A Spectral Method via Orthogonal Polynomial Expansions on Sparse Grids for Solving Stochastic Partial Differential Equations

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1 Instruction

Most mathematical models contain uncertainties that may be originated from various sources such as initial and boundary conditions, geometry representation of the domain and input parameters. When these sources are expressed as random processes or random fields, partial differential equations describing the underlying models become stochastic partial differential equations (SPDEs). Stochastic models are more complex than deterministic ones; as part of this complexity, the solution of an SPDE is not simply a function, but rather a random field which expresses the implicit variability of the system. This is the reason that SPDE are able to more fully capture the behavior of interesting phenomena.

Numerical solutions for stochastic partial differential equations have received much attention in recent years. Several competitive methods have emerged, including generalized polynomial chaos method [21, 22, 23, 24], high order finite element method [6, 11, 12, 14], gPC based stochastic collocation method, spectral Galerkin method [3, 10, 16] , spectral collocation method [1, 2] and spectral collocation method on sparse grids [17, 18].

High dimensionality is the main bottleneck of any numerical methods for SPDEs. When the random inputs of a SPDE are represented by a finite number of random variables, the SPDE can be reformulated as a high dimensional deterministic problem where the dimension of the problem is the sum of the spatial dimension and the number of random variables. In practically applicable problems, the number of random variables is usually large. In such cases numerical methods based on the tensor products of one dimensional numerical algorithms suffer the so called “curse of dimensionality” since the computing complexity increases exponentially as the dimension grows. Efforts have been made to overcome this difficulty, with the spectral collocation method on sparse grid as one of most prevailing approaches. Still obstacles remain. For instance, in the spectral collocation method on sparse grid, one must solve a very larger number of deterministic problems ([5]), which can be difficult when the complexity of solving the deterministic problem is large.

In this paper we attempt to construct an efficient numerical algorithm for solving stochastic partial differential equations (SPDEs) through the construction of a fast algorithm for orthogonal polynomial expansions. Here the orthogonality is in terms of inner product of $L_\rho^2$ where $\rho$ is a probability density weight. When solving SPDEs, the “curse of dimensionality” is due to not only the exponential increase of the number of terms in the orthogonal expansion, but also the equal number of multiple dimensional integrals one must evaluate in order to compute the generalized Fourier coefficients. In order to overcome these obstacles, we first use the sparse grid idea to reduce the number of terms in the orthogonal polynomial expansion. Then we use the idea of sparse grid and fast Fourier transform to construct an efficient algorithm to evaluate the Fourier coefficients in the orthogonal polynomial expansion with Chebyshev weight. Through a basis transform on sparse grid we obtain the coefficients of orthogonal polynomial expansion with an arbitrary probability density weight. We shall demonstrate that the overall complexity of our algorithm is $O(n \ln^{d+1} n)$ where $n$ is the number of terms of one dimensional expansion while the convergence rate of the algorithm is quasi-optimal. The total
complexity also include the computing cost of solving the linear system of equations resulted from the spectral Galerkin finite element approximation which is only possible because of the use of the orthogonal polynomials.

Orthogonal polynomial expansion approach has been used to study high dimensional partial differential equations and stochastic partial differential equations in a number of literatures [2, 3, 19, 20]. Our study distinguishes the existing work in the following three aspects.

• In comparison with the fast algorithm of introduced in [20], our algorithm is applicable to orthogonal polynomial expansions with arbitrary probability density weight $\rho$. This is essential for solving SPDEs when $\rho$ is the joint density function of the input random variables.

• In comparison with the stochastic Galerkin method with polynomial chaos expansion and non-orthogonal polynomials, the use of orthogonal polynomial basis will result in a sparse coefficient matrix in the spectral Galerkin approximation. This is extremely important since a non-sparse coefficient matrix can make it prohibitively expensive to solve the linear system of equations for high dimensional problems.

• Our study contains a rigorous proof of exponential convergence of orthogonal polynomial expansion on sparse grid and hence the exponential convergence of our spectral Galerkin method in probability space.

The rest of the paper is organized as follows. In Section 2, we describe the spectral Galerkin method with orthogonal polynomial expansions on sparse grid. In Section 3, we construct a fast algorithm for computing the coefficients of orthogonal polynomial expansions and numerical evaluations of the related multi-dimensional integrals. Section 4 is devoted to error estimates of orthogonal polynomial expansions on sparse grid and spectral Galerkin approximations to SPDEs. Finally in Section 5 we conduct numerical experiments to verify our theoretical results and demonstrate the efficiency of our numerical algorithm.

2 Sparse Galerkin finite element method

We derive a weakly form of elliptic SPDE with random diffusion coefficient by sparse Galerkin finite element methods. We also show the scheme for computing the coefficient matrix of the weakly form of elliptic SPDE.

First we define a set of notations that will be used throughout the rest of the paper. Let $N := \{1, 2, \ldots\}$, $N_0 := \{0, 1, \ldots\}$, $Z_n := \{0, 1, \ldots, n - 1\}$ for $n \in N$, and $I := [-1, 1]$. Let $D$ is a convex bounded polygonal domain in $\mathbb{R}^2$. Denote by $H^1(D)$ the usual Sobolev space of order 1 and $H^1_0(D)$ the subspace of $H^1(D)$ consisting of functions vanishing on $\partial D$.

Though our methodology is applicable to general SPDEs with random input parameters, in this paper, we focus on the following elliptic SPDE with random diffusion coefficient

$$-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) = f(\omega, x), \quad x \in D, \quad \omega \in \Omega,$$

$$u(\omega, x) = 0, \quad x \in \partial D, \quad \omega \in \Omega$$

(2.1)

where $(\Omega, \mathcal{F}, P)$ is a complete probability space, and $a = a(\omega, x)$ is a continuous random field. We assume that $a$ has the following finite Karhunen–Loeve (K-L) expansion

$$a(\omega, x) = E[a](x) + \sum_{k \in \mathbb{Z}_d} b_k(x)y_k(\omega)$$

(2.2)

where $d \in \mathbb{N}$, $\{y_k : k \in \mathbb{Z}_d\}$ are real valued and independent random variables with zero mean value and unit variance, and $I = y_k(\Omega)$. Then the exact solution $u$ of equation (2.1) can be express as
\[ u = u(y_0, \ldots, y_{d-1}, x) \] which satisfies that

\[
-\nabla \cdot (a(y, x) \nabla u(y, x)) = f(y, x) \quad x \in D, \quad y \in I^d, \tag{2.3}
\]
\[ u(y, x) = 0 \quad x \in \partial D, \quad y \in I^d, \tag{2.4}
\]
where \( y := [y_k : k \in \mathbb{Z}_d] \). To ensure that equation (2.1) admits a unique solution, we assume that

\[(A1) \text{ There exist two positive constants } a_{\text{min}} \text{ and } a_{\text{max}} \text{ such that for } x \in D \text{ and for almost all } \omega \in \Omega \]
\[ a_{\text{min}} \leq a(\omega, x) \leq a_{\text{max}}. \tag{2.5}\]

Next we define the variational equation and weak solution of (2.1). Let \( w \) be the probability density function of \( y_k, k = 0, 1, \ldots, d - 1 \). Then the joint probability density function of \((y_0, y_1, \ldots, y_{d-1})\) is

\[ w(y) := \prod_{k \in \mathbb{Z}_d} w(y_k), \quad y := [y_k : k \in \mathbb{Z}_d] \in I^d. \]

Define Hilbert spaces

\[ L_w^2(I^d) := \{ f : \int_{I^d} w(y) |f(y)|^2 dy < \infty \} \]
with inner product

\[ (f, g) := \int_{I^d} w(y) f(y)g(y) dy, \quad \text{for } f, g \in L_w^2(I^d) \]
and

\[ L_{w}^2(I^d, H_0^1(D)) := \{ f : I^d \to H_0^1(D) \text{ and } \int_{I^d} w(y) ||f(y, \cdot)||_{H_0^1(D)}^2 dy < \infty \} \]
with the norm \( ||f||_{L_w^2(I^d, H_0^1(D))} := (\int_{I^d} w(y) ||f(y, \cdot)||_{H_0^1(D)}^2 dy)^{\frac{1}{2}} \). For convenient, we denote \( L_{w}^2(I^d, H_0^1(D)) \) by \( L_{w}^2(I^d, H_0^1(D)) \). For a given \( f \in L_{w}^2(I^d) \) and all \( v \in L_{w}^2(I^d) \), let

\[ V_f(v) := \int_{I^d} w(y) \int_D f(y, x)v(y, x) dx dy. \]

For all \( v_1 \in L_{w}^2(I^d) \) and all \( v_2 \in L_{w}^2(I^d) \), we define the bilinear form

\[ A(v_1, v_2) := \int_{I^d} w(y) \int_D a(y, x) \nabla v_1(y, x) \cdot \nabla v_2(y, x) dx dy. \]

We say that \( u \in L_{w}^2(I^d) \) is a weak solution of equation (2.1) if \( u \) satisfies the variational equation

\[ A(u, v) = V_f(u) \quad \text{for all } v \in L_{w}^2(I^d). \tag{2.6}\]

It follows from the Lax-Milgram theorem [3] that the weak solution exists and is unique.

We now describe the spectral Galerkin method for solving equation (2.6). To this aim, we first present the orthogonal polynomial expansions on sparse grid which will be used to define spectral Galerkin approximations for equation (2.6).

The orthonormal polynomials defined on \( I^d \) are defined by tensor products of univariate orthonormal polynomials. Let \( p_r, r \in \mathbb{N} \) be orthonormal polynomials of order \( r \) with respect to the inner product of \( L_w^2(I^d) \). We assume that \( \{p_r : r \in \mathbb{N}\} \) satisfy the three term recurrence formula

