quad \text{and} \quad \frac{U_\nu(a,b) - Q_\nu(a,b)}{Q_\nu(a,b)}$$

tend to zero as b approaches infinity. Observe that to prove this it is enough to show that $L_\nu(a,b)/Q_\nu(a,b)$ and $U_\nu(a,b)/Q_\nu(a,b)$ tend to 1 as b tends to infinity. On the other hand, it is known that for $a > 0$ and $\nu > 0$ fixed the asymptotic formula [3, p. 81]

$$Q_\nu(a,b) \sim \frac{(b/a)^{\nu-1/2}}{\sqrt{2\pi}}\int_{b-a}^{\infty} e^{-\frac{u^2}{2}}\mathrm{du} \tag{22}$$

is valid for large values of b. Thus, we have

$$\lim_{b\to\infty}\frac{L_\nu(a,b)}{Q_\nu(a,b)} = \lim_{b\to\infty}\frac{\sqrt{2\pi ab}}{e^{ab}}I_{\nu-1}(ab) = 1$$

where we have used the above asymptotic formula for the modified Bessel function of the first kind. Moreover, using this result and the l'Hospital rule we have

$$\begin{aligned} &\lim_{b\to\infty}\frac{U_\nu(a,b)}{Q_\nu(a,b)} \\ &\quad = \lim_{b\to\infty}\frac{\int_{b-a}^{\infty}(u+a)^{2\nu-1}e^{-\frac{u^2}{2}}\mathrm{du}}{b^{2\nu-1}\int_{b-a}^{\infty}e^{-\frac{u^2}{2}}\mathrm{du}} \\ &\quad = \lim_{b\to\infty}\frac{-b^{2\nu-1}e^{-\frac{(b-a)^2}{2}}}{(2\nu-1)b^{2\nu-2}\int_{b-a}^{\infty}e^{-\frac{u^2}{2}}\mathrm{du} - b^{2\nu-1}e^{-\frac{(b-a)^2}{2}}} \\ &\quad = 1. \end{aligned}$$

Here, we have used that from l'Hospital's rule

$$\lim_{b\to\infty} b^{2\nu-1}\int_{b-a}^{\infty}e^{-\frac{u^2}{2}}\mathrm{du} = \frac{1}{2\nu-1}\lim_{b\to\infty}\frac{b^{2\nu}}{e^{\frac{(b-a)^2}{2}}} = 0$$

and

$$\lim_{b\to\infty}\frac{\int_{b-a}^{\infty}e^{-\frac{u^2}{2}}\mathrm{du}}{be^{-\frac{(b-a)^2}{2}}} = 0.$$

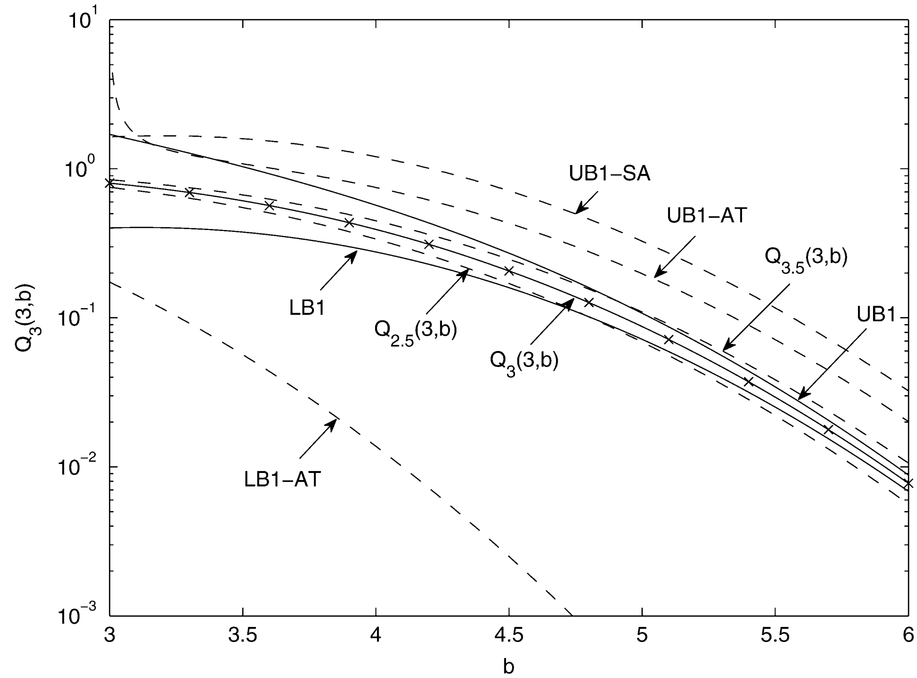

Fig. 1. Numerical results for $Q_\nu(a,b)$ of integer order ν and its upper and lower bounds versus b for the case $b > a = 3$ and $\nu = 3$. x: exact. Dashed line: previous bounds. Solid line: our proposed bounds, i.e., LB1 and UB1.

