Tight Bounds of the Generalized Marcum Q-Function Based on Log-concavity

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***Abstract*—In this paper, we manage to prove the log-concavity of the generalized Marcum Q-function $Q_\nu(a,b)$ with respect to its order ν on $[1,\infty)$. The proof relies on a powerful mathematical concept named total positivity. Based on the recursion relation of the generalized Marcum Q-function, a new intuitive formula for $Q_\nu(a,b)$ is proposed, where ν is an odd multiple of 0.5. After these results, we derive upper and lower bounds for the generalized Marcum Q-function of positive integer order m. Numerical results show that in most of the cases our proposed bounds are much tighter than the existing bounds in the literature. It is surprising to see that the relative errors of the proposed bounds converge to 0 when b approaches infinite.**

I. INTRODUCTION

The generalized Marcum Q-function was firstly used to calculate target detection probability in radar communications [1]-[2]. Recently, it has gained much attention for its important applications in digital communications, such as error performance analysis of multichannel dealing with partially coherent, differential coherent and non-coherent detection [3]-[7], etc. It is defined as

$$Q_\nu(a,b) = \begin{cases} \dfrac{1}{a^{\nu-1}}\displaystyle\int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)dt, & \text{if } \ a>0 \\ \dfrac{1}{2^{\nu-1}\Gamma(\nu)}\displaystyle\int_b^\infty t^{2\nu-1} e^{-\frac{t^2}{2}}dt, & \text{if } \ a=0 \end{cases}, \quad b\geq 0, \nu>0 \quad (1)$$

where I_ν is the modified Bessel function of the first kind of order ν. We allow ν to be any positive real number, because the integrand in (1) is a probability density function for each $a \geq 0$ and $\nu > 0$ (the interested reader is referred to [10]). Therefore, we use ν to denote positive real number, and m for positive integer throughout the paper.

The precise computation of the generalized Marcum Q-function is quite difficult, mainly because I_ν is involved in the integrand in (1). In the last few decades, several authors established approximation formulas and bounds for the function $Q_m(a,b)$. Corazza and Ferrari [7] obtained tight bounds for the 1st-order Marcum Q-function, the relative errors of their bounds converge to 0 when $b \to \infty$. Recently, Li and Kam [8], using an interesting geometric interpretation of the generalized Marcum Q-function, derived a closed-form representation of $Q_{m+0.5}(a,b)$. They pointed out that $Q_{m-0.5}(a,b)$ and $Q_{m+0.5}(a,b)$ are lower and upper bounds for $Q_m(a,b)$, when m is positive integer and $a,b>0$. It is important to note that their bounds are tighter than other bounds in the literature in most cases, but are still loose for large a and b in terms of relative error.

Simon and Alouini [5, pp. 81] presented an asymptotic formula (but not bound) for $Q_\nu(a,b)$ when b tends to infinite

$$Q_\nu(a,b) \sim \left(\frac{b}{a}\right)^{\nu-0.5} Q(b-a), \quad (2)$$

where $Q(\cdot)$ is the Gaussian Q-function and $\sim$ means that these two functions tend to be equal as b increases. This implies that $\sqrt{Q_{m-0.5}(a,b)Q_{m+0.5}(a,b)}$ can be used to estimate $Q_m(a,b)$ for very large b, and the relative error of this estimation converges to 0 when $b\to\infty$. Moreover, further investigation shows that $\sqrt{Q_{m-0.5}(a,b)Q_{m+0.5}(a,b)}$ is also a lower bound of $Q_m(a,b)$ of integer order m, when $m\geq 2$. This fact relies on the log-concavity of the function $\nu \mapsto Q_\nu(a,b)$ on $[1,\infty)$, where we use $\mapsto$ to denote mapping. As far as we know, this problem has not been discussed in the literature. In the course of the proof, we use a powerful mathematical concept called total positivity. More details about total positivity are introduced in Section II. Making use of the recursion relation of the generalized Marcum Q-function, we propose a new intuitive formula for $Q_\nu(a,b)$, where ν is of the form $(2m-1)/2$. After these results, upper and lower bounds for $Q_m(a,b)$ are derived. Numerical results show that the proposed bounds are much tighter than the existing bounds in the literature in most cases. Moreover, it is very interesting that the relative errors of the proposed bounds converge to 0 when $b\to\infty$.