\[ p_{r+1} = (\alpha_r y + \beta_r) p_r - \gamma_r p_{r-1}, \quad r \in \mathbb{N}, \tag{2.7}\]
where \( \alpha_r, \beta_r, \gamma_r \in \mathbb{R} \) and \( p_{-1} = 0 \). It is well-known that such a formula always exists and is available for some of the most well-known orthogonal polynomial bases. For \( r := [r_k : k \in \mathbb{Z}_d] \in \mathbb{N}^d \), define \( p_r := \prod_{k \in \mathbb{Z}_d} p_{r_k} \). Then \( \{p_r : r \in \mathbb{N}^d\} \) is an orthonormal polynomial basis of \( L_w^2(I^d) \).
Next we define the finite dimensional orthogonal polynomial approximations of $f$ on anisotropic sparse grid. For $j \in \mathbb{N}_0$, let
\[
\mathbb{I}_j := \\begin{cases} 
\{ l \in \mathbb{Z} : 2^j - 1 \leq l \leq 2^j \} , & j \neq 0, \\
\{0, 1\}, & j = 0.
\end{cases}
\]
and for $j = [j_k : k \in \mathbb{Z}_d] \in \mathbb{N}_0^d$, denote
\[
\mathbb{I}^d_j := \mathbb{I}_{j_0} \otimes \mathbb{I}_{j_1} \otimes \cdots \otimes \mathbb{I}_{j_{d-1}}.
\]
Let $a := [a_k : k \in \mathbb{Z}_d] \in \mathbb{R}^d$ with $a_k \geq 1$, and $\underline{a} := \min\{a_k : k \in \mathbb{Z}_d\}$. We define that for all $\alpha := [\alpha_k : k \in \mathbb{Z}_d] \in \mathbb{N}_0^d$, $|\alpha|_a := \sum_{k \in \mathbb{Z}_d} a_k \alpha_k$. We let for all $x \in \mathbb{R}$ with $x \geq 0$,
\[
\mathcal{S}^d_{a,x} := \left\{ j \in \mathbb{N}_0^d : |j|_a \leq \underline{a} x \right\}.
\]
For all $x \in \mathbb{R}$ with $x \geq 0$, the index set $\mathbb{J}^d_{a,x}$ of orthonormal polynomials on sparse grid is defined as
\[
\mathbb{J}^d_{a,x} := \bigcup_{j \in \mathcal{S}^d_{a,x}} \mathbb{I}^d_j.
\]
The orthogonal polynomial approximation of $f \in \mathbb{L}^2_w(\mathcal{I}^d)$ on anisotropic sparse grid is defined as for all $N \in \mathbb{N}_0$,
\[
\tilde{f}_N = \sum_{r \in \mathbb{J}^d_{a,N}} (f, p_r) p_r.
\] (2.8)
It is clear that when $a_0 = a_1 = \cdots = a_{d-1}$, $\tilde{f}_N$ is the approximation of $f$ on the isotropic sparse grid. The strategy for determining the parameter vector $a$ is discussed in Section 3 according to either a priori knowledge or a posteriori information. We shall prove in Section 4 that as in the full grid case, the anisotropic sparse grid approximation (2.8) converges to $f$ exponentially. The use of anisotropic sparse grid substantially reduces the computational complexity comparing with the isotropic sparse grid whose cardinality is well known, $O(N^{d-1}2^N)$. We show the estimate of the cardinality of index set $\mathbb{J}^d_{a,N}$ for other cases in the next proposition. For any finite set $\mathcal{A}$, we denote the cardinality of $\mathcal{A}$ by $\#(\mathcal{A})$. For $\alpha := [\alpha_k : k \in \mathbb{Z}_d] \in \mathbb{N}_0^d$, set $|\alpha| := \sum_{k \in \mathbb{Z}_d} \alpha_k$ and $\bar{\alpha} := [a_k : k \in \mathbb{Z}_{d-1}]$. Define that for all $N \in \mathbb{N}_0$,
\[
\mathbb{J}^d_{a,N} := \left\{ j \in \mathbb{N}_0^d : \prod_{k \in \mathbb{Z}_d} (j_k + 1)^{a_k} \leq 2^{|\alpha|+\bar{\alpha}} \right\},
\]
and for all $M \in \mathbb{Z}_{d+1} \setminus \{0\}$, $N \in \mathbb{N}_0$,
\[
\mathbb{J}^d_{a,N} := \left\{ j \in \mathbb{N}_0^d : \prod_{k \in \mathbb{Z}_M} (j_k + 1)^{a_k} \leq 2^{|\alpha|+\bar{\alpha}} \right\},
\]
We also define that for all $M \in \mathbb{Z}_{d+1} \setminus \{0\}$ and $j \in \mathbb{N}_0^d$,
\[
\lambda_{a,M}(j) := \frac{2^{|\alpha|}}{\prod_{k \in \mathbb{Z}_{d-M}} (j_k + 1)^{\alpha_k}}.
\]

**Proposition 2.1** Let $d \in \mathbb{N}$ and $a := [a_k : k \in \mathbb{Z}_d] \in \mathbb{R}^d$ with $a_k \geq 1$. If there exists a positive integer $M \leq d$ such that $a_{k_0} = a_{k_1} = \cdots = a_{k_{M-1}} = \underline{a}$ and $\underline{a} < a_k$ for all $k \in \mathbb{Z}_d \setminus \{k_0, \ldots, k_{M-1}\}$, there exists a positive constant $c_a$ such that for all $N \in \mathbb{N}_0$,
\[
\#(\mathbb{J}^d_{a,N}) \leq c_a N^{M-1}2^N,
\] (2.9)
where $c_{d,a}$ depends on $a$.  

\[4\]
Proof: It is shown in [2] that (2.9) holds when $M = d$. To prove (2.9) with $M < d$, we first show that for all $N \in \mathbb{N}$, $\mathcal{J}^d_{a,N} \subset \mathcal{J}^d_{a,N}$.

Then, estimate (2.9) can be obtained by showing that there exists a positive constant $c_a$ such that for all $N \in \mathbb{N}_0$, $\#(\mathcal{J}^d_{a,N}) \leq c_a2^N$.

Let $N \in \mathbb{N}_0$ and $j \in \mathcal{J}^d_{a,N}$. Thus, we know that there exists $i \in \mathcal{J}^d_{a,N}$ such that $j \in \mathcal{J}_i$. Therefore, from the definition of $\mathcal{J}_i$ we have that for all $k \in \mathbb{Z}_d$, $j_k \leq 2^i$. This leads to the following inequality

$$\prod_{k \in \mathbb{Z}_d} (j_k + 1)^{a_k} \leq \prod_{k \in \mathbb{Z}_d} (2^i + 1)^{a_k}. \tag{2.10}$$

It is clear that for all $i \in \mathbb{N}_0$, $2^i + 1 \leq 2^{i+1}$. Thus, by noting that $|i|_a \leq \underline{a}N$, from (2.10) we have that

$$\prod_{k \in \mathbb{Z}_d} (j_k + 1)^{a_k} \leq 2^{|a|+\underline{a}N}. \tag{2.11}$$

Inequality (2.11) shows that $\mathcal{J}^d_{a,N} = \mathcal{J}^d_{a,N}$. Thus, we know that for all $N \in \mathbb{N}$, $\mathcal{J}^d_{a,N} \subset \mathcal{J}^d_{a,N}$.

We now estimate $\#(\mathcal{J}^d_{a,N})$. It is clear that when $d = 1$, there holds that for all $N \in \mathbb{N}_0$,

$$\#(\mathcal{J}^1_{a,1}) \leq 2^{N+1} - 1. \tag{2.12}$$

This means that (2.9) holds when $d = 1$. We next consider the case when $d > 1$.

Without loss of generality, we let $a_{d-1} = a_{d-2} = \cdots = a_{d-M} = \underline{a}$. By the definition of $\lambda_{a,M}$, we rewrite $\mathcal{J}^d_{a,N}$ into the form as

$$\mathcal{J}^d_{a,N} = \bigcup_{j \in \mathcal{J}^{d-M}_{a,N}} \left\{[j, \tilde{j}] : \tilde{j} := [j_k : k \in \mathcal{J}_M] \in \mathbb{N}_0^M \text{ and } \prod_{k \in \mathcal{J}_M} (\tilde{j}_k + 1) \leq \lambda_{a,M}(j)2^N \right\}. \tag{2.13}$$

Thus, we have that for all $N \in \mathbb{N}_0$,

$$\#(\mathcal{J}^d_{a,N}) \leq \sum_{j \in \mathcal{J}^{d-M}_{a,N}} \lambda_{a,M}(j)2^N \log^{M-1}(\lambda_{a,M}(j)2^N). \tag{2.14}$$

From (2.13), we can see that for all $N \in \mathbb{N}_0$,

$$\#(\mathcal{J}^d_{a,N}) \leq N^{M-1}2^N \sum_{j \in \mathcal{J}^{d-M}_{a,N}} \lambda_{a,M}(j) \log^{M-1}(2\lambda_{a,M}^{1/N}(j)). \tag{2.15}$$

Noting that for all $N \in \mathbb{N}_0$ and $j \in \mathcal{J}^{d-M}_{a,N}$,

$$\lambda_{a,M}^{1/N}(j) = \left(\frac{2^{|a|}}{\prod_{k \in \mathcal{J}_{d-M}} (j_k + 1)^{a_k}}\right) \leq 2^{|a|},$$

from (2.13) we know that there exists a positive constant $\tilde{c}_a$ depended on $a$ such that for all $N \in \mathbb{N}_0$,

$$\#(\mathcal{J}^d_{a,N}) \leq \tilde{c}_aN^{M-1}2^N \sum_{j \in \mathcal{J}^{d-M}_{a,N}} \lambda_{a,M}(j) \tag{2.16}$$

Since for all $k \in \mathbb{Z}_{d-M}$, $a_k > \underline{a}$, there holds that for all $k \in \mathbb{Z}_{d-M}$, $\frac{a_k}{\underline{a}} > 1$. Thus, we know that

$$\sum_{j \in \mathbb{N}_0^M} \lambda_{a,M}(j) < +\infty \tag{2.17}$$

Note that for all $N \in \mathbb{N}_0$, $\mathcal{J}^{d-M}_{a,N} \subset \mathbb{N}_0^{d-M}$. From (2.13) and (2.15), we obtain (2.9). \[\square\]}
We now describe the spectral Galerkin method for solving equation (2.6). Define the finite dimensional space \( Y^{d}_{a,N} := \text{span}\{p_{r} : \mathbf{r} \in J^{d}_{a,N}\} \). Let \( X_{h} \) be a subspace of \( H^{1}_{0}(D) \) consisting of piecewise linear continuous functions on a regular triangulation of \( D \) with a mesh size \( h > 0 \). We assume that \( \{\varphi_{i} : i = 1, 2, \ldots, M_{h}\} \) be a basis of \( X_{h} \). Then the spectral Galerkin approximation for the solution of (2.6) is to find \( u^{d}_{N,h} \in Y^{d}_{a,N} \otimes X_{h} \) such that

\[
A(u^{d}_{N,h}, v) = V_{f}(v), \quad \text{for all } v \in Y^{d}_{a,N} \otimes X_{h}.
\]  

(2.16)

Let \( A^{d}_{N,h} := [A_{r,l,r',i} : \mathbf{r} \in J^{d}_{a,N}, i \in Z_{M_{h}}, \mathbf{r}' \in J^{d}_{a,N} \text{ and } l \in Z_{M_{h}}] \) where \( A_{r,l,r',i} := A(p_{r} \otimes \varphi_{i}, p_{r} \otimes \varphi_{l}) \). Also let \( f^{d}_{N,h} := [f_{r,l} : \mathbf{r}' \in J^{d}_{a,N} \text{ and } l \in Z_{M_{h}}] \) where \( f_{r,l} := V_{f}(p_{r} \otimes \varphi_{l}) \). Then finding \( u^{d}_{N,h} \in Y^{d}_{a,N} \otimes X_{h} \) satisfied (2.16) is equivalent to finding \( u^{d}_{N,h} := [u_{r,i} \in \mathbb{R} : \mathbf{r} \in J^{d}_{a,N} \text{ and } i \in Z_{M_{h}}] \) such that

\[
A^{d}_{N,h} u^{d}_{N,h} = f^{d}_{N,h}.
\]  

(2.17)

Theoretically, to evaluate each entry of the coefficient matrix \( A^{d}_{N,h} \), one needs to evaluate a \( d \)-dimensional integral, which may be even far more costly then solving the linear system (2.17) itself. We next show that with the orthogonal basis, this can be avoided. Moreover, the use of the orthogonal basis will result in the sparsity of the coefficient matrix \( A^{d}_{N,h} \).

Computing the entries of \( A^{d}_{N,h} \) involves multi-dimensional integrations on \( I^{d} \otimes D \). In this paper, we focus on the effect of quadrature formulas on \( I^{d} \) to solving equation (2.6). Thus, we only consider quadrature formulas on \( I^{d} \), and assume that the integral on \( D \) can be estimated with sufficient precision.

