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Abstract

The numerical solution of backward doubly stochastic differential equations (BDS-DES) and the related stochastic partial differential equations (Zakai equations) are considered. First order algorithms are constructed using a generalized Itô-Taylor formula for two-sided stochastic differentials. The convergence order is proved through rigorous error analysis. Numerical experiments are carried out to demonstrate the efficiency of the proposed numerical scheme.

Keywords: SPDEs, Zakai equation, backward SDEs, SPDEs, conditional expectation

1. Introduction

We consider the following forward-backward doubly stochastic differential equation (FBDSDE),

\begin{align*}
X_{t}^{s,x} &= x + \int_{s}^{t} b(X_{r}^{s,x}) dr + \int_{s}^{t} \sigma(X_{r}^{s,x}) dW_{r}, \quad t \leq s \leq T, \\
Y_{t}^{s,x} &= h(X_{T}^{s,x}) + \int_{t}^{T} f(r,X_{r}^{s,x},Y_{r}^{s,x}) dr \\
&\quad + \int_{s}^{T} g(r,X_{r}^{s,x},Y_{r}^{s,x}) d\mathcal{B}_{r} - \int_{s}^{T} Z_{r}^{s,x} dW_{r}, \quad t \leq s \leq T,
\end{align*}

where \( f, g, b \) and \( \sigma \) are given functions, and \( W \) and \( \mathcal{B} \) are two independent Brownian motions. The first equation in (1) is the forward stochastic differential equation (SDE) with the standard Itô integral \( \int_{s}^{t} dW_{r} \), while the second equation in (1) contains both...
the standard Itô integral and the backward Itô integral $\int_s^T \cdot \, dB_r$, which will be defined in Section 2.

Without the backward Itô integral term, (1) becomes a forward-backward stochastic differential equation (FBSDE). On the other hand, if $b = 0$ and $\sigma = 0$, (1) becomes a standard backward doubly stochastic differential equation (BDSDE). The existence and uniqueness of the solution of BDSDEs was established by Peng and Pardoux in ([22]). We refer to [2, 15, 26] for further theoretical results on BDSDEs and [13] for a study of BDSDEs related to optimal control problems.

One of the most important properties of an FBDSDE is its equivalence to the following parabolic stochastic partial differential equation (SPDE) [22]

$$u_t(x) = h(x) + \int_t^T (Lu_s(x) + f(s, x, u_s(x))) \, ds$$
$$+ \int_t^T g(s, x, u_s(x)) \, dB_s, \quad t \in [0, T],$$

where $L$ is the elliptic partial differential operator defined as

$$(Lu_t)(x) := (L(u_t))(x), \quad 1 \leq i \leq k,$$

with

$$L := \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}.$$

The SPDE (2) given above is also known as a Zakai-type equation, and has been widely used in solving nonlinear filtering problems ([27]). A number of numerical methods have been proposed to solve Zakai equations ([4, 5, 7, 8, 9, 11, 12, 10, 14]).

In this paper, we attempt to develop high order efficient numerical methods for solving BDSDEs and thus high order methods for solving SPDEs through the equivalence between SPDEs and BDSDEs. Numerous efficient algorithms for numerical solutions of SDEs have been developed (see [16, 24, 25]). Several effective numerical approaches for BSDEs and FBSDEs have also been proposed in the last decade, including primary schemes for BSDEs ([1, 6, 17, 28]), a four-step scheme for FBSDEs by Ma, Protter and Yong([18, 19, 21]), and the $\theta$-scheme with high convergence rate for BSDEs by Zhao et. al([29, 31]).

Obviously solving BDSDEs numerically is more difficult than solving forward stochastic differential equations (SDEs) since it involves solving both forward SDEs and backward stochastic differential equations (BSDEs) with both forward and backward Brownian motions. In comparison to SDEs and FBSEs, efficient high order numerical algorithms for BDSDEs have not been well developed. A half-order numerical scheme with Euler approximation for the backward stochastic integrals was proposed by the authors in [3]. The bottleneck of constructing accurate numerical algorithms for BDSDEs is the approximation of the backward Itô integrals involved in (1) by a high order quadrature rule. For standard SDEs, one may obtain such a quadrature rule through the Itô-Taylor expansion for the Itô integrals. However, this approach does not apply for the case of the backward Itô integrals in (1).
In this research, we propose to use an Itô-Taylor formula for two-sided stochastic integrals [22, 23] to obtain high order quadrature rules for the backward Itô integrals and hence a class of high order numerical schemes for BDSDEs. To obtain rigorous error estimates, we first derive error estimates with respect to truncation errors. This is equivalent to a stability analysis for the numerical algorithm. Then we derive the error estimates of the approximate solutions through the estimation of truncation errors. We will focus on the construction of a first order scheme in this paper. However, our methodology can be extended to higher order numerical schemes using higher order Itô-Taylor expansions for two-sided stochastic integrals.