II. LOG-CONCAVITY OF $\nu \mapsto Q_\nu(a,b)$

The generalized Marcum Q-function $Q_\nu(a,b)$ has an important interpretation in probability theory: it is the complementary cumulative distribution function (ccdf) of the normalized non-central chi-square distribution with 2ν degrees of freedom [5, pp. 82]. For this, let $X_1, X_2, \ldots, X_\mu$ be random variables that are normally distributed with unit variance and nonzero mean γ_i, where $i=1,2,\ldots,\mu$. It is known that $X_1^2+X_2^2+\ldots+X_\mu^2$ has the non-central chi-square

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distribution with μ degrees of freedom and non-centrality parameter $\lambda = \gamma_1^2 + \gamma_2^2 + \ldots + \gamma_\mu^2$. The probability density function (pdf) $f_{\chi^2_{\nu,\lambda}} : (0,\infty) \to (0,\infty)$ of the non-central chi-square distribution is defined as

$$f_{\chi^2_{\mu,\lambda}}(x) = \frac{e^{-(x+\lambda)/2}}{2} \left(\frac{x}{\lambda}\right)^{\mu/4-1/2} I_{\mu/2-1}\left(\sqrt{\lambda x}\right). \quad (3)$$

Although μ is an integer in our description of the non-central chi-square distribution above, the distribution defined by (3) is a proper distribution for any positive μ [12, pp. 436].

Consequently, $Q_\nu(a,b)$ can be expressed as

$$Q_\nu\left(\sqrt{a}, \sqrt{b}\right) = \int_b^\infty f_{\chi^2_{2\nu,a}}(x)dx, \quad (4)$$

which is equivalent to [3, eq. (2.1-124)].

If the non-centrality parameter $\lambda = 0$, the above distribution reduces to the classical central chi-square distribution, whose pdf is

$$f_{\chi^2_\mu}(x) = f_{\chi^2_{\mu,0}}(x) = \frac{x^{\mu/2-1}e^{-x/2}}{2^{\mu/2}\Gamma(\mu/2)}, \quad (5)$$

and hence (4) becomes

$$Q_\nu\left(0, \sqrt{b}\right) = \int_b^\infty f_{\chi^2_{2\nu}}(x)dx. \quad (6)$$

It is known that the integrand of (1) and (4) are log-concave in ν, when $\nu \in (0,\infty)$ [11]. But, this result does not help too much in the study of the log-concavity of the function $\nu \mapsto Q_\nu(a,b)$, because we may need the integrand of (1) and (4) to be log-concave in two variables, i.e. ν and x [13, pp. 106], which is not true. However, we have found a powerful mathematical concept, namely total positivity, which can help us to conquer this difficulty. We offer a brief introduction of total positivity, which is necessary for our proof. More details about total positivity can be found in Karlin's monograph [14].

A. A Brief Introduction to Total Positivity

The definition of totally positive function is [14, pp. 11]

Definition 1: A function $f(x,y)$ of two real variables, x ranging over Δ_1 and y ranging over Δ_2, is said to be totally positive of order r (TP_r), if for all $x_i \in \Delta_1$ and $y_i \in \Delta_2$, $i \in \{1,2,\ldots,m\}$ such that $x_1 < x_2 < \ldots < x_m$, $y_1 < y_2 < \ldots < y_m$, where $m \in \{1,2,\ldots,r\}$, we have the inequalities

$$f\begin{pmatrix} x_1, x_2, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{pmatrix} = \begin{vmatrix} f(x_1,y_1) & f(x_1,y_2) & \ldots & f(x_1,y_m) \\ \vdots & \vdots & \vdots & \vdots \\ f(x_m,y_1) & f(x_m,y_2) & \ldots & f(x_m,y_m) \end{vmatrix} \geq 0.$$

A related concept is the sign reverse regularity.