With this, actually we have proved that the relative errors of the bounds of Corazza and Ferrari [7] (see Remark 1) tend also to zero as b approaches infinity. This is in the agreement with [7, Table IV, p. 3007]. Summarizing, our bounds have preserved the property that their relative errors tend to zero as b tends to infinity. But the previously introduced bounds $Q_{\nu-0.5}(a,b)$ and $Q_{\nu+0.5}(a,b)$ by Li and Kam do not possess this property. For example, by using the asymptotic formula (22) it is clear that $Q_{\nu+0.5}(a,b)/Q_\nu(a,b) \to \infty$ as $b \to \infty$ and $a, \nu > 0$ are fixed. In other words, the relative error of the upper bound stated in [8] does not tends to zero as b approaches infinity. Similarly, using (22) we have that $Q_{\nu-0.5}(a,b)/Q_\nu(a,b) \to 0$ as $b \to \infty$ and $a, \nu > 0$ are fixed. Thus, the relative error of the lower bound stated in [8] does not tend to zero as b approaches infinity. To the extent of the authors' knowledge, our proposed bounds $L_\nu(a,b)$ and $U_\nu(a,b)$ are the first bounds of $Q_\nu(a,b)$ with order $\nu > 1$ that tend to be equal to the exact value as b increases.

Now let us focus on tightness of the bounds (12), (13), (18), and (19). Let us fix $a > 0$ and $\nu \geq 1$. Observe first that, since $b \mapsto Q_\nu(a,b)$ is a reliability function, if $b \to 0$ then $Q_\nu(a,b) \to 1$. Moreover, it is easy to see that the second terms on the right-hand sides of (12) and (13) tend also to zero as b tends to zero. These show that the bounds (12), (13), (18), and (19) are sharp as $b \to 0$. A similar, but somewhat more technical, argument to that presented above for the case $b \geq a > 0$ yields that the ratio of the bounds in (13) and (12) tends to 1 as $b \to 0$, which shows that the relative errors of the bounds (12), (13), (18), and (19) tend also to zero as $b \to 0$.

B. Numerical Results

First, let us consider the case when $b \geq a > 0$. For this case, we have derived one lower bound for real order ν, i.e., (4) and (8) in the integer order form, and also one upper bound for integer order ν, i.e., (9). We denote them as LB1 and UB1, respectively. The previous lower bounds for $Q_\nu(a,b)$ of integer order ν include LB1-AT in [5, the first line in eq. 18],[1] $Q_{\nu-0.5}(a,b)$ in [8, eqs. 11 and 14]. The previous upper bounds include UB1-SA in [4, eq. 8], UB1-AT in [5, eq. 17], $Q_{\nu+0.5}(a,b)$ in [8, eqs. 11 and 14]. Fig. 1 presents numerical results for the case $b > a = 3$ and $\nu = 3$ in a logarithmic scale. It shows that our proposed bounds are looser than the bounds proposed in [8] when b is relative small. However, our bounds are tighter than the other bounds for large b when ν and a are fixed, and the tightness of our bounds improves quickly for increasing b. Moreover, in the previous subsection we have proved that the relative errors of our bounds converge to zero as b tends to infinity, while the bounds of Li and Kam do not possess this property. While LB1 is also the lower bound of the generalized Marcum Q-function with noninteger order, the simulation results are similar with the integer order case, and therefore are omitted.

For the case $b < a$, we derived a pair of lower and upper bounds for integer order generalized Marcum Q-function, i.e., (18) and (19), which are denoted by LB2 and UB2, respectively. Moreover, we also proposed a pair of lower and upper bounds for the generalized Marcum Q-function with noninteger order ν, i.e., (14) and (15), which are denoted by LB3 and UB3, respectively.

We first consider the case of integer order ν. The previous lower bounds include LB2-SA in [4, eq. 12], LB1-AT in [5, the first line in eq. 18], LB2-AT in [5, eq. 20], LB3-AT in [5, eq. 21], $Q_{\nu-0.5}(a,b)$ in [8, eqs. 11 and 14]. And there is an upper bound previously introduced in the literature, i.e., $Q_{\nu+0.5}(a,b)$ in [8, eqs. 11 and 14]. The numerical results for the case $b < a = 3$ and $\nu = 3$ are illuminated in Fig. 2. We can see that our bounds

[1]There is a mistake in the formula of $Q_\nu(0,b)$ given in [5]. For a correct version, the readers are referred to [3].