Next we employ the three term recurrence formula to compute the entries of \( A^{d}_{N,h} \). For all \( k \in Z_{d+1} \setminus \{0\}, i \in Z_{M_{h}} \text{ and } l \in Z_{M_{h}} \), define

\[
A^{0}_{l,i} := \int_{D} b_{k-1}(x) \nabla \varphi_{i}(x) \cdot \nabla \varphi_{l}(x) dx,
\]

and for all \( i \in Z_{M_{h}} \text{ and } l \in Z_{M_{h}} \),

\[
A^{0}_{l,i} := \int_{D} E[a](x) \nabla \varphi_{i}(x) \cdot \nabla \varphi_{l}(x) dx.
\]

It follows from (2.2) that for all \( \mathbf{r} \in J^{d}_{a,N}, i \in Z_{M_{h}} \), the entries \( A_{r,l,r',i} \) of \( A^{d}_{N,h} \) is given by

\[
A_{r,l,r',i} := A^{0}_{l,i} + \sum_{k \in Z_{d+1} \setminus \{0\}} A^{k}_{l,i} \int_{I^{d}} w(y) y_{k-1} p_{r}(y) p_{r'}(y) dy.
\]  

(2.18)

Since \( p_{r}, \mathbf{r} \in J^{d}_{a,N}, \text{ are orthonormal, there holds that for all } k \in Z_{d}, \mathbf{r}, \mathbf{r}' \in J^{d}_{a,N}, \)

\[
\int_{I^{d}} w(y) y_{k} p_{r}(y) p_{r'}(y) dy = \int_{I^{d}} w_{k}(y) y_{k} p_{r}(y) p_{r'}(y) dy \prod_{\kappa \in Z_{d} \setminus \kappa \neq k} \delta_{r_{\kappa}, r'_{\kappa}}.
\]  

(2.19)

For all \( k \in Z_{d}, \mathbf{r}, \mathbf{r}' \in J^{d}_{a,N}, \) let

\[
\alpha_{r,r'}^{k} := \frac{1}{\alpha_{r_{k}}} \left( \delta_{r_{k+1},r'_{k}} - \beta_{r_{k}} \delta_{r_{k},r'_{k}} + \gamma_{r_{k}} \delta_{r_{k-1},r'_{k}} \right) \prod_{\kappa \in Z_{d} \setminus \kappa \neq k} \delta_{r_{\kappa}, r'_{\kappa}}.
\]

Thus, substituting (2.7) into (2.19), we have that for all \( k \in Z_{d}, \mathbf{r}, \mathbf{r}' \in J^{d}_{a,N}, \)

\[
\int_{I^{d}} w(y) y_{k} p_{r}(y) p_{r'}(y) dy = \alpha_{r,r'}^{k}.
\]  

(2.20)
From (2.18) and (2.20), we have that for all $r \in J_{d,a,N}$, $i \in Z_{M_h}$, $r' \in J_{d,a,N}$, and $l \in Z_{M_h}$,

$$A_{r',l,i} = A_{i,i}^0 + \sum_{k \in Z_d} A_{i,i}^{k+1} a_{r,r'}^k. \quad (2.21)$$

Using (2.21), we can compute the elements of matrix $A_{d,N,h}^d$ without numerical integrations in the probability dimensions.

Now the only high dimensional integrals one must evaluate in (2.17) are the entries of the right hand side $f_{d,N,h}^d$. These are the generalized Fourier coefficients of $f$ on sparse grid. Each one of them is $d$ dimensional integral on $I^d$ and evaluations of these coefficients with standard methods based the tensor product of a one dimensional quadrature will be prohibitively expensive. In the next section, we introduce a fast algorithm for computing these orthogonal coefficients.

### 3 A fast algorithm for orthogonal polynomial expansions on anisotropic sparse grids

In this section, we construct a fast algorithm to compute the coefficients of the orthogonal polynomial expansion defined in (2.8). Though the main purpose of this algorithm is to compute the right hand side of the linear system of equation efficiently, the algorithm itself is a powerful tool of computing generalized Fourier series approximations for a given function $f \in L^2_{w}(I^d)$.

To compute the coefficients of the orthogonal polynomial expansion defined in (2.8), one needs high dimensional quadrature formulas to evaluate $(f,p_r)$, for all $r \in J_{d,a,N}$. When $d >> 1$, quadrature formulas based on the tensor product of a one dimensional quadrature will be prohibitively expensive.

In our fast algorithm, we first develop polynomial interpolations on anisotropic sparse grid and fast Fourier transform to evaluate the coefficients of the Chebyshev orthogonal polynomial expansion. Then we use a basis transform to evaluate the coefficients of orthogonal polynomial expansions with an arbitrary density weight $w$.

We first discuss orthogonal polynomial expansions with an arbitrary density weight $w$ in the one dimensional case. The construction of high dimensional case will be based on a sequence of one dimensional algorithms.

The aim is to evaluate the coefficients $(f,p_r)$, $r = 1, 2 \ldots, n$. The basic idea is to replace $f$ with its polynomial interpolation $P_N f$ on the Chebyshev points and use $(P_N f, p_r)$ to approximate $(f, p_r)$. To define the Chebyshev polynomial interpolation, we recall the definition of the Chebyshev polynomials: for all $l \in \mathbb{Z}$ and $x \in I$, $T_l(x) := \cos(l \arccos(x))$. For $j \in \mathbb{N}_0$, let

$$V_j := \left\{ v_{j,r} := \cos\left(\frac{r \pi}{2^j}\right) : r \in \mathbb{Z}_{2^j+1} \right\} \quad (3.1)$$

denote the zeros of $T_i$ and $W_j := \text{span}\{T_l : l \in \mathbb{Z}_{2^j+1}\}$. We let $P_j$ be a interpolation projection from $C(I)$ to $W_j$ such that $(P_j f)(v) = f(v)$, $v \in V_j$. It is well-known that

$$P_j f = \sum_{l \in \mathbb{Z}_{2^j+1}} c_{j,l} T_l \quad (3.2)$$

where

$$c_{j,l} := \frac{1}{2^{j-1+q_j,r_0+\delta_{j,l}}} \left( \sum_{r=0}^{2^j} \frac{1}{2^{q_j,r_0+\delta_{j,l}+r_0}} f(v_{j,r}) T_l(v_{j,r}) \right).$$

Notice that $c_{j,l}$ can be evaluated using the fast cosine transform.

Next, we apply the fast expansion and decomposition algorithms presented in [7] to (3.2) to obtain the expansion of $P_N f$ with the basis $p_r$, $r \in \mathbb{Z}_{n+1}$. Specifically, we employ the algorithms named...
“Expand” and “Decomp” in [7] to transform the coefficients of the Chebyshev polynomials into the coefficients of the orthogonal polynomials \( p_r, r \in \mathbb{Z}_{n+1} \). This is achieved by two steps. We first use the algorithm “Expand” transform the coefficients of the Chebyshev polynomials into the coefficients of the monomial basis. Secondly, we employ the algorithm “Decomp” transform the coefficients of the monomial basis into the coefficients of the arbitrary orthogonal polynomials \( p_r, r \in \mathbb{Z}_{n+1} \) as follows,

\[
P_N = \sum_{r=1}^{n} a_{Nr} p_r.
\]

Then \( (f, p_r) \approx a_{Nr}, r = 1, \ldots, n \).

We next present a hierarchical interpolation scheme in the one-dimensional case as a preparation for the polynomial expansions on sparse grids. We define \( \Delta_0 := \mathcal{P}_0 \) and for all \( j \in \mathbb{N} \),

\[
\Delta_j := \mathcal{P}_j - \mathcal{P}_{j-1}.
\]

Then for each \( f \in C(I) \) the hierarchical Chebyshev polynomial interpolation scheme is defined as

\[
\mathcal{P}_N f = \sum_{j \in \mathcal{N}_{n+1}} \Delta_j f.
\]

Let \( \mathbb{I}_0 := \mathbb{Z}_2 \) and for all \( j \in \mathbb{N} \), \( \mathbb{I}_j := \mathbb{Z}_{2^{j+1}} \setminus \mathbb{Z}_{2^{j-1}+1} \). For all \( j \in \mathbb{N} \) and \( l \in \mathbb{I}_j \), denote \( \chi_{j,l} := T_l - T_{2^{j-1}+l} \) and \( \chi_{0,l} := T_l \) for all \( l \in \mathbb{I}_0 \). It is shown in [13, 20] that for each \( f \in C(I) \) and \( j \in \mathbb{N}_0 \), there holds

\[
\Delta_j f = \sum_{l \in \mathbb{I}_j} c_{j,l}(f) \chi_{j,l}.
\]

For all \( N \in \mathbb{N}_0 \), given \( \xi_N^1 := [\xi_{j,l} \in \mathbb{R} : j \in \mathbb{Z}_{N+1} \text{ and } l \in \mathbb{I}_j] \) and \( r \in \mathbb{Z}_{2^{j+1}} \) we define

\[
\mathcal{E}_{\xi_N^1}(r) := \sum_{j \in \mathbb{Z}_{N+1}} \sum_{l \in \mathbb{I}_j} \xi_{j,l} \psi_{j,l}(r),
\]

where \( \psi_{j,l}(r) := (\chi_{j,l}, p_r) \). It is easy to see that when \( \xi_{j,l} = c_{j,l}(f), (\mathcal{P}_N f, p_r) = \mathcal{E}_{\xi_N^1}(r) \). By substituting (3.5) into (3.4) we obtain a hierarchical interpolation scheme in the one-dimensional case.

Replacing \( f \) in \((f, p_r)\) by \( \mathcal{P}_N f \) leads to a quadrature formula for evaluating the generalized Fourier coefficient \((f, p_r)\) of \( f \). To deliver a fast algorithm for computing \((\mathcal{P}_N f, p_r)\), we need to rewrite \( \mathcal{E}_{\xi_N^1} \) as a linear combination of \((T_l, p_r)\) for all \( l \in \mathbb{Z}_0 \) and \( r \in \mathbb{Z}_0 \). Then we can and apply the fast algorithms “Expand” and “Decomp” to evaluate \((\mathcal{P}_N f, p_r)\) for all \( r \in \mathbb{Z}_{n+1} \). Let \( \mathbb{Z}_0 := \emptyset \) and \( \mathbb{Z}_{1/2} := \emptyset \). For all \( N \in \mathbb{N}_0 \), given \( \xi_N^1 := [\xi_{j,l} \in \mathbb{R} : j \in \mathbb{Z}_{N+1} \text{ and } l \in \mathbb{I}_j] \), \( \tilde{j} \in \mathbb{Z}_{N+2} \) and \( \tilde{l} \in \mathbb{Z}_{n+1} \), let

\[
\tilde{\xi}_{j,l} := \begin{cases} 
0, & \tilde{j} = N + 1, \\
\xi_{j,l} + \tilde{\xi}_{j+1,l}, & \tilde{j} \leq N \text{ and } \tilde{l} \in \mathbb{I}_j, \\
-\xi_{j,2^{j-1}+l} + \tilde{\xi}_{j+1,l}, & \tilde{j} \leq N \text{ and } \tilde{l} \in \mathbb{Z}_{2^{j-1}}, \\
\tilde{\xi}_{j+1,l}, & \tilde{j} \leq N \text{ and } \tilde{l} \notin \mathbb{I}_j \cup \mathbb{Z}_{2^{j-1}}.
\end{cases}
\]

It is shown in Lemma 2.1, [9], that for all \( r \in \mathbb{Z}_{n+1} \),

\[
\mathcal{E}_{\xi_N^1}(r) = \sum_{l \in \mathbb{Z}_{n+1}} \tilde{\xi}_{0,l}(T_l, p_r).
\]

We now review the algorithm for evaluating \( \mathcal{E}_{\xi_N^1} \) on \( \mathbb{Z}_{n+1} \) with given \( \xi_N^1 \) which is developed in [9]. For all \( N \in \mathbb{N}_0 \) and a given \( \xi_N^1 \), define \( \tilde{\xi}_N^1 := [\tilde{\xi}_{0,l} : l \in \mathbb{Z}_{n+1}] \) and \( \mathcal{E}_{\xi_N^1} := [\mathcal{E}_{\xi_N^1}(r) : r \in \mathbb{Z}_{n+1}] \). We also define that

\[
\mathcal{P}_{\xi_N^1} := \sum_{l \in \mathbb{Z}_{n+1}} \tilde{\xi}_{0,l} T_l.
\]
Lemma 2.1 in [9] indicates that we can obtain $E_{\xi_N}$ by expanding the polynomial $P_{\xi_N}$ on the basis \{\(p_l : l \in \mathbb{Z}_{n+1}\)\}, which can be achieved by the algorithms “Expand” and “Decomp” in [7]. We review the algorithm for evaluating $E_{\xi_N}$ on $\mathbb{Z}_{n+1}$ for given $\xi_N$.