Definition 2: A function $f(x,y)$ is said to be sign reverse regular of order k (SRR_k), if for every $x_1 < x_2 < \ldots < x_m$ and $y_1 < y_2 < \ldots < y_m$, where $m \in \{1,2,\ldots,k\}$, the sign of

$$f\begin{pmatrix} x_1, x_2, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{pmatrix}$$

is $(-1)^{m(m-1)/2}$.

The following composition theorem is very important for our proof [14, pp. 130].

Lemma 1: Let $f(x+y)$ be SRR_r for $x, y > 0$. Suppose that $\phi(t,x)$ is TP_r for $t, x > 0$, and satisfies

$$\phi(t+s,x) = \int_0^x \phi(t,\xi)\phi(s,x-\xi)d\xi.$$

If c is defined by

$$c(t) = \int_0^\infty \phi(t,x)f(x)dx,$$

then $c(t+s)$ is SRR_r for $t, s > 0$.

Some other results which are useful in the sequel are included in the following lemma. The proof of parts (a)-(d) can be found in [14] and part (e) is stated in [13, pp. 79].

Lemma 2: The following assertions are true:

(a) $f(x,y) = x^y$ is TP_2 for $\Delta_1 = (0,\infty)$, $\Delta_2 = (-\infty,\infty)$.
(b) If $f(x,y)$ is TP_r, let $\phi(x), \varphi(y)$ maintain the same constant sign on Δ_1 and Δ_2, respectively, then $\phi(x)\varphi(y)f(x,y)$ is also TP_r.
(c) Let f be a strictly positive second order differentiable function. Then $(x,y) \mapsto f(x-y)$ is TP_2, if and only if $x \mapsto f(x)$ is log-concave on its domain.
(d) $(x,y) \mapsto f(x+y)$ is SRR_2, if and only if $x \mapsto f(x)$ is also log-concave on its domain.
(e) Let $f : \mathbb{R}^n \mapsto \mathbb{R}$, $A \in \mathbb{R}^{n\times m}$, and $b \in \mathbb{R}^n$, and consider the function $g : \mathbb{R}^m \mapsto \mathbb{R}$, defined by

$$g(x) = f(Ax+b),$$

with $\mathbf{dom} g = \{x \mid Ax + b \in \mathbf{dom} f\}$. If f is log-concave, then g is log-concave too.

B. Log-Concavity of $\nu \mapsto Q_\nu(a,b)$

Recently, Sun and Baricz [10], motivated by the work of Li and Kam [8], proved that the generalized Marcum Q-function is increasing with respect to its order, i.e. the function $\nu \mapsto Q_\nu(a,b)$ is strictly increasing on $(0,\infty)$ for each $b > 0$ and $a \geq 0$. Moreover, based on some preliminary results, the authors of [10] conjectured that the function $\nu \mapsto Q_\nu(a,b)$ is strictly log-concave on $(0,\infty)$ for all $a \geq 0$ and $b > 0$. Now, we are able to verify this conjecture partially.

Due to the simplicity of the pdf of chi-square distribution in (5), we consider the log-concavity of $\nu \mapsto Q_\nu(0,b)$ first.

Theorem 1: The function $\nu \mapsto Q_\nu(0,b)$ is log-concave on $(0,\infty)$ for all $b \geq 0$.

Proof: The proposed result is equivalent with the log-concavity of $\nu \mapsto Q_\nu(0,\sqrt{b})$. According to (6), we have

$$Q_\nu\left(0, \sqrt{b}\right) = \int_0^\infty f_{\chi^2_{2\nu}}(x)\mathcal{L}_{[b,\infty)}(x)dx, \quad (7)$$

where $\mathcal{L}_{[b,\infty)}$ is the indicator function for the interval $[b,\infty)$.