The rest of the paper is organized as follows. In Section 2, we carefully define BDSDEs and discuss the relation between BDSDEs and SPDEs. In Section 3 we construct a first order numerical scheme for the BDSDE (1) using the Itô-Taylor formula. In Section 4, we carry out rigorous error analysis and show the first order convergence of the numerical scheme constructed in Section 3. Section 4 contains numerical experiments demonstrating the accuracy and effectiveness of our scheme. Section 5 presents some concluding remarks.

2. BDSDEs and SPDEs

In this section, we provide a brief introduction to forward-backward doubly stochastic differential equations (FBDSDEs) and the relationship between FBDSDEs and the SPDE (2).

Let \((\Lambda, \mathcal{F}, P)\) be a complete probability space, \(T > 0\) a terminal time, while \(\{W_t, 0 \leq t \leq T\}\) and \(\{B_t, 0 \leq t \leq T\}\) are two mutually independent standard Brownian motions defined on \((\Lambda, \mathcal{F}, P)\) with their values in \(\mathbb{R}^d\) and in \(\mathbb{R}^l\), respectively. Let \(N\) denote the class of \(P\)-null sets of \(\mathcal{F}\). For each \(t \in [0, T]\), we define \(\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^B\), where \(\mathcal{F}_{s,t}^\eta = \sigma(\eta_r - \eta_s; s \leq r \leq t) \vee N\) is the \(\sigma\)-field generated by \(\{\eta_r - \eta_s; s \leq r \leq t\}\), and \(\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta\) for a stochastic process \(\eta\). Note that the collection \(\{\mathcal{F}_t, t \in [0, T]\}\) is neither increasing nor decreasing and it is not a filtration. We use \(E[X]\) to denote the mathematical expectation of the random variable \(X\). For any positive integer \(n \in \mathbb{N}\), we define spaces \(M^2(0, T; \mathbb{R}^n)\) and \(S^2([0, T]; \mathbb{R}^n)\) as follows.

\[
M^2(0, T; \mathbb{R}^n) := \{\varphi_t | \varphi_t \in \mathbb{R}^n, E \int_0^T |\varphi_t|^2 dt < \infty, \varphi_t \in \mathcal{F}_t, \text{ a.e. } t \in [0, T]\}
\]

and

\[
S^2([0, T]; \mathbb{R}^n) := \{\varphi_t | \varphi_t \in \mathbb{R}^n, E(\sup_{0 \leq t \leq T} |\varphi_t|^2) < \infty, \varphi_t \in \mathcal{F}_t, \text{ } t \in [0, T]\}.
\]

Let

\[
f : \Lambda \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^k
\]

and

\[
g : \Lambda \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^{k \times l}
\]
be jointly measurable such that for any \((x, y) \in \mathbb{R}^d \times \mathbb{R}^k\),
\[
f(\cdot, x, y) \in M^2(0, T; \mathbb{R}^k),
\]
\[
g(\cdot, x, y) \in M^2(0, T; \mathbb{R}^{d \times l}).
\]
We assume that there exists a constant \(c > 0\) such that for any \(x_1, x_2 \in \mathbb{R}^d\), \((\omega, t, x) \in \Lambda \times [0, T] \times \mathbb{R}^d\) and \(y_1, y_2 \in \mathbb{R}^k\),
\[
|b(x_1) - b(x_2)| \leq c|x_1 - x_2|,
\]
\[
||\sigma(x_1) - \sigma(x_2)|| \leq c|x_1 - x_2|,
\]
\[
|f(t, x, y_1) - f(t, x, y_2)| \leq c|y_1 - y_2|,
\]
\[
|g(t, x, y_1) - g(t, x, y_2)| \leq c|y_1 - y_2|.
\]