**Algorithm 3.1** $FE1d(N, \xi_N^1)$

**Input:** $N \in \mathbb{N}$, $\xi_N^1 := \{\xi_{j,r} : j \in \mathbb{Z}_{N+1} \text{ and } r \in \mathbb{I}_j\}$.

**Step 1** Compute $\xi_N^1$ according to (3.6).

**Step 2** Compute the expansion coefficients of $P_{\xi_N^1}$ on the monomial basis by applying the algorithm “Expand” in [7] to $\xi_N^1$.

**Step 3** Compute $E_{\xi_N}$ by applying the algorithm “Decomp” in [7] to $m$.

**Output:** $E_{\xi_N}$.

We now present a fast algorithm for $d$-dimensional orthogonal polynomial expansions with probability density weight $w$ on anisotropic sparse grids by modifying Algorithm 2.4 in [9]. For this purpose we first derive the Chebyshev polynomial interpolation scheme on anisotropic sparse grid.

Let $G_0 := V_0$. For $j \in \mathbb{N}$, define $G_j := V_j \setminus V_{j-1}$ and $j \in \mathbb{N}_0^d$.

$$G_j := G_{j_0} \otimes G_{j_1} \otimes \cdots \otimes G_{j_{d-1}}.$$  

The spare grid $S_N^d$ is defined as

$$S_N^d := \bigcup_{j \in \mathbb{N}_0^d} G_j.$$  

For $j \in \mathbb{N}_0^d$, let

$$\Delta_j := \Delta_{j_0} \otimes \Delta_{j_1} \otimes \cdots \otimes \Delta_{j_{d-1}}.$$  

We define the polynomial interpolation of $f \in C(I^d)$ on the anisotropic sparse grid as

$$S_N(f) := \sum_{j \in \mathbb{N}_0^d} \Delta_j(f).$$  

(3.8)

For $j \in \mathbb{N}_0^d$ and $l \in \mathbb{I}_j^d$, define $c_{j,l} := c_{j_0,l_0} \otimes c_{j_1,l_1} \otimes \cdots \otimes c_{j_{d-1},l_{d-1}}$ and $\chi_{j,l} := \chi_{j_0,l_0} \otimes \chi_{j_1,l_1} \otimes \cdots \otimes \chi_{j_{d-1},l_{d-1}}$.

It follows from (3.5) that for $f \in C(I^d)$ and $j \in \mathbb{N}_0^d$,

$$\Delta_j f = \sum_{l \in \mathbb{I}_j^d} c_{j,l}(f) \chi_{j,l}.$$  

(3.9)

and

$$S_N(f) := \sum_{j \in \mathbb{N}_0^d} \sum_{l \in \mathbb{I}_j^d} c_{j,l}(f) \chi_{j,l}.$$  

(3.10)

The coefficients $c_{j,l}(f)$ in above equation can be computed through the fast cosine transform. In what follows we construct an quadrature formula to evaluate $(f, p_r)$ by computing a sequence of one dimensional fast algorithm we constructed in the last subsection. For all $j \in \mathbb{N}_0^d$ and all $l \in \mathbb{I}_j^d$, define $\psi_{j,l} := \psi_{j_0,l_0} \otimes \psi_{j_1,l_1} \otimes \cdots \otimes \psi_{j_{d-1},l_{d-1}}$.

Replacing $f$ in $(f, p_r)$, $r \in \mathbb{N}_0^d$, by $S_N(f)$, from (3.10) we obtain the following quadrature formula,

$$\eta_{N,r}(f) := \sum_{j \in \mathbb{N}_0^d} \sum_{l \in \mathbb{I}_j^d} c_{j,l}(f) \psi_{j,l}(r).$$  

(3.11)
We let $\eta^d_N(f) := \{\eta_{N,r}(f) : r \in \mathbb{J}^d_{a,N}\}$.

We now construct a fast algorithm for computing $\eta^d_N(f)$ by simplifying a general algorithm developed in [15] to this special case. Specifically, in [15] we present a fast evaluation scheme for dimension-reducible sums, which is taken over a source set, at points of a target set where the source and target sets are both dimension-reducible in the following sense. A set of $d$ dimensions is called dimension-reducible if it can be represented as a union of sets, each of which is the tensor product (which we will call a cell) of a one dimensional set and a $d-1$ dimensional set that has the same structure as the original $d$ dimensional set, with the one dimensional sets being disjoint and the $d-1$ dimensional sets being a nested sequence. Since a dimension-reducible set share the same structure of a tensor product set, we can evaluate a dimension-reducible sum in the same way as a tensor product sum. The details of this general algorithm is presented in [15]. It is easy to show that $\mathcal{J}^d_{a,N}$ and $\{(j,1) : j \in S^d_{a,N} \text{ and } l \in I^d_j\}$ are both dimension-reducible sets. Hence, we can apply the fast evaluation scheme in [15] to evaluate $\eta_{N,r}(f)$. Moreover, by employing the property that $\psi_{j,l}(r) = 0$ if $r > 2^j$, we can simplify the fast evaluation scheme in [15] in this special case. For $N \in \mathbb{N}_0$ and $\xi^d_N := \{\xi_{j,1} \in \mathbb{R} : j \in S^d_{a,N} \text{ and } l \in I^d_j\}$, we define the partial sum by

$$\mathcal{E}_{\xi^d_N} := \sum_{j \in S^d_{a,N}} \sum_{l \in I^d_j} \xi_{j,1} \psi_{j,l}.$$  

For all $N \in \mathbb{N}_0$ and $f \in C(I^d)$, we let $c^d_N(f) := \{c_{j,1}(f) : j \in S^d_{a,N} \text{ and } l \in I^d_j\}$. We can see that for and for $f \in C(I^d)$ and $r \in \mathcal{J}^d_{a,N}$ there holds

$$\eta_{N,r}(f) = \mathcal{E}_{c^d_N(f)}(r).$$

For all $k \in \mathbb{Z}_{d-1}$, let $a_k := (a^k : k' \in \mathbb{Z}_{d-1} \setminus \{k\})$. We also let for all $k \in \mathbb{Z}_d$ and $R \in \mathbb{R}$ with $R \geq 0$,

$$\tilde{S}^{d-1}_{a_k,R} := \{j \in \mathbb{N}_0^{d-1} : |j|_{a_k} \leq aR\}.$$  

For all $x \in \mathbb{R}$, denote the largest integer which is not greater than $x$ by $\lfloor x \rfloor$. For all $N \in \mathbb{N}_0$, $k \in \mathbb{Z}_d$ and $j \in \tilde{S}^{d-1}_{a_k,N}$, let $Z_N,k(j) := \lfloor (aN - |j|_{a_k}) / a_k \rfloor$. For all $N \in \mathbb{N}_0$ and $j \in \mathbb{Z}_{N+1}$, we also let $W_{N,j} := N - a^{-d-1} \bar{l}$. For all $N \in \mathbb{N}_0$, $j \in \tilde{S}^{d-1}_{a_{d-1},N}$, $l \in I^d_j$ and $r \in \mathbb{Z}_{2^dN,d-1}(0)+1$,

$$\tilde{\xi}_{j,l}(r) := \sum_{j \in \mathbb{Z}_{2^dN,d-1}(0)+1} \sum_{l \in I^d_j} \xi_{j,[l],[l]} \psi_{j,l}(r).$$  

With these notations, we have the following technical lemma.

**Lemma 3.2** Let $d > 1$ and $N \in \mathbb{N}$. Then for $r := [r, \bar{r}] \in \mathbb{J}^d_{a,N}$ and $j \in \mathbb{Z}_{[aN/a_d-1]+1}$ with $r \in I^d_j$, there holds

$$\mathcal{E}_{\xi^d_N}(r) = \sum_{j \in \tilde{S}^{d-1}_{a_{d-1},N}} \sum_{l \in I^d_j} \tilde{\xi}_{j,l}(r) \psi_{j,l}(\bar{r}). \quad (3.12)$$

**Proof:** Let $r := [r, \bar{r}] \in \mathbb{J}^d_{a,N}$ and $j \in \mathbb{Z}_{[aN/a_d-1]+1}$ with $r \in I^d_j$. Note that $\tilde{S}^{d-1}_{a_{d-1},N} = \{[j, \bar{j}] : j \in \tilde{S}^{d-1}_{a_{d-1},N} \text{ and } \bar{j} \in \mathbb{Z}_{2^dN,d-1}(0)+1\}$. Thus, from the definition of $\mathcal{E}_{\xi^d_N}(r)$, we have that

$$\mathcal{E}_{\xi^d_N}(r) = \sum_{j \in \tilde{S}^{d-1}_{a_{d-1},N}} \sum_{l \in I^d_j} \left( \sum_{j \in \mathbb{Z}_{2^dN,d-1}(0)+1} \sum_{l \in I^d_j} \xi_{j,[l],[l]} \psi_{j,l}(r) \right) \psi_{j,l}(\bar{r}). \quad (3.13)$$
Also note that for all \( j \in Z_j \), the degree of polynomial \( \chi_{j,l} \), \( l \in \mathbb{I}_j \), is less than \( p_r \). Thus, due to the orthogonality of \( \{ p_r : r \in \mathbb{N}_0 \} \) and the definition of \( \psi_{j,d} \), if \( j \in \tilde{Z}_{a,j}^{d-1} \) with \( |j|_{a,d} > aN - a_{d-1} \), then \( Z_{N,d-1}(j) + 1 \leq j \) and terms in the bracket of (3.13) vanish. Therefore

\[
E_{\psi_{j,d}}(r) = \sum_{j \in \tilde{Z}_{a,j}^{d-1}} \sum_{l \in \mathbb{I}_j} \left( \sum_{j \in Z_{N,d-1}(j) + 1} \xi_{j,l} \psi_{j,l}(r) \right) \psi_{j,1}(r). \tag{3.14}
\]

Also note that for all \( j \in \tilde{Z}_{a,j}^{d-1} \), there holds that \( j \leq Z_{N,d-1}(j) \). Thus, form \( r \in \mathbb{I}_j \), we know that \( r \in Z_{N,d-1}(j) + 1 \). This means that for all \( j \in \tilde{Z}_{a,j}^{d-1} \) and \( l \in \mathbb{I}_j \), \( \xi_{j,1}(r) \) is well-defined. Thus, from the definition of \( \xi_{j,1}(r) \) and (3.14), we obtain (3.12).