According to the characteristic function of chi-square distribution, we acquire the relation for chi-square random variables [12, pp. 450]

$$\chi^2_{\mu_1+\mu_2} = \chi^2_{\mu_1} + \chi^2_{\mu_2}. \quad (8)$$

This implies that

$$f_{\chi^2_{2(\nu_1+\nu_2)}}(x) = \int_0^x f_{\chi^2_{2\nu_1}}(t) f_{\chi^2_{2\nu_2}}(x-t)dt. \quad (9)$$

It is easy to verify that $\mathcal{L}_{[b,\infty)}(x+y)$ is SRR_2 for $x, y > 0$ from the definition. Applying parts (a) and (b) of Lemma 2 for the pdf of (central) chi-squared distribution given in (5), we get that $(x,\nu) \mapsto f_{\chi^2_{2\nu}}(x)$ is TP_2 for $x, \nu > 0$. Then, from Lemma 1, we can easily obtain that $Q_{\nu_1+\nu_2}(0,\sqrt{b})$ is SRR_2 for $\nu_1 > 0$ and $\nu_2 > 0$. This implies that $\nu \mapsto Q_\nu(0,\sqrt{b})$ is log-concave for $\nu > 0$ due to part (d) of Lemma 2. Substituting b with b^2, the proposed theorem is proved. ■

The log-concavity of $\nu \mapsto Q_\nu(a,b)$ is more interesting for the case $a > 0$.

Theorem 2: The function $\nu \mapsto Q_\nu(a,b)$ is log-concave on $[1,\infty)$ for all $a > 0$ and $b \geq 0$.

Proof: In view of (4), we have

$$\begin{aligned} & Q_{\nu+1}\left(\sqrt{a},\sqrt{b}\right) \\ &= P(\chi^2_{2\nu+2,a} \geq b) \\ &= P(\chi^2_{2\nu} + \chi^2_{2,a} \geq b) \\ &= E_{\chi^2_{2\nu}}(P(\chi^2_{2,a} + x \geq b) | \chi^2_{2\nu} = x)) \\ &= \int_0^\infty f_{\chi^2_{2\nu}}(x) P(\chi^2_{2,a} \geq b-x)dx, \end{aligned} \quad (10)$$

where

$$P(\chi^2_{2,a} \geq b-x) = \begin{cases} Q_1\left(\sqrt{a},\sqrt{b-x}\right) & \text{, if } x \leq b, \\ 1 & \text{, if } x > b. \end{cases} \quad (11)$$

Recently, Sun and Baricz [10] proved that $b \mapsto Q_\nu\left(a,\sqrt{b}\right)$ is strictly log-concave on $(0,\infty)$ when $a \geq 0$ and $\nu > 1$. By the same proof process in [10], we can also obtain that $b \mapsto Q_1\left(a,\sqrt{b}\right)$ is log-concave on $(0,\infty)$. In view of part (e) of Lemma 2, we have that $x \mapsto Q_1\left(a,\sqrt{b-x}\right)$ is log-concave on $(-\infty,b)$. Therefore, it is easy to prove that $x \mapsto P(\chi^2_{2,a} \geq b-x)$ is log-concave on $(-\infty,\infty)$, since $x \mapsto Q_1\left(a,\sqrt{b-x}\right)$ is increasing and $P(\chi^2_{2,a} \geq b-x)$ is continuous on $(-\infty,\infty)$. Hence, $(x,y) \mapsto P\left(\chi^2_{2,a} \geq b-(x+y)\right)$ is SRR_2 due to part (d) of Lemma 2. In view of (9) and the fact that $(x,\nu) \mapsto f_{\chi^2_{2\nu}}(x)$ is TP_2 for $x,\nu > 0$, we can apply Lemma 1 in (10) and obtain that $(\nu_1,\nu_2) \mapsto Q_{\nu_1+\nu_2+1}\left(\sqrt{a},\sqrt{b}\right)$ is SRR_2 for $\nu_1,\nu_2 > 0$. This implies that $\nu \mapsto Q_\nu\left(\sqrt{a},\sqrt{b}\right)$ is log-concave for $\nu > 1$ due to part (d) of Lemma 2. Since the function $\nu \mapsto Q_\nu\left(\sqrt{a},\sqrt{b}\right)$ is continuous at $\nu = 1$, we can prove that $\nu \mapsto Q_\nu\left(\sqrt{a},\sqrt{b}\right)$ is log-concave for $\nu \geq 1$. Substituting a with a^2 and b with b^2, Theorem 2 is also proved. ■

III. An New Intuitive Formula of $Q_{n+0.5}(a,b)$

Recently, Li and Kam proposed a closed-form formula of $Q_{n+0.5}(a,b)$ for non-negative integer n [8, eq. 11]. Their method is based on the computation of a multi-dimensional integral of the pdf of central chi-square distribution.