From [22], we know that there exists a process triple \(\{(X^{\omega, x}_t, Y^{\omega, x}_t, Z^{\omega, x}_t); (t, x) \in [0, T] \times \mathbb{R}^d\}\) which is the unique solution to the following FBDSDEs: For \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\)
\[
X^{\omega, x}_s = x + \int_s^t b(X^{\omega, x}_r)dr + \int_s^t \sigma(X^{\omega, x}_r)dW_r, \quad t \leq s \leq T,
\tag{3}
\]
\[
Y^{\omega, x}_s = h(X^{\omega, x}_T) + \int_s^T f(r, X^{\omega, x}_r, Y^{\omega, x}_r)dr
+ \int_s^T g(r, X^{\omega, x}_r, Y^{\omega, x}_r)d\widehat{B}_r - \int_s^T Z^{\omega, x}_r dB_r, \quad t \leq s \leq T,
\tag{4}
\]
where \((Y^{\omega, x}_s, Z^{\omega, x}_s) \in S^2([0, T]; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{d \times l})\). Here \(d\widehat{B}_r\) denotes the backward Itô integration, i.e., for an \(\mathcal{F}^B_{t, r}\) adapted process \(V_r\), and quasi-uniform time partitions \(\Delta: 0 = t_0 < t_1 < t_2 < \cdots < t_{N_r - 1} < t_{N_r} = T\),
\[
\int_0^T V_r d\widehat{B}_r := \lim_{\Delta \to 0} \sum_{n=0}^{N_r} V_{t_{n+1}}(B_{t_{n+1}} - B_n)
\]
where \(\Delta t = \max_{0 \leq s \leq N_r - 1} (t_{s+1} - t_s)\). According to Pardoux and Peng ([22]), we have the following nonlinear Feynman-Kac formula.
\[
Y^{\omega, x}_s = u_s(X^{\omega, x}_s), \quad Z^{\omega, x}_s = (\nabla u_s, \sigma)(X^{\omega, x}_s); (t, x) \in [0, T] \times \mathbb{R}^d,
\tag{5}
\]
where \(u = u_s(x) \in \mathcal{R}^k\) is the unique solution of the system of backward stochastic partial differential equations (2).

For the BDSDE considered in this paper, we assume
\[
X^{\omega, x}_s = x + W_s, \quad x \in \Omega, s \in [0, T],
\]
which is a Brownian motion starting at space point \(x\). Then the elliptic partial differential operator \(L\) becomes
\[
L = \frac{1}{2} \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}.
\]
3. Numerical algorithm

To derive a numerical algorithm for BDSDE (4), we introduce the following time partition on \([0, T]\)
\[\mathcal{R}_{sh} = \{t_n | t_n \in [0, T], t_n < t_{n+1}, n = 0, 1, \ldots, N_T - 1, t_0 = 0, t_{N_T} = T\}\]
and denote \(\Delta t_n = t_{n+1} - t_n, \Delta B_{t_n} = B_{t_{n+1}} - B_{t_n}, \Delta W_{t_n} = W_{t_{n+1}} - W_{t_n}, n = 0, \ldots, N_T - 1, \Delta t = \max_{0 \leq n \leq N_T - 1} \Delta t_n\).

For simplicity of presentation, we consider the one-dimensional case; similar results for multi-dimensional cases can be derived through a straightforward generalization.

In the sequel, we assume that \(g\) in BDSDE (4) is a function of time \(t\), the Brownian motions \(W_t\) and \(B_s\), and the solution \(Y^{t,x}_s\) of the BDSDE. As in the SDE case, the Euler scheme for the numerical solution of the BDSDE (2) leads to only half-order accuracy \([3]\). In such a scheme, the stochastic integral
\[\int_t^T g(r, W_r, B_r, Y^{t,x}_r)dB_r\]
is approximated by a right point formula for the explicit Euler scheme. To improve the convergence order, our numerical method shall apply a special Itô-Taylor expansion \([22, 23]\) for the doubly stochastic differential to derive a higher order quadrature for (6). This is the essential idea for obtaining higher convergence orders in our numerical schemes.