Because of the expression (3.12) in Lemma 3.2, we can compute \( E_{\psi_{j,d}}(r), r \in \mathbb{J}_{a,N}^d \) using the one-dimensional transforms along each dimension. Specifically, to compute \( E_{\psi_{j,d}}(r), r \in \mathbb{J}_{a,N}^d \), we first compute \( [\tilde{\xi}_{j,1}(r) : r \in Z_{N,d-1}(j) + 1] \) for all \( j \in \tilde{Z}_{a,j}^{d-1} \) and \( l \in \mathbb{I}_j \) which can be done by applying a one-dimensional transform to \( [\tilde{\xi}_{j,l} : j \in Z_{N,d-1}(j) + 1 \text{ and } l \in \mathbb{I}_j] \). Then for all \( r \in \mathbb{I}_{j+1} \), we compute \( E_{\psi_{j,d}}(r), r \in \mathbb{J}_{a,j}^d \), \( l \in \mathbb{I}_j \), where \( \xi_{j,r} := [\tilde{\xi}_{j,l}(r) : j \in \tilde{Z}_{a,j}^{d-1} \text{ and } l \in \mathbb{I}_j] \) and \( j \in \mathbb{J}_{a,N}^{d-1} \) with \( r \in \mathbb{I}_j \). Note that we also can apply Lemma 3.2 to compute \( E_{\psi_{j,d}}(r), r \in \mathbb{J}_{a,j}^d \). Thus we can compute \( E_{\psi_{j,d}}(r), r \in \mathbb{J}_{a,N}^d \) by applying one-dimensional transforms to \( \xi_{j,d} \) along each dimension.

We summarize the above computation procedure the following algorithm.

**Algorithm 3.3**

**Input:** \( d \in \mathbb{N}, N \in \mathbb{N} \) and \( f \in C(\mathbb{I}^d) \).

**Step 1** Compute \( c_{\psi_{j,d}}(f) \) by the fast discrete cosine transform.

**Step 2** Compute \( \eta_{\psi_{j,d}}(f) \) by evaluating \( E_{\psi_{j,d}}(f) \) on \( \mathbb{J}_{a,N}^d \) which is achieved by applying Algorithm 3.1 to each dimension.

**Output:** \( \eta_{\psi_{j,d}}(f) \).

We denote \( \mathcal{N}_N^d \) by the number of operations used in Algorithm 3.3.

**Theorem 3.4**

Let \( d \in \mathbb{N} \) and \( a := [a_k : k \in \mathbb{Z}_d] \in \mathbb{R}_d^d \) with \( a_k \geq 1 \). If for all \( x \in \mathbb{I}^d \), evaluating \( f(x) \) uses \( O(1) \) arithmetic operations, and there exists a positive integer \( M < d \) such that \( a_{k_0} = a_{k_1} = \cdots = a_{k_{M-1}} = \frac{a}{d} \) and \( a < a_k \) for all \( k \in \mathbb{Z}_d \setminus \{a_{k_0}, \ldots, a_{k_{M-1}}\} \), then there exists a positive constant \( c_{d,a} \) such that for all \( N \in \mathbb{N} \),

\[
\mathcal{N}_N^d \leq c_{d,a}N^{M+1}2^N, \tag{3.15}
\]

where \( c_{d,a} \) depends on \( d \) and \( a \).

**Proof:** We use \( \mathcal{N}^{(i)}, i \in \{1, 2\} \), denote the number of operations used in Step \( i \) of Algorithm 3.3. We recall that \( c_{\psi_{j,d}}(f) = [c_{j,1}(f) : j \in \mathbb{J}_{a,N}^d \text{ and } l \in \mathbb{I}_j] \) and for each \( j \in \mathbb{J}_{a,N}^d \), computing \( [c_{j,1}(f) : l \in \mathbb{I}_j] \) requests \( O(||f||_\infty 2^j) \) operations. Thus, we have that there exists a positive constant \( c_1 \) such that for all \( N \in \mathbb{N} \),

\[
\mathcal{N}^{(1)} \leq c_1 \sum_{j \in \mathbb{J}_{a,N}^d} ||f||_\infty 2^j. \tag{3.16}
\]

Note that for all \( j \in \mathbb{J}_{a,N}^d \), \( ||f||_\infty \leq N \). From (3.16) we know that for all \( N \in \mathbb{N} \),

\[
\mathcal{N}^{(1)} \leq cN \sum_{j \in \mathbb{J}_{a,N}^d} 2^j. \tag{3.17}
\]
It is clear that for all $N \in \mathbb{N}$, $\sum_{j \in \mathbb{Z}^{d-1}_{a,N}} 2^{|j|} \leq 2^d \#(J_{a,N}^d)$. Thus, from Proposition 2.1 we have that there exists a positive constant $\tilde{c}_{d,a}$ such that for all $N \in \mathbb{N}$,

$$\mathcal{N}^{(1)} \leq \tilde{c}_{d,a} N^M 2^N,$$  

(3.18)

where $\tilde{c}_{d,a}$ only depends on $d$ and $a$.

Let $\mathcal{N}^{(2)}_k$ be the number of the arithmetic operations requested for applying Algorithm 3.1 to dimension $k$, $k \in \mathbb{Z}_d$. The number of operations used in Algorithm 3.1 is $\mathcal{O}(N^2 2^N)$ which is proved in [7]. Thus, we have that there exists a positive constant $c$ such that for all $N \in \mathbb{N}$ and $k \in \mathbb{Z}_d$,

$$\mathcal{N}^{(2)}_k \leq c \left( \sum_{j \in \mathbb{Z}^{d-1}_{a,N}} \sum_{l \in l_j} (Z_{N,k}(j))^2 2^{\Delta_{N,k}(j)} \right).$$  

(3.19)

Note that for all $k \in \mathbb{Z}_d$ and $j \in \mathbb{Z}^{d-1}_{a,N}$, there hold $Z_{N,k}(j) \leq N$ and $\#(\bigcup_{j \in \mathbb{Z}^{d-1}_{a,N}} \{j\}) = 2^{\Delta_{N,k}(j)} + 1$. Thus, from (3.19) we have that for all $N \in \mathbb{N}$ and $k \in \mathbb{Z}_d$,

$$\mathcal{N}^{(2)}_k \leq cN^2 \left( \sum_{j \in \mathbb{Z}^{d-1}_{a,N}} \sum_{l \in l_j} \sum_{j \in \mathbb{Z}^{d-1}_{a,N}} \sum_{l \in l_j} 1 \right) \leq cN^2 \#(J_{a,N}^d).$$  

(3.20)

From (2.9) and (3.20), we have that there exists a positive constant $\tilde{c}_{a,d}$ such that for all $N \in \mathbb{N}$,

$$\mathcal{N}^{(2)} \leq \tilde{c}_{a,d} N^M 2^{N^2},$$  

(3.21)

where $\tilde{c}_{a,d}$ depends on $d$ and $a$. Combing (3.18) and (3.21) leads to estimate (3.15). □

We now estimate $\|S_N(f) - f\|_{L_\infty^2(I^d)}$ when $f \in C^\infty(I^d)$ has a holomorphic extension on $\mathbb{C}^d$, where $\mathbb{C}$ is the complex plane. Let $2^d := [2^k : k \in \mathbb{Z}_d]$, for all $j \in \mathbb{N}_0^d$. For $r > 1$ define the Bernstein’s regularity ellipse $E_r := \{z \in \mathbb{C} : |z| + |z + 1| \leq r + r^{-1}\}$. For $r := [r_k > 1 : k \in \mathbb{Z}_d]$ also define

$$E_r := E_{r_0} \otimes \cdots \otimes E_{r_{d-1}}.$$

For $f$ defined on $E_r$, denote $M_r(f) := \max_{z \in E_r} |f(z)|$. With these notations, we have the following lemma.

**Lemma 3.5** Let $d \in \mathbb{N}$, $r := [r_k > 1 : k \in \mathbb{Z}_d]$. If function $f \in C^\infty(I^d)$ has a holomorphic extension to $E_r$, then there exists a positive constants $c$ such that for all $j \in \mathbb{N}_0^d$,

$$\|\Delta_j f\|_{\infty} \leq c M_r(f) r^{-d-1}.$$  

(3.22)

Proof: The proof of this lemma is presented in Lemma 3.4 of [9]. □

With inequality (3.22), we are ready to estimate $\|S_N(f) - f\|_{\infty}$ in the next theorem. For all $j \in \mathbb{N}_0^d$, let $I_j := \{y \in \mathbb{R}^d : y_k \in [j_k - 1, j_k] \text{ for all kin } \mathbb{Z}_d\}$. For all $N \in \mathbb{N}$, we define

$$\tilde{S}^d_{a,N} := \bigcup_{j \in \mathbb{N}_0^d \setminus \mathbb{Z}^d_{a,N}} I_j.$$

We also define that $\rho(y) := \sum_{k \in \mathbb{Z}_d} a_k 2^{y_k - 1}$, $y \in \mathbb{R}^d$. Let $\tilde{y} := \frac{a_N}{|a|}$ and $\tilde{y} := [\tilde{y}, \tilde{y}, \ldots, \tilde{y}]$. 12
Theorem 3.6 Let \( d \in \mathbb{N}, r := [r_k > 1 : k \in \mathbb{Z}_d] \). If \( f \in C^\infty(I^d) \) has a holomorphic extension to \( E_r \) and for all \( k \in \mathbb{Z}_d, a_k = \ln r_k \), then there exists a positive constant \( c \) such that for all \( N \in \mathbb{N}_0, \)

\[
\|S_N(f) - f\|_\infty \leq cM_r(f)e^{-\varrho(\bar{y})},
\]  

(3.23)

where \( c \) depends on \( d \) and \( a \).

Proof: Note that \( f = \sum_{j \in \mathbb{N}_0^d} \Delta_j f \). It follows from the definition of \( S_N(f) \) that for all \( N \in \mathbb{N}_0, \)

\[
\|S_N(f) - f\|_\infty \leq \sum_{j \in \mathbb{N}_0^d \setminus S_{a,N}^d} \|\Delta_j f\|_\infty.
\]  

(3.24)

Applying Lemma 3.5 to (3.24), we have that for all \( N \in \mathbb{N}_0, \)

\[
\|S_N(f) - f\|_\infty \leq cM_r(f) \sum_{j \in \mathbb{N}_0^d \setminus S_{a,N}^d} r^{-2j-1}.
\]  

(3.25)

Note that for all \( k \in \mathbb{Z}_d, r_k > 1 \). Thus, for all \( j \in \mathbb{N}_0^d \setminus S_{a,N}^d \) and \( y \in I_j \), there holds that \( r^{-2j-1} \leq r^{-2y-1} \). Moreover, we have that for all \( j \in \mathbb{N}_0^d \setminus S_{a,N}^d, \)

\[
r^{-2y-1} \leq \int_{I_j} r^{-2y-1} \, dy.
\]  

(3.26)

Substituting (3.26) into (3.25) leads to

\[
\|S_N(f) - f\|_\infty \leq cM_r(f) \int_{S_{a,N}^d} r^{-2y-1} \, dy.
\]  

(3.27)

Note that for all \( k \in \mathbb{Z}_d, a_k = \ln r_k \). Thus, from the definition of \( \varrho \) we rewrite (3.27) into the form as

\[
\|S_N(f) - f\|_\infty \leq cM_r(f) \int_{S_{a,N}^d} e^{-\varrho(y)} \, dy.
\]  

(3.28)

From the Taylor expansion, we know that for all \( y \in \hat{S}_{a,N}^d, \)

\[
\varrho(y) = \varrho(\bar{y}) + \nabla \varrho(\bar{y}) \cdot \Delta y + \frac{1}{2} \Delta y^T \nabla^2 \varrho(\bar{y} + \zeta \Delta y) \Delta y,
\]

where \( \Delta y = y - \bar{y} \) and \( \zeta \in [0, 1] \). It has be shown in the proof of Lemma 5.2 [17] that for all \( y \in \hat{S}_{a,N}^d, \)

\[
\nabla \varrho(\bar{y}) \cdot \Delta y \geq 0,
\]

and

\[
\frac{1}{2} \Delta y^T \nabla^2 \varrho(\bar{y} + \zeta \Delta y) \Delta y \geq \frac{(\ln 2)^4}{8} \sum_{k \in \mathbb{Z}_d} a_k^2 (y_k - \bar{y})^2.
\]