In this section we will propose a new intuitive formula of $Q_{n+0.5}(a,b)$. It can be deduced simply by using the recursion formula of $Q_\nu(a,b)$, which is expressed as [5, pp. 82]

$$Q_\nu(a,b) = \left(\frac{b}{a}\right)^{\nu-1} e^{-(a^2+b^2)/2} I_{\nu-1}(ab) + Q_{\nu-1}(a,b). \quad (12)$$

Since [15, pp. 202]

$$I_{-0.5}(x) = \sqrt{\frac{2}{\pi x}} \cosh(x), \quad (13)$$

we can obtain that

$$\begin{aligned} Q_{0.5}(a,b) &= \sqrt{\frac{2}{\pi}} \int_b^\infty e^{-(x^2+a^2)/2} \cosh(ax)dx \\ &= \frac{1}{2}\text{erfc}\left(\frac{a+b}{\sqrt{2}}\right) + \frac{1}{2}\text{erfc}\left(\frac{b-a}{\sqrt{2}}\right). \end{aligned} \quad (14)$$

Moreover, the modified Bessel function of the first kind of order $n+0.5$ has closed-form formula, when n is non-negative integer [15, pp. 202], given by

$$\begin{aligned} & I_{n+0.5}(x) = \left(\frac{1}{2\pi x}\right)^{0.5} \\ & \cdot \left\{ \sum_{r=0}^{n} \frac{(n+r)!}{r!(n-r)!(2x)^r} \cdot \left[(-1)^r e^x + (-1)^{n+1} e^{-x}\right] \right\}. \end{aligned} \quad (15)$$

Therefore, $Q_{n+0.5}(a,b)$ can be expressed as

$$\begin{aligned} & Q_{n+0.5}(a,b) \\ &= e^{-(a^2+b^2)/2} \sum_{k=0}^{n-1} \left(\frac{b}{a}\right)^{k+0.5} I_{k+0.5}(ab) + Q_{0.5}(a,b) \\ &= \frac{1}{2}\text{erfc}\left(\frac{a+b}{\sqrt{2}}\right) + \frac{1}{2}\text{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \\ & \quad + \left(\frac{1}{2\pi ab}\right)^{0.5} e^{-(a^2+b^2)/2} \sum_{k=0}^{n-1} \left(\frac{b}{a}\right)^{k+0.5} \\ & \quad \cdot \left\{ \sum_{r=0}^{k} \frac{(k+r)!}{r!(k-r)!(2ab)^r} \left[(-1)^r e^{ab} + (-1)^{k+1} e^{-ab}\right] \right\}. \end{aligned} \quad (16)$$

The proposed formula utilizes the powerful recursion formula of $Q_\nu(a,b)$. Thus, it has much simpler derivation than its analogue deduction with Li and Kam's method.

IV. Upper and Lower Bounds for $Q_m(a,b)$

Recall that in the previous section, we have derived the closed-form formula of $Q_{n+0.5}(a,b)$ for non-negative integer n. Based on the log-concavity of $\nu \mapsto Q_\nu(a,b)$ on $[1,\infty)$, we can easily get a lower bound for $Q_m(a,b)$ with positive integer order m, given by

$$\begin{aligned} Q_m(a,b) &\geq Q_{m-LB}(a,b) \\ &= \left[Q_{m-0.5}(a,b) Q_{m+0.5}(a,b)\right]^{0.5}, \ m \geq 2. \end{aligned} \quad (17)$$