3.1. Reference equations

In this subsection we derive the so-called reference equations (see \([29]\) for the original definition) which will be used to obtain our numerical scheme. For this purpose, we define \(\mathcal{F}^{W_t}_{s}: = \sigma(W_r; t \leq r \leq s) \vee \sigma(W_t) \vee \sigma(B_s; 0 \leq t \leq T)\). Thus the conditional expectation, given \(\mathcal{F}^{W_t}_{s}\) and the path \(B_s(0 \leq s \leq T)\), is a random variable determined by \(W_t\). We use \(E^s_t[X]\) to denote the conditional expectation \(E[X|\mathcal{F}^{W_t}_{s}]\) of the random variable \(X\) in the case that the value of \(W_t\) is equal to \(x\).

Let \((y_t, z_t)\) be the solution of the BDSDE. To further simplify the presentation, we adopt the following notations throughout the rest of the paper.
\[f(s, y_s) := f(s, W_s, B_s, y_s), \quad g(s, y_s) := g(s, W_s, B_s, y_s), \quad g'_B(s, y_s) := g'_B(s, W_s, B_s, y_s), \quad g'_y(s, y_s) := g'_y(s, W_s, B_s, y_s).\]

With these notations, from (4), we have the following identity
\[y_n = y_{n+1} + \int_{t_n}^{t_{n+1}} f(s, y_s)ds - \int_{t_n}^{t_{n+1}} z_s dW_s + \int_{t_n}^{t_{n+1}} g(s, y_s)dB_s. \quad (7)\]
Taking the conditional expectation \( \mathbb{E}_t^{y_{t_n}}[\cdot] \) on both sides of (7), we obtain

\[
y_{t_n}^x = \mathbb{E}_t^{y_{t_n}}[y_{t_n}] + \int_t^{t_n} \mathbb{E}_s^{y_{t_n}}[f(s, y_s)]ds + \int_t^{t_n} \mathbb{E}_s^{y_{t_n}}[g(s, y_s)]dB_s, \tag{8}
\]

where \( y_{t_n}^x = \mathbb{E}_t^{y_{t_n}}[y_{t_n}] \) is the value of \( y_{t_n} \) at point \( x \) on time level \( t_n \).

For the second term on the right hand side of (8), we use the right point formula to obtain

\[
\int_t^{t_n} \mathbb{E}_s^{y_{t_n}}[f(s, y_s)]ds = \Delta_n \mathbb{E}_t^{y_{t_n}}[f(t_{n+1}, y_{t_{n+1}})] + R_{y_n}^{W_n}
\]

where

\[
R_{y_n}^{W_n} = \int_t^{t_n} \mathbb{E}_s^{y_{t_n}}[f(s, y_s) - f(t_{n+1}, y_{t_{n+1}})]ds. \tag{9}
\]

For the backward Itô integral in (7), the application of the right point formula results in a half-order numerical scheme. To obtain a first order or higher order numerical scheme, one needs a more accurate quadrature rule to approximate the backward Itô integral. To achieve this goal, we use Itô’s formula for the doubly stochastic differential( [22] ). To proceed, we denote \( f_x = f(s, y_s) \) and \( g_x = g(s, y_s) \) and let \( \alpha_t = (t, W_t, B_t, y_t) \), \( \beta_t = (1, 0, 0, -f) \), \( \gamma_t = (0, 0, 1, -g) \), and \( \delta_t = (0, 1, 0, z) \). From (4), the process \( \alpha_t \) satisfies the stochastic differential equation

\[
\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t \delta_s dW_s.
\]

Using Itô’s formula for the doubly stochastic differential( see Lemma 1.3, [22] ), we have that

\[
g(t, W_t, B_t, y_t) = g(0, 0, 0, y_0(0)) + \int_0^t (g_x'(s, W_s, B_s, y_s) - g_x'(s, W_s, B_s, y_s) \cdot f_s)ds + \int_0^t (g_x''(s, W_s, B_s, y_s) - g_x''(s, W_s, B_s, y_s) \cdot g_s)dB_s + \int_0^t (g_x''(s, W_s, B_s, y_s) + g_x''(s, W_s, B_s, y_s) \cdot z_s)dW_s - \frac{1}{2} \int_0^t (g_x'''(s, W_s, B_s, y_s) + g_x'''(s, W_s, B_s, y_s) \cdot g_s^2)ds + \frac{1}{2} \int_0^t (g'''(s, W_s, B_s, y_s) + g'''(s, W_s, B_s, y_s) \cdot z_s^2)ds,
\]