Thus, from (3.28) we have that

\[
\|S_N(f) - f\|_\infty \leq cM_r(f)e^{-\varrho(\bar{y})} \int_{\hat{S}_{a,N}^d} e^{-\frac{(\ln 2)^4}{8} \sum_{k \in \mathbb{Z}_d} a_k^2 (y_k - \bar{y})^2} \, dy.
\]  

(3.29)

To derive estimate (3.23), we next show that the integration term in (3.29) is up bounded by a constant which does not depend on \( N \). Note that for all \( k \in \mathbb{Z}_d, a_k = \frac{a_k (\ln 2)^2}{2\sqrt{2^{2d}}} e^{-\frac{a_k^2 (y_k - \bar{y})^2 (\ln 2)^4}{8}}, \, y \in \mathbb{R}, \) is the probability density function of a normal random variable with mean \( \bar{y} \) and variance \( \frac{4}{a_k^2 (\ln 2)^4} \). Then, we have that

\[
\int_{\hat{S}_{a,N}^d} e^{-\frac{\ln 2}{8} \sum_{k \in \mathbb{Z}_d} a_k^2 (y_k - \bar{y})^2} \, dy \leq \int_{\mathbb{R}^d} e^{-\frac{\ln 2}{8} \sum_{k \in \mathbb{Z}_d} a_k^2 (y_k - \bar{y})^2} \, dy = \left(\frac{8\pi}{(\ln 2)^2}\right)^{d/2} \prod_{k \in \mathbb{Z}_d} \frac{1}{a_k}.
\]  

(3.30)

Combing (3.29) and (3.30) we have estimate (3.23).
Corollary 3.7 Let $d \in \mathbb{N}$, $r := |r_k > 1 : k \in \mathbb{Z}_d|$. If $f \in C^\infty(I^d)$ has a holomorphic extension to $E_r$ and for all $k \in \mathbb{Z}_d$, $a_k = \ln r_k$, and $|\int_{I^d} w(x) dx| < +\infty$, then there exists a positive constants $c$ such that for all $N \in \mathbb{N}_0$,
\[ \|\mathcal{S}_N(f) - f\|_{L^2(I^d)} \leq c M_r(f) e^{-c(x)}, \] (3.31)
where $c$ depends on $d$ and $a$.

Proof: The result follows directly from Theorem 3.6. \hfill \Box

4 Convergence analysis

With Algorithm 3.3, we can estimate the entries of $\mathbf{f}_{d,N}^d$ in (2.17) and deliver a full discrete form of equation (2.17). The formula for estimating the entries of $\mathbf{f}_{d,N}^d$ is shown as follows. Define
\[ C_H(I^d) := \{ g : g \in C(I^d \otimes D), \text{ and for each fixed } y \in I^d, g(y, \cdot) \in H_0^1(D) \}. \]

For all $g \in C_H(I^d)$ and $l \in \mathbb{Z}_{M_h}$, we let $g_l$ be a function defined on $I^d$ by $g_l(y) := \int_D g(y, x) \varphi_l(x) dx$, $y \in I^d$. For given $g \in C_H(I^d)$, we define that for all $v \in Y_{a,N}^d \otimes X_h$,
\[ V_{g,N,h}^d(v) := \sum_{r \in \mathcal{J}_{a,N}^d} \sum_{l \in \mathbb{Z}_{M_h}} v_{r,l} \eta_{N,h}(g_l), \]
where $\{v_{r,l} : r \in \mathcal{J}_{a,N}^d \text{ and } l \in \mathbb{Z}_{M_h}\}$ satisfies that $v = \sum_{r \in \mathcal{J}_{a,N}^d} \sum_{l \in \mathbb{Z}_{M_h}} v_{r,l} \eta_{r} \otimes \varphi_l$. For a given $g \in C_H(I^d)$, all $r \in \mathcal{J}_{a,N}^d \text{ and } l \in \mathbb{Z}_{M_h}$, we let
\[ \bar{g}_{r,l} = V_{g,N,h}^d(p_r \otimes \varphi_l). \]

Let $\tilde{f}_{d,N,h}^d := [\tilde{f}_{r,l} : r \in \mathcal{J}_{a,N}^d \text{ and } l \in \mathbb{Z}_{M_h}]$. By replacing in $f_{N,h}^d$ in (2.17) by $\tilde{f}_{N,h}^d$, we obtain the following linear system
\[ A_{d,N,h}^d \tilde{u}_{N,h}^d = \tilde{f}_{N,h}^d, \]
where $\tilde{u}_{N,h}^d$ is unknown. It is clear that solving the linear system (4.1) equals to find $\tilde{u}_{N,h}^d \in Y_{a,N}^d \otimes X_h$ such that
\[ A(\tilde{u}_{N,h}^d, v) = V_{f,N,h}^d(v) \quad \text{for all } v \in Y_{a,N}^d \otimes X_h. \] (4.2)

We now estimate the error between $\tilde{u}_{N,h}^d$ and $u$.

Lemma 4.1 Assume that (A1). Then for all $N \in \mathbb{N}$ and $h > 0$, the equation (4.2) has a unique solution. Moreover, there exists a positive constant $c$ such that for all $N \in \mathbb{N}$ and $h > 0$,
\[ \|u - \tilde{u}_{N,h}^d\|_{L^2_{H,w}(I^d)} \leq c \left( \inf_{v \in Y_{a,N}^d \otimes X_h} \|u - v\|_{L^2_{H,w}(I^d)} + \sup_{v \in Y_{a,N}^d \otimes X_h} \frac{|A_f(v) - V_{f,N,h}^d(v)|}{\|v\|_{L^2_{H,w}(I^d)}} \right). \] (4.3)

Proof: Let $N \in \mathbb{N}$ and $h > 0$. From Lemma 4.6 we know that $V_{f,N,h}^d$ is a bounded linear functional on $Y_{a,N}^d \otimes X_h$. Under the assumption (A1), we have that for all $v \in L^2_{H,w}(I^d)$ and $v' \in L^2_{H,w}(I^d)$
\[ |A(v, v')| \leq a_{\max}\|v\|_{L^2_{H,w}(I^d)}\|v'\|_{L^2_{H,w}(I^d)}, \] (4.4)
and for all $v \in L^2_{H,w}(I^d)$

$$a_{\min}\|v\|^2_{L^2_{H,w}(I^d)} \leq A(v,v),$$

(4.5)

(refer to Section 2.2 of [3]). By Lax-Milgram lemma, the equation (4.2) has one and only one solution. Furthermore, by the first Strang lemma we obtain (4.3).

From Lemma 4.1, in order to estimate the different between $\tilde{u}_{N,k}^d$ and $u$ we only need to estimate the two terms on the right hand side of (4.3). To provide an estimate of first term in the right hand side of (4.3), we show the estimate of regularity of $u$ by the following lemma. For any function $g$ defined on $I^d \otimes D$ and $\alpha \in \mathbb{N}_0^d$, define

$$\partial^\alpha y g := \frac{\partial^{\alpha} y g}{\partial y_0 \partial y_1 \cdots \partial y_{d-1}}.$$ 

We also define $\alpha! = \prod_{k \in \mathbb{Z}_d} \alpha_k!$. For all $k \in \mathbb{Z}_d$, let $\mathbf{e}_k := [e_{k'} : k' \in \mathbb{Z}_d] \in \mathbb{N}_0^d$ where $e_k = 1$ and $e_{k'} = 0$ for all $k \in \mathbb{Z}_d$ with $k' \neq k$. Let $\vartheta_k \in (0, +\infty)$, $k \in \mathbb{Z}_d$, and $\vartheta := [\vartheta_k : k \in \mathbb{Z}_d]$. Denote by $L^2(D)$ the usual $L^2$ space defined on $D$ with norm $\| \cdot \|_{L^2(D)}$.

**Lemma 4.2** If there hold that for each $k \in \mathbb{Z}_d$,

$$\sup_{x \in D} |b_k(x)| \leq \frac{a_{\min}}{d} \vartheta_k$$

(4.6)

and there exits a positive constant $c$ such that for all $y \in I^d$ and $\alpha \in \mathbb{N}_0^d$,

$$\|\partial^\alpha y f(y, \cdot)\|_{L^2(D)} \leq c|\alpha|! \vartheta |\alpha| \|f(y, \cdot)\|_{L^2(D)},$$

(4.7)

then there holds that for all $y \in I^d$ and $\alpha \in \mathbb{N}_0^d$,

$$\|\partial^\alpha y u(y, \cdot)\|_{H^1_0(D)} \leq c|\alpha| \frac{a_{\min}}{d} \vartheta |\alpha| \|f(y, \cdot)\|_{L^2(D)}.$$ 

(4.8)

**Proof:** We prove estimate (4.8) by induction on $|\alpha|$. It is easy to see that (4.8) holds for $|\alpha| = 0$. Assume it holds for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$ where $m \in \mathbb{N}$. We next consider $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = m + 1$. Applying $\partial^\alpha y$ to (2.3) yields to

$$-\nabla \cdot (a(y, \cdot) \nabla \partial^\alpha y u(y, \cdot)) = \sum_{k \in \mathbb{Z}_d} \alpha_k \nabla \cdot (b_k(\cdot) \nabla \partial^\alpha y - e_{\alpha_k} u(y, \cdot)) + \partial^\alpha y f(y, \cdot).$$

(4.9)

It follows from (4.6) and (4.7) that

$$a_{\min}\|\partial^\alpha y u(y, \cdot)\|_{H^1_0(D)} \leq \frac{a_{\min}}{d} \sum_{k \in \mathbb{Z}_d} \alpha_k \vartheta_k \|\partial^\alpha y - e_{\alpha_k} u(y, \cdot)\|_{H^1_0(D)} + \|\partial^\alpha y f(y, \cdot)\|_{L^2(D)}.$$ 

(4.10)

Using inequality (4.7) and (4.8) for all $\alpha - e_k, k \in \mathbb{Z}_d$, from (4.10) we know that (4.8) holds for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = m + 1$. Then, it follows that (4.8) holds for all $y \in I^d$ and $\alpha \in \mathbb{N}_0^d$.

Using Lemma 4.2, we estimate $\inf_{v \in Y_{N,K}^d(\mathbb{Z}_d)} \|u - v\|_{L^2_{H,w}(I^d)}$ in the following lemma. For $r > 1$ define

$$D_r := \{z \in C : r^{-1} \leq |z| \leq r\}.$$ 

For $r := [r_k : k \in \mathbb{Z}_d]$ also define $D_r := D_{r_0} \otimes \cdots \otimes D_{r_{d-1}}$. For $r := [r_k > 0 : k \in \mathbb{Z}_d]$, denote $\Gamma_r := \{z \in C^d : |z_k| = r_k \text{ for all } k \in \mathbb{Z}_d\}$. Let $\tilde{u}(y) := \prod_{k \in \mathbb{Z}_d} (1 - y_k^2)^{-\frac{1}{2}}$ for all $y \in I^d$. For $l \in \mathbb{Z}$ with
$l < 0$, let $T_l := T_{-l}$, $1 := [l_k : k \in \mathbb{Z}_d] \in \mathbb{Z}_d$, define $T_l := \prod_{k \in \mathbb{Z}_d} T_{l_k}$ and $\tilde{T}_l := \frac{T_l}{||T_l||_{L^2(\mu_d)}}$. We also define that for all $g \in L^2_{H, w}(\mu_d)$ and $l \in \mathbb{N}_0^d$,

$$\langle g, \tilde{T}_l \rangle := \int_{\mathbb{R}^d} \hat{w}(y) g(y) \tilde{T}_l(y) dy.$$  

Note that the mapping $(\zeta_k + \zeta_k^{-1})/2 = z_k$, $k \in \mathbb{Z}_d$; maps $D_r$ in $E_r$. Thus, for given mapping $g : E_r \rightarrow H^1_0(D)$ and $\zeta \in D_r$, the mapping

$$v_g(\zeta) := 2^d g \left( \frac{\zeta_0 + \zeta_0^{-1}}{2}, \ldots, \frac{\zeta_d + \zeta_d^{-1}}{2} \right).$$

is well defined. Let $\gamma_k := \sqrt{\gamma_k^{-2} + 1}$ for all $k \in \mathbb{Z}_d$. We also let

$$\mathcal{M}(f) := \sup_{y \in \mathbb{R}^d} \{ ||f(y, \cdot)||_{L^2(D)} \}.$$

With these notations, we have the following lemmas.