The log-concavity of $\nu \mapsto Q_\nu(a,b)$ also implies the following two inequalities

$$Q_{m+0.5}(a,b) \geq Q_m(a,b)^{2/3} Q_{m+1.5}(a,b)^{1/3}, \ m \geq 2, \quad (18)$$
$$Q_{m-0.5}(a,b) \geq Q_m(a,b)^{2/3} Q_{m-1.5}(a,b)^{1/3}, \ m \geq 3. \quad (19)$$

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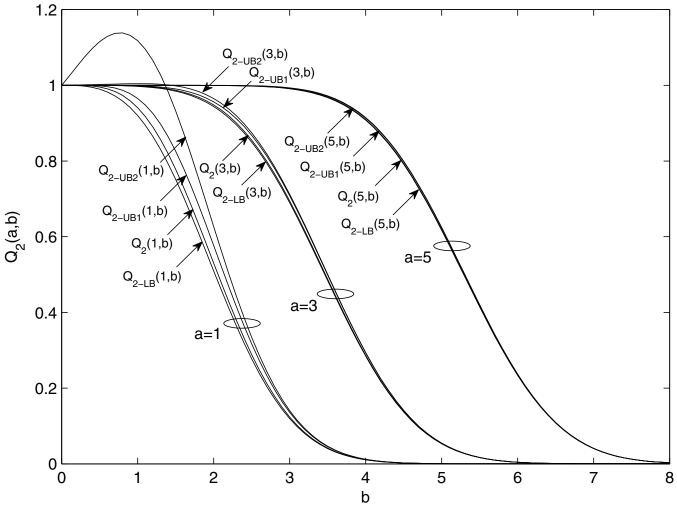

Fig. 1. Numerical results for $Q_m(a, b)$ and the proposed bounds versus b for $a \in \{1, 3, 5\}$ and $m = 2$.

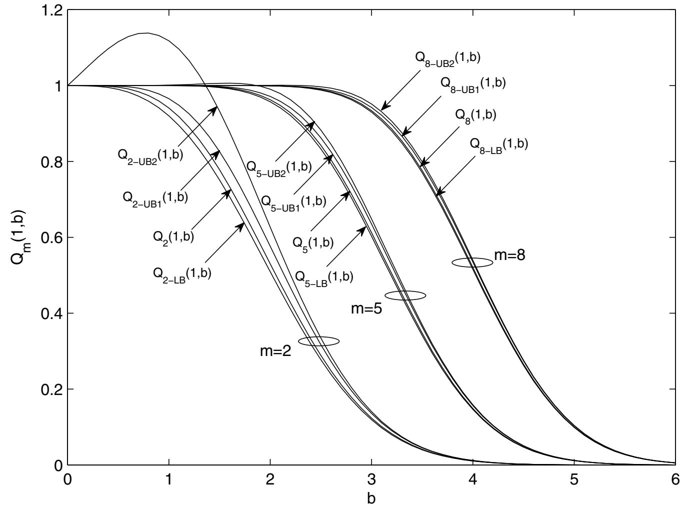

Fig. 2. Numerical results for $Q_m(a, b)$ and the proposed bounds versus b for $m \in \{2, 5, 8\}$ and $a = 1$.

After some simple algebraic manipulations, two upper bounds for $Q_m(a, b)$ are obtained, given by

$$Q_m(a,b) \le Q_{m-UB1}(a,b) = \left(Q_{m+0.5}(a,b)^3 / Q_{m+1.5}(a,b)\right)^{0.5}, m \ge 2, \quad (20)$$

and

$$Q_m(a,b) \le Q_{m-UB2}(a,b) = \left(Q_{m-0.5}(a,b)^3 / Q_{m-1.5}(a,b)\right)^{0.5}, m \ge 3. \quad (21)$$