where \( g_x', g_x'', g_x''' \) and \( g_x''' \) are first order partial derivatives with respect to variables \( t, B_t, W_t \) and \( y_t \), respectively; \( g_x'''' \) and \( g_x'''' \) are second order partial derivatives with respect to variables \( B_t, W_t \) and \( y_t \), respectively. For \( s < t_{n+1} \), the above equation leads to
we deduce using the identities
Taking the conditional expectation where
\[ R_{g1}(s) = \int_s^{t_n+1} (g'_y(r, W_r, B_r, y_r) - g'_y(r, W_r, B_r, y_r) \cdot f_r) dr \]

\[ + \frac{1}{2} \int_s^{t_n+1} (g''_{yy}(r, W_r, B_r, y_r) + g''_{yy}(r, W_r, B_r, y_r) \cdot g_y^2) dr \]
\[ - \frac{1}{2} \int_s^{t_n+1} (g''_{yy}(r, W_r, B_r, y_r) + g''_{yy}(r, W_r, B_r, y_r) \cdot \zeta^2) dr. \]

Taking the conditional expectation \( \mathbb{E}^{t_n-}[\cdot] \) for a given \( x \in \mathbb{R} \) on both side of (10) and using the identities
\[ \mathbb{E}^{t_n-}_{t_n}[\int_s^{t_n+1} g'_y(r, W_r, B_r, y_r) dW_r] = 0, \quad \mathbb{E}^{t_n-}_{t_n}[\int_s^{t_n+1} g'_y(r, W_r, B_r, y_r) \cdot \zeta dW_r] = 0, \]
we deduce
\[ \mathbb{E}^{t_n-}_{t_n}[g(s, W_s, B_s, y_s)] = \mathbb{E}^{t_n-}_{t_n}[g(t_{n+1}, W_{t_{n+1}}, B_{t_{n+1}}, y_{t_{n+1}})] \]
\[ - g'_{B}(t_{n+1}, W_{t_{n+1}}, B_{t_{n+1}}, y_{t_{n+1}}) \int_s^{t_{n+1}} d\bar{B}_r \]
\[ + g'_y(t_{n+1}, W_{t_{n+1}}, B_{t_{n+1}}, y_{t_{n+1}}) \int_s^{t_{n+1}} d\bar{B}_r \]
\[ \cdot g(t_{n+1}, W_{t_{n+1}}, B_{t_{n+1}}, y_{t_{n+1}}) \int_s^{t_{n+1}} d\bar{B}_r \]
\[ + \mathbb{E}^{t_n-}_{t_n}[R_{g1}^n] + \mathbb{E}^{t_n-}_{t_n}[R_{g2}^n], \]

where
\[ R_{g2}^n(s) = \int_s^{t_n+1} g'_y(r, W_r, B_r, y_r) d\bar{B}_r \]
\[ + \int_s^{t_n+1} g'_y(r, W_r, B_r, y_r) \cdot g(r, W_r, B_r, y_r) d\bar{B}_r \]
\[ - \left( - g'_{B}(t_{n+1}, W_{t_{n+1}}, B_{t_{n+1}}, y_{t_{n+1}}) \int_s^{t_{n+1}} d\bar{B}_r \right) \]
\[ + g'_y(t_{n+1}, W_{t_{n+1}}, B_{t_{n+1}}, y_{t_{n+1}}) \]
\[ \cdot g(t_{n+1}, W_{t_{n+1}}, B_{t_{n+1}}, y_{t_{n+1}}) \int_s^{t_{n+1}} d\bar{B}_r \). \]
By equation (12), we have the identity

\[
\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{g} \left[ g(s, W_s, B_s, y_s) \right] d\overline{B}_s
\]

\[
= \mathbb{E}_{t_n}^{g} \left[ g(t_{n+1}, W_{t_{n+1}}, B_{t_{n+1}}, y_{t_{n+1}}) \right] \int_{t_n}^{t_{n+1}} d\overline{B}_s
\]

\[
- \left. g_s(t_{n+1}, W_{t_{n+1}}, B_{t_{n+1}}, y_{t_{n+1}}) \right|_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} d\overline{B}_r d\overline{B}_s
\]

\[
+ \left. g_s'(t_{n+1}, W_{t_{n+1}}, B_{t_{n+1}}, y_{t_{n+1}}) \right|_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} d\overline{B}_r d\overline{B}_s
\]

\[
= \mathbb{E}_{t_n}^{g} \left[ R_n^g + R_n^g(s) \right] d\overline{B}_s
\]