**Lemma 4.3** If the conditions in Lemma 4.2 hold, then for each $r := [r_k \in (1, \gamma_k) : k \in \mathbb{Z}_d]$, there exists a positive constant $c_r$ such that for all $l \in \mathbb{N}_0^d$,

$$\left\| \left\langle u, \tilde{T}_l \right\rangle \right\|_{H^1_0(D)} \leq \frac{c_r \mathcal{M}(f)}{2^{3d/2\pi^2 / 2} \min} \left| l \right|^{-1}.$$  

(4.11)

Proof: Let $r := [r_k \in (1, \gamma_k) : k \in \mathbb{Z}_d]$. It follows from Lemma 4.2 that $u$ can be extended to be a weakly holomorphic vector-valued function on $E_r$ (see Section 2 of [4] for the definition). Thus, $v_u$ is also weakly holomorphic on $D_r$. Moreover, $v$ has the Laurent expansion (see the proof of Theorem 5 in [4])

$$v_u(\zeta) = \sum_{l \in \mathbb{N}_0^d} \frac{1}{(2\pi i)^d} \int_{\Gamma_r} \frac{v_u(t)}{t^{l+1}} dt \zeta^l.$$  

(4.12)

Let $\zeta = [e^{i\theta_k} : k \in \mathbb{Z}_d]$ with $\theta_k \in [0, 2\pi]$. The definition of $v$ shows that

$$v_u(\zeta) = 2^d u(\cos(\theta_0), \ldots, \cos(\theta_{d-1})).$$  

(4.13)

Thus, applying the Chebyshev expansion to the right-side term of (4.13) leads to

$$v_u(\zeta) = 2^d \sum_{l \in \mathbb{N}_0^d} \left\langle u, \tilde{T}_l \right\rangle \left( \prod_{k \in \mathbb{Z}_d} \cos(l_k \theta_k) \right) / ||T_l||_{L^2(\mu_d)}.$$  

(4.14)

Note that for all $l \in \mathbb{N}_0^d$, $\prod_{k \in \mathbb{Z}_d} \cos(l_k \theta_k) = 2^{Z(l)} \prod_{k \in \mathbb{Z}_d} (e^{i l_k \theta_k} + e^{-i l_k \theta_k})/2^d$, where $Z(l)$ equals to the number of zeros in $l$. Thus, equality (4.14) can be rewritten into the following form

$$v_u(\zeta) = \sum_{l \in \mathbb{Z}_d} \frac{2^{Z(l)}}{||T_l||_{L^2(\mu_d)}} \left\langle u, \tilde{T}_l \right\rangle \zeta^{|l|}.$$  

(4.15)

Also note that $||T_l||_{L^2(\mu_d)} = \left( \frac{2d}{2\pi^{-d/2} \pi^2} \right)^{1/2}$ and $||\tilde{T}_l||_{L^2(\mu_d)} = \left( \frac{2d}{2\pi^{-d/2} \pi^2} \right)^{1/2}$. Comparing (4.12) and (4.15), we obtain that for all $l \in \mathbb{N}_0^d$,

$$\left\langle u, \tilde{T}_l \right\rangle = \frac{1}{2^{3d/2\pi^2 / 2} 2^d} \int_{\Gamma_r} \frac{v_u(t)}{t^{l+1}} dt.$$  

(4.16)
Applying \( \| \cdot \|_{H^1(I)} \) to both side of (4.16), we have that for all \( l \in \mathbb{N}_0^d \),
\[
\left\| \left\langle u, \bar{T}_1 \right\rangle \right\|_{H^1(I)} \leq \frac{\sup_{x \in E_r} \{\| u(z, \cdot) \|_{H^1(I)} \}}{2^{3d/2} \pi^{d/2} \min r[l]}.
\] (4.17)

From Lemma 4.2 and Taylor expansions, we know that
\[
\sup_{x \in E_r} \{\| u(z, \cdot) \|_{H^1(I)} \} \leq \frac{M(f)}{a_{\min}} \sum_{\alpha \in \mathbb{N}_0^d} |\alpha| |\theta(\alpha)| \prod_{k \in \mathbb{Z}_+} \left( \frac{|r_k - r_k^{-1}|}{2} \right)^{\alpha_k}
\] (4.18)

Note that for all \( k \in \mathbb{Z}_+ \), \( \frac{|r_k - r_k^{-1}|}{2} = \vartheta_k^{-1} \). Combining (4.17) and (4.18) leads to (4.11).

\[\square\]

**Lemma 4.4** Let \( r := [r_k \in (1, \gamma_k) : k \in \mathbb{Z}_+] \). If the conditions in Lemma 4.2 hold and \( a_k = \ln r_k \) for all \( k \in \mathbb{Z}_d \), and \( \int_{I_d} |w(\mathbf{y})|d\mathbf{y} < +\infty \), then there exists a positive constant \( c_r \) such that for all \( N \in \mathbb{N} \),
\[
\inf_{v \in Y_{a,N} \otimes H^1(D)} \| u - v \|_{L^2_{H,w}(I^d)} \leq \frac{c_r M(f)}{2^{3d/2} \pi^{d/2} a_{\min}} e^{-\theta(\bar{y})}.
\] (4.19)

**Proof:** Let \( r := [r_k \in (1, \gamma_k) : k \in \mathbb{Z}_+] \) and for all \( N \in \mathbb{N} \)
\[
\bar{u}_N := \sum_{j \in \mathbb{N}_0^d} \sum_{l \in \mathbb{N}_0^d} \left\langle u, \bar{T}_1 \right\rangle T_1.
\]

It is sufficient to prove (4.19) by showing that there exist a positive constant \( c \) such that for all \( N \in \mathbb{N} \),
\[
\| u - \bar{u}_N \|_{L^2_{H,w}(I^d)} \leq \frac{c_r M(f)}{2^{3d/2} \pi^{d/2} a_{\min}} e^{-\theta(\bar{y})}.
\]

Since \( \int_{I_d} |w(\mathbf{y})|d\mathbf{y} < +\infty \), there exists a positive constant \( c_0 \) such that for all \( N \in \mathbb{N} \),
\[
\| u - \bar{u}_N \|_{L^2_{H,w}(I^d)} \leq c_0 \sum_{j \in \mathbb{N}_0^d} \sum_{l \in \mathbb{N}_0^d} \left\| \left\langle u, \bar{T}_1 \right\rangle \right\|_{H^1(I)}.
\] (4.20)

From (4.20), doing the similar computation in Lemma 3.4 of [9], we know that there exists a positive constant \( c_r \) such that for all \( N \in \mathbb{N} \),
\[
\| u - \bar{u}_N \|_{L^2_{H,w}(I^d)} \leq \frac{c_r M(f)}{2^{3d/2} \pi^{d/2} a_{\min}} \sum_{j \in \mathbb{N}_0^d} \sum_{l \in \mathbb{N}_0^d} r^{[-2^{j-1}]}.
\] (4.21)

By conducting the similar computation in Theorem 3.6, from (4.21) we obtain (4.19).

\[\square\]

We now ready to estimate \( \inf_{v \in Y_{a,N} \otimes X_h} \| u - v \|_{L^2_{H,w}(I^d)} \).

**Lemma 4.5** Let \( r := [r_k \in (1, \gamma_k) : k \in \mathbb{Z}_+] \). If the conditions in Lemma 4.4 hold and for all \( \mathbf{y} \in I_d \), \( u(\mathbf{y}, \cdot) \in H^2(D) \), then there exists a constant \( c_r > 0 \) such that for all \( N \in \mathbb{N} \) and \( h > 0 \),
\[
\inf_{v \in Y_{a,N} \otimes X_h} \| u - v \|_{L^2_{H,w}(I^d)} \leq c_r \left( h \| u \|_{L^2_{H,w}(I^d)} + e^{-\theta(\bar{y})} M(f) \right).
\] (4.22)
Proof: Note that that for all \( N \in \mathbb{N} \) and \( h > 0 \),
\[
\inf_{v \in Y_{a,N}^d \otimes X_h} \| u - v \|_{L^2_{H,w}(I^d)} \leq \inf_{v \in L^2_{H,w}(I^d) \otimes X_h} \| u - v \|_{L^2_{H,w}(I^d)} + \inf_{v \in Y_{a,N}^d \otimes H^1_0(D)} \| u - v \|_{L^2_{H,w}(I^d)}.
\]
\( (4.23) \)
Since \( u(y, \cdot) \in H^2(D) \) for all \( y \in I^d \), it follows from the standard finite element estimate that (cf. \[8\]) there exists a constant \( c > 0 \) such that for all \( h > 0 \) and all \( y \in I^d \),
\[
\inf_{v \in L^2_{H,w}(I^d)} \| u - v \|_{L^2_{H,w}(I^d)} \leq ch\| u \|_{L^2_{H,w}(I^d)}.
\]
\( (4.24) \)
Substituting (4.19) and (4.24) into (4.23), we obtain (4.22). \( \square \)

Next we estimate second term on the right hand side of (4.3). For a given function \( g \) defined on \( I^d \otimes D \) denote
\[
\| f \|_\infty := \sup_{y \in I^d, x \in D} |f(y, x)|.
\]
To this aim, we need the following lemmas. For all \( g \in C_H(I^d) \), \( y \in I^d \) and \( x \in D \), let \( \tilde{S}_Ng(y, x) := (S_N(g(\cdot, x)))(y) \).

**Lemma 4.6** If \( f \in C_H(I^d) \), then for \( v \in Y_{a,N}^d \otimes X_h^2 \), there holds
\[
V^d_{f,N,h}(v) = \int_{I^d} w(y) \int_D \tilde{S}_Nf(y, x)v(y, x)dydx.
\]
\( (4.25) \)
Moreover, if \( f \in C_H(I^d) \) and \( \int_{I^d} |w(y)|dy < \infty \), then \( V^d_{f,N,h} \) is bounded on \( Y_{a,N}^d \otimes X_h \).

Proof: Let \( N \in \mathbb{N} \), \( h > 0 \) and \( v := \sum_{r \in \mathbb{R}^d_{a,N}} \sum_{l \in \mathbb{Z}_{M_h}} v_{r,l}p_r \otimes \varphi_l \), where \( v_{r,l} \in \mathbb{R} \) for all \( r \in \mathbb{R}^d_{a,N} \) and \( l \in \mathbb{Z}_{M_h} \). From the definitions of \( V^d_{f,N,h} \) and \( \eta_{N,r} \), we have that for all \( v \in Y_{a,N}^d \otimes X_h^2 \),
\[
V^d_{f,N,h}(v) = \sum_{r \in \mathbb{R}^d_{a,N}} \sum_{l \in \mathbb{Z}_{M_h}} v_{r,l}(\tilde{S}_N(f_l), p_r).
\]
\( (4.26) \)
Note that for all \( y \in I^d \), \( (\tilde{S}_N(f_l))(y) = \int_D (\tilde{S}_N(f(\cdot, x)))(y)\varphi_l(x)dx \). Thus, from the definition of \( \tilde{S}_Nf \) and (4.26) we obtain (4.25).