Recall that the function $\nu \mapsto Q_\nu(a, b)$ is strictly increasing for $\nu \in (0, \infty)$ [10], which has been verified also recently by Mihos, Kapinas and Karagiannidis [16], using a completely different approach. Using this result, we can easily obtain that

$$\begin{aligned} Q_{m-0.5}(a,b) &\le Q_{m-LB}(a,b) \le Q_m(a,b) \\ &\le Q_{m-UB1}(a,b) \le Q_{m+0.5}(a,b), \end{aligned} \quad (22)$$

which means that our bounds $Q_{m-LB}(a, b)$ and $Q_{m-UB1}(a, b)$ are tighter than $Q_{m-0.5}(a, b)$ and $Q_{m+0.5}(a, b)$. Our numerical results show that $Q_{m-UB2}(a, b)$ is looser than $Q_{m-UB1}(a, b)$, but we were not able to find an analytic proof for this.

Now, researches, who are concerned with new tight bounds for the Marcum Q-function and generalized Marcum Q-function are in active progress. Readers can refer to [16]-[19].

V. Comparison and Numerical Results

We first use some numerical results to show the tightness of our new bounds on $Q_m(a, b)$, i.e. one lower bound denoted as $Q_{m-LB}(a, b)$ and two upper bounds denoted as $Q_{m-UB1}(a, b)$ and $Q_{m-UB2}(a, b)$. Fig. 1 shows the results for different values of a, i.e. $a \in \{1, 3, 5\}$, when $m = 2$; Fig. 2 shows the results for different values of m, i.e. $m \in \{2, 5, 8\}$, when $a = 1$. The tightness of each of the proposed bounds improves as b increases. We can see that for a given a and m, $Q_{m-UB1}(a, b)$ and $Q_m(a, b)$ are very tight, but $Q_{m-UB2}(a, b)$ can be loose for small a and m.

By comparing the curves for different choice of a and m, we find that our proposed bounds improve the previously known bounds as a and m increase. For relatively small a and m, e.g. $a = 5$ and $m = 2$, the differences between our proposed bounds and the exact value of $Q_m(a, b)$ are hard to recognize. Thus, the proposed bounds are pretty tight.

Next, we compare the tightness of our proposed bounds with other existing bounds in the literature. Since most of the existing bounds are valid for only either $a > b$ or $a < b$, we choose to show the comparisons case by case.

For the case $a > b$, the existing lower bounds include LBm2_SA in [4, eq. (12)], LBm3_AT in [6, eq. (20)], $Q_{m-0.5}(a, b)$ [8, eq. (11) and (14)], GLBm2_KL [9, eq. (9)]. The existing upper bounds include $Q_{m+0.5}(a, b)$ [8, eq. (11) and (14)], GUBm2_KL [9, eq. (8)]. The results for the case $b < a = 4$ and $m = 4$ are illuminated in Fig. 3. The numerical results show that for a given integer m, our new bounds are much tighter than the other bounds in most cases.

For the case $b > a$, the existing lower bounds include LBm1_AT in [6, the first line of eq. (18)], $Q_{m-0.5}(a, b)$ [8, eq. (11) and (14)], GLBm1_KL [9, eq. (6)]. The existing upper bounds include UBm1_AT in [6, eq.(17)],$Q_{m+0.5}(a, b)$ [8, eq. (11) and (14)], GUBm1_KL [9, eq. (5)]. Fig. 4 shows the results for the case $b > a = 2$ and $m = 4$ in a logarithmic scale. We can see that our new bounds are much tighter than the other bounds. The relative errors of the bounds, i.e. $\varepsilon\% = 100\% \cdot \frac{\text{bound}-Q_m(a,b)}{Q_m(a,b)}$, are presented in Table I when $a = 5$ and $m = 2$. For the limitation of space, we only give the results for the bounds that are tighter than the other bounds in the literature. It is interesting that the relative errors of the proposed bounds converge to 0 when $b \to \infty$, which is not true for the other bounds. Previous bounds with similar tightness like this can only be found for the case $m = 1$ [7], [17].