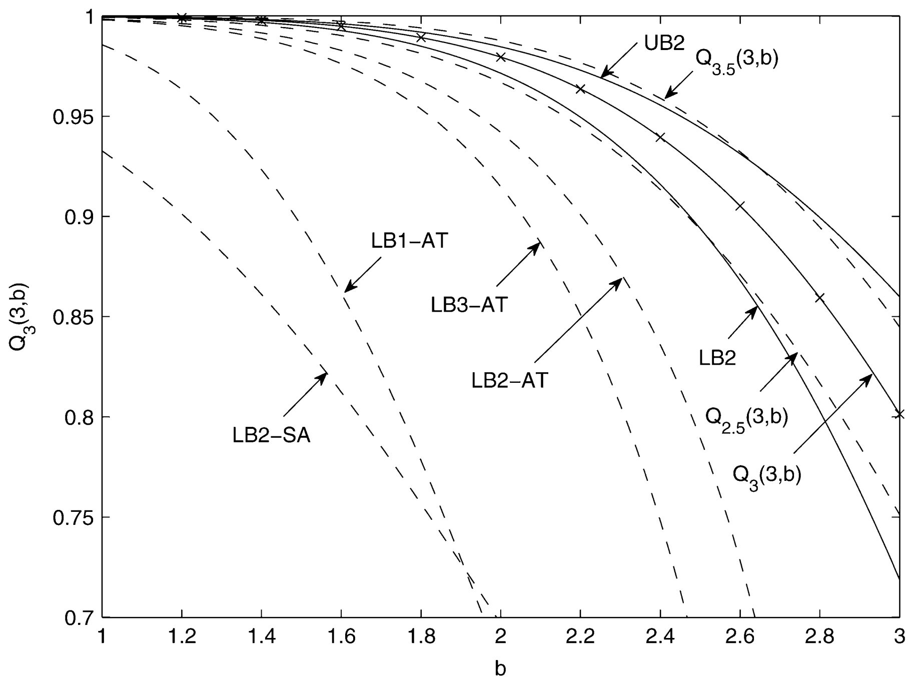

Fig. 2. Numerical results for $Q_\nu(a,b)$ of integer order ν and its upper and lower bounds versus b for the case $b < a = 3$ and $\nu = 3$. x: exact. Dashed line: previous bounds. Solid line: our proposed bounds, i.e., LB2 and UB2.

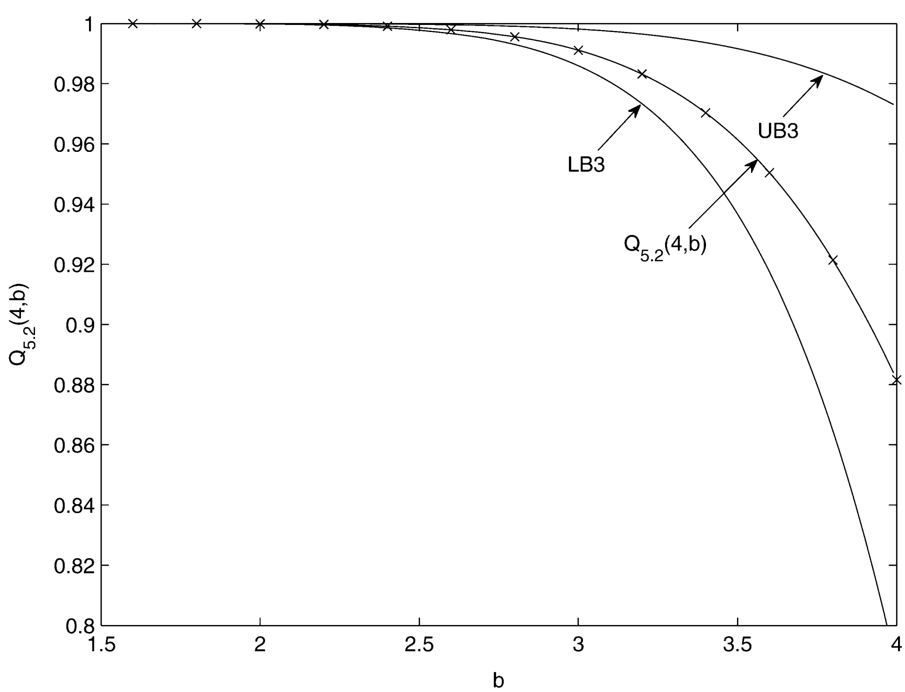

Fig. 3. Numerical results for $Q_\nu(a,b)$ of noninteger order ν and its upper and lower bounds versus b for the case $b < a = 4$ and $\nu = 5.2$. x: exact. Solid line: our proposed bounds, i.e., LB3 and UB3.

are tighter than the other bounds for small b region, and looser than the bounds proposed in [8] when b is near to a. For different sets of ν and a, our simulation results suggest that we can always find an interval $(0, \xi) \in (0, a)$ for b in which our proposed bounds are tighter than the bounds proposed in [8]. Our proposed bounds LB3 and UB3 can be also used for $Q_\nu(a,b)$ of integer order ν. But LB3 and UB3 are clearly weaker than the bounds LB2 and UB2. Therefore, the numerical results of LB3 and UB3 for integer order $Q_\nu(a,b)$ are not shown here. For the case of noninteger order ν, we provide the numerical results of $Q_\nu(a,b)$ and bounds LB3 and UB3 in Fig. 3.

V. Conclusion

This paper extends the work [7] to the case of the generalized Marcum Q-function by exploring the monotonicity of some functions that involves the modified Bessel function of the first kind. New upper and lower bounds for integer order and non-integer order generalized Marcum Q-function are derived via rigorous proofs. These bounds are shown to be very tight when b approaches infinity, i.e., the relative errors of our bounds converge to zero as b increases. As far as we know, no previous bound introduced in the literature has such tightness for large b. The tightness of our bounds is shown by both theoretical analysis and numerical results.

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