We next show that \( V^d_{f,N,h} \) is bounded on \( Y_{a,N}^d \otimes X_h \). Since \( f \) is continuous on \( I^d \otimes \overline{D} \), from the definitions of \( \tilde{S}_Nf(y, x) \) and \( \tilde{S}_Nf \) we know that \( \tilde{S}_Nf(y, x) \) is continuous which means that
\[
\max_{y \in I^d, x \in D} |\tilde{S}_Nf(y, x)| < \infty.
\]
\( (4.27) \)
Thus
\[
\left| \int_D w(y) \int_{I^d} (\tilde{S}_Nf(y, x))^2 dydx \right| < \infty.
\]
\( (4.28) \)
The boundness of \( V^d_{f,N,h} \) on \( Y_{a,N}^d \otimes X_h \) then follows from (4.25) and (4.28) and the Hölder inequality. \( \square \)

**Lemma 4.7** Let \( r := [r_k \in (1, \gamma_k) : k \in \mathbb{Z}_d] \). If \( a_k = \ln r_k \) for all \( k \in \mathbb{Z}_d \), and there exists a positive constant \( c \) such that for all \( x \in D \), \( y \in I^d \) and \( \alpha \in \mathbb{N}^d \),
\[
|\partial^\alpha_y f(y, x)| \leq c\alpha! \| \alpha \| \| f \|_\infty,
\]
\( (4.29) \)
then there exists a positive constant \( c_r \) such that for all \( N \in \mathbb{N} \) and all \( h > 0 \),
\[
\sup_{v \in Y_{a,N}^d \otimes X_h} \frac{|V_f(v) - V^d_{f,N,h}(v)|}{\|v\|_{L^2_{H,w}(I^d)}} \leq c_r e^{-\epsilon(S)} \| f \|_\infty.
\]
\( (4.30) \)
Proof: From the definition of $V_f$ and Lemma 4.6, we know that for all $N \in \mathbb{N}$, all $h > 0$ and all $v \in Y^{d}_{a,N} \otimes X^{2}_{h}$,

$$|V^{d}_{f}(v) - V^{d}_{f,N,h}(v)| \leq \int_{I^{d}} w(y) \int_{D} |f(y,x) - \widehat{S}_{N} f(y,x)| \ |v(y,x)| \ dx \ dy.$$  \hspace{1cm} (4.31)

It follows from (4.29) that, function $f(\cdot, x) \in C^{\infty}(I^{d})$ admits a holomorphic extension no $E_{r}$. Thus, from Theorem 3.6, there exists a positive constant $c_{1}$ such that for all $x \in D$ and $N \in \mathbb{N}$,

$$\left\| f(\cdot, x) - \widehat{S}_{N} f(\cdot, x) \right\|_{\infty} \leq c_{1} M_{r}(f(\cdot, x)) e^{-\theta(\gamma)}.$$ \hspace{1cm} (4.32)

Note that applying (4.29) to the Taylor expansions of $f(\cdot, x)$ yields that there exits a positive constant $\tilde{c}_{r}$ such that for all $x \in D$, $M_{r}(f(\cdot, x)) \leq \tilde{c}_{r} \ |f|_{\infty}$. Thus, combining (4.31) and (4.32) leads to

$$|V^{d}_{f}(v) - V^{d}_{f,N,h}(v)| \leq c_{r} \ |f|_{\infty} e^{-\theta(\gamma)} \int_{I^{d}} w(y) \int_{D} |v(y,x)| \ dx \ dy.$$ \hspace{1cm} (4.33)

It follows from (4.33) and Hölder inequality that there exists a positive constant $c_{r}$ such that for all $N \in \mathbb{N}$, all $h > 0$ and all $v \in Y^{d}_{a,N} \otimes X^{2}_{h}$,

$$|V^{d}_{f}(v) - V^{d}_{f,N,h}(v)| \leq c_{r} \ |f|_{\infty} e^{-\theta(\gamma)} \ |v|_{L^{2}_{H,w}(I^{d})}.$$ \hspace{1cm} (4.34)

(4.30) then follows from (4.34). \hspace{1cm} $\Box$

Substituting the inequalities (4.22) and (4.30) into (4.3), we obtain the estimate of $\ |u - \tilde{u}^{d}_{N,h}|_{L^{2}_{H,w}(I^{d}, H^{1}_{b}(D))}$ which we present in the following theorem.

**Theorem 4.8** Let $r := [r_{k} \in (1, \gamma_{k}) : k \in \mathbb{Z}_{d}]$, $a_{k} = \ln r_{k}$ for all $k \in \mathbb{Z}_{d}$ and $f \in C_{H}(I^{d})$. If the assumption (A1) holds, and for all $y \in I^{d}$, $u(y, \cdot) \in H^{2}(D)$, and for each $k \in \mathbb{Z}_{d}$,

$$\sup_{x \in D} |b_{k}(x)| \leq \frac{a_{\min}}{d} \theta_{k},$$ \hspace{1cm} (4.35)

and there exists a positive constant $c$ such that for all $x \in D$, $y \in I^{d}$ and $\alpha \in \mathbb{N}_{0}^{d}$,

$$|\partial_{y}^{\alpha} f(y,x)| \leq \ c \ |\alpha| \ |\theta|^{\alpha} \ |f|_{\infty},$$ \hspace{1cm} (4.36)

then there exists a positive constants $c_{r}$ such that for all $N \in \mathbb{N}$ and $h > 0$,

$$\ |u - \tilde{u}^{d}_{N,h}|_{L^{2}_{H,w}(I^{d})} \leq c_{r} \left( h + e^{-\theta(\gamma)} \right) \ |f|_{\infty}.$$ \hspace{1cm} (4.37)

Proof: Substituting (4.22) and (4.30) into (4.3), we know that there exists a positive constant $\tilde{c}_{r}$ such that for all $N \in \mathbb{N}$ and $h > 0$,

$$\ |u - \tilde{u}^{d}_{N,h}|_{L^{2}_{H,w}(I^{d})} \leq \tilde{c}_{r} \left( h \ |u|_{L^{2}_{H,w}(I^{d})} + e^{-\theta(\gamma)} M(f) \right) + c_{r} \ |f|_{\infty} e^{-\theta(\gamma)}.$$ \hspace{1cm} (4.38)

From the assumption (A1), we know that

$$a_{\min} \ |u|_{L^{2}_{H,w}(I^{d})}^{2} \leq A(u, u).$$ \hspace{1cm} (4.39)

Note that $A(u, u) = V_{f}(u)$ and

$$V_{f}(u) \leq \ |f|_{\infty} \int_{I^{d}} \ dx \ dy \ |u|_{L^{2}_{H,w}(I^{d})}.$$
It follows from (4.39) that there is a positive constant \( c_2 \) such that
\[
\|u\|_{L_{H,w}^2(I^d)} \leq c_2 \|f\|_\infty. \tag{4.40}
\]
Combining (4.38) and (4.40), we have that for all \( N \in \mathbb{N} \) and \( h > 0 \),
\[
\|u - \tilde{u}_{N,h}^d\|_{L_{H,w}^2(I^d)} \leq \tilde{c}_r (c_2 h \|f\|_\infty + \mathcal{M}(f)e^{-\varrho(S)}) + \tilde{c}_r \|f\|_\infty e^{-\varrho(S)}. \tag{4.41}
\]
Since \( \mathcal{M}(f) \leq \|f\|_\infty \int_D dx \), (4.37) then follows from (4.41) with \( c_r = \max\{\tilde{c}_r c_2, \tilde{c}_r \int_D dx, \tilde{c}_r\} \).

\[\square\]

5 Numerical examples

In this section we conduct numerical experiments to verify the theoretical results on computing complexity and error estimates of our algorithm. Let
\[
\text{Err} := \|u - \tilde{u}_{N,h}^d\|_{L_{H,w}^2(I^d)},
\]
and
\[
\text{Err}^\prime := \|\tilde{u}_{N+1,h}^d - \tilde{u}_{N,h}^d\|_{L_{H,w}^2(I^d)}.
\]

We shall use ‘Time’ to denote the computing time measured in seconds spent in generating the matrix \( \mathbf{K}_N^d \), vector \( \tilde{\mathbf{f}}_{N,h}^d \) and solving the linear system (4.1) by the conjugate gradient method. We denote the number of elements in vector \( \tilde{\mathbf{u}}_{N,h}^d \) by ‘Num’.

**Example 1:** In this example, we consider the elliptic stochastic partial differential equation problem (2.1) with random coefficient
\[
a(y, x) := 1 + \sum_{k \in \mathbb{Z}_d} (k + 1)e^{-((x_0^2 + x_1^2)/(k+1))}y_k, \quad y \in I^d \text{ and } x \in D,
\]
and forcing term
\[
f(y, x) = 2\pi^2 (e^{2\Pi_k \in \mathbb{Z}_d} y_k^k + 1) \sin(\pi x_0) \sin(\pi x_1)
- \sum_{k \in \mathbb{Z}_d} (e^{2\Pi_k \in \mathbb{Z}_d} y_k^k + 1) y_k e^{-((x_0^2 + x_1^2)/(k+1))} [4 \sin(\pi x_0) \sin(\pi x_1)
+ 2 \sin(\pi x_0) \cos(\pi x_1) x_1/\pi + 2 \cos(\pi x_0) \sin(\pi x_1) x_0/\pi].
\]

The joint probability density function of \((y_1, \ldots, y_{d-1})\) is given by
\[
u(y) := 1, \quad y := [y_k : k \in \mathbb{Z}_d] \in I^d.
\]

In this case, the exact solution \( u \) is given by
\[
u(y, x) = (e^{2\Pi_k \in \mathbb{Z}_d} y_k^k + 1) \sin(\pi x_0) \sin(\pi x_1).
\]

In Figure 1 we plot the computing time versus the partition level \( n \) for \( d = 4, 6 \) and 8. The graph verifies that the computing complexity of our algorithm is bounded by \( O(N \ln^{d+1} N) \). The numerical errors are plotted in Figure 1. In the images (a) and (b) of Figure 1, we plot the values of Err’ with \( d = 4, 6, 8, 10, N = 2, 3, 4, 5, 6, h = 1/8 \). These graphs show the exponential convergence in the probability space. In the images (c) and (d) of Figure 1, we plot the values of Err with \( d = 4, 6, 8, N = 2, 3, 4, 5, 6, h = 1/8 \). These graphs show the overall error is eventually dominated by the finite element error in spatial dimension.
Example 2: In this example, all the input data are the same as in Example 1 except that the joint density probability density function of \((y_1, \ldots, y_{d-1})\) which is given by
\[
w(y) := \prod_{k \in \mathbb{Z}_d} (1 - y_k^2), \quad y := [y_k : k \in \mathbb{Z}_d] \in I^d.
\]
Tables ?? and ?? list the values Num, Time, Err and Err’ for the cases \(d = 4, d = 6\) and \(d = 8, d = 10\), respectively. These numerical results are shown in Figure 2. As in Example 1, Figure 2 verifies the computing complexity of our algorithm and the theoretical error estimates derived in last section.

References


Figure 2: Numerical results of Example 2