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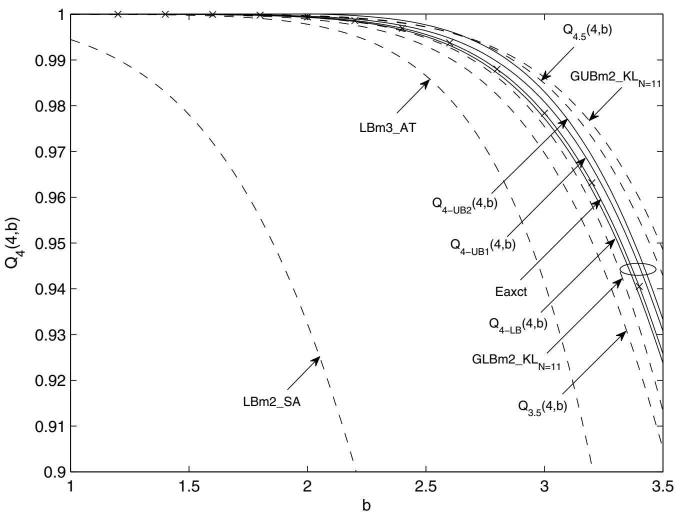

Fig. 3. Numerical results for $Q_m(a,b)$ and its upper and lower bounds versus b for the case $b < a = 4$ and $m = 4$. 'x': exact. Dashed line: previous bounds. Solid line: our bounds.

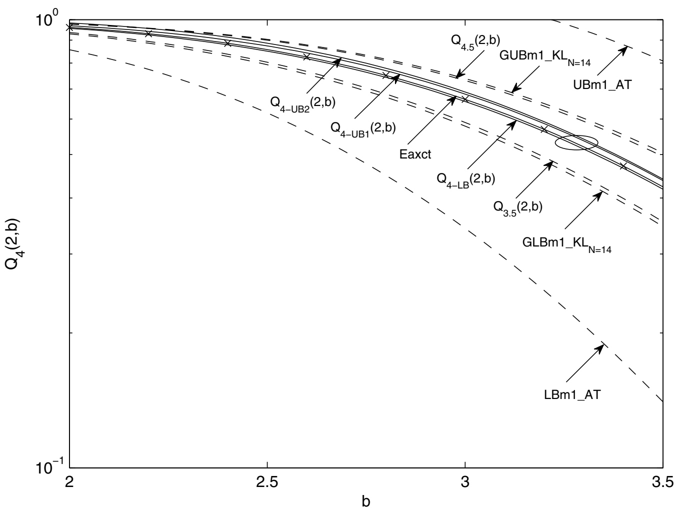

Fig. 4. Numerical results for $Q_m(a,b)$ and its upper and lower bounds versus b for the case $b > a = 2$ and $m = 4$. 'x': exact. Dashed line: previous bounds. Solid line: our bounds.

VI. Conclusion

We proposed tight upper and lower bounds for the generalized Marcum Q-function $Q_m(a,b)$ of positive integer order m. The derivation of these bounds relies on the log-concavity of $\nu \mapsto Q_\nu(a,b)$ on $[1,\infty)$. The numerical results show that our proposed bounds are much tighter than the existing bounds in the literature for most of the cases. In [20], the authors will present a comprehensive study on the monotonicity and log-concavity of the generalized Marcum and Nuttall $Q-$functions. Some more tight bounds for these two functions will be also given.

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TABLE I
Relative Error $\varepsilon\%$ of the Bounds of $Q_m(a,b)$ when $a = 5$, $m = 2$

b	LB	UB1	UB2	$Q_{m-0.5}(a,b)$	$Q_{m+0.5}$(a,b)
2	-0.004	0.007	0.024	-0.020	0.012
4	-0.172	0.441	0.600	-2.17	1.87
6	-0.323	0.953	0.997	-11.6	12.4
8	-0.291	0.874	0.882	-21.6	26.8
10	-0.243	0.733	0.734	-29.4	41.0
12	-0.206	0.619	0.620	-35.4	54.2
14	-0.178	0.534	0.535	-40.2	66.5
16	-0.156	0.469	0.469	-44.0	78.0
18	-0.139	0.417	0.417	-47.2	88.8
20	-0.125	0.376	0.376	-49.9	99.1

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