(14)

where

\[
R_n^g = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{g} \left[ R_n^g(s) \right] d\overline{B}_s
\]

(15)

is the error term. Here \( R_n^g \) and \( R_n^g(s) \) are respectively defined by (11) and (13).

We need the following lemma about the double integral appearing in (14) in the derivation of our numerical algorithm.

**Lemma 1.**

\[
\int_s^t \int_r^t d\overline{B}_r d\overline{B}_r = \frac{1}{2} (\Delta B_t \cdot \Delta B_t - (t - s)),
\]

(16)

where \( \Delta B_t = B_t - B_s \).

**Proof:** Under the backward filtration \( \mathcal{F}_{t,T}^g \) and by the Itô-Taylor formula for \( (B_t)^2 \), we have

\[
\int_s^t B_r d\overline{B}_r = B_t \cdot (B_t - B_s)
\]

and

\[
(B_t)^2 = (B_s)^2 + \int_s^t 2B_r d\overline{B}_r - \frac{1}{2} \int_s^t 2dr,
\]

which leads to

\[
- \int_s^t B_r d\overline{B}_r = \frac{1}{2} ((B_s)^2 - (B_t)^2) - \frac{1}{2} (t - s).
\]

Thus

\[
\int_s^t \int_r^t d\overline{B}_r d\overline{B}_r = \int_s^t B_r d\overline{B}_r - \int_s^t B_r d\overline{B}_r
\]

\[
= B_t \cdot (B_t - B_s) + \frac{1}{2} ((B_s)^2 - (B_t)^2) - \frac{1}{2} (t - s)
\]

\[
= \frac{1}{2} (B_t - B_s)^2 - \frac{1}{2} (t - s),
\]

which is identity (16). □
Now letting \( s = t_n \) and \( t = t_{n+1} \) in (16), by (14) and Lemma 1 we obtain

\[
\int_{t_n}^{t_{n+1}} \mathbb{E}_t^{y_{t_n}} [g(s, y_s)] dB_s = \mathbb{E}_t^{y_{t_n}} [g(t_{n+1}, y_{t_{n+1}}) \Delta B_{t_{n+1}} - g'_B(t_{n+1}, y_{t_{n+1}})] \frac{1}{2} (\Delta B_{t_{n+1}} \cdot \Delta B_{t_{n+1}} - \Delta t_n) \tag{17}
\]

where \( R^{\cdot, n}_Y \) is still defined by (15), but with \( s \) and \( t \) respectively replaced by \( t_n \) and \( t_{n+1} \).

Inserting (9) and (17) into (8) yields

\[
y^{t_{n+1}}_{t_n} = \mathbb{E}_t^{y_{t_n}} [y_{t_{n+1}}] + \Delta t_n \cdot (\mathbb{E}_t^{y_{t_n}} [f(t_{n+1}, y_{t_{n+1}})] + \frac{1}{2} \mathbb{E}_t^{y_{t_n}} [g'(t_{n+1}, y_{t_{n+1}})])
\]

\[
- \frac{1}{2} \mathbb{E}_t^{y_{t_n}} [g'(t_{n+1}, y_{t_{n+1}})] + \Delta B_{t_{n+1}} \cdot (\mathbb{E}_t^{y_{t_n}} [g(t_{n+1}, y_{t_{n+1}})] - \frac{1}{2} \mathbb{E}_t^{y_{t_n}} [g'(t_{n+1}, y_{t_{n+1}})]) \Delta B_{t_{n+1}}
\]

\[
+ \frac{1}{2} \mathbb{E}_t^{y_{t_n}} [g'(t_{n+1}, y_{t_{n+1}})] \cdot (g(t_{n+1}, y_{t_{n+1}}) \Delta B_{t_{n+1}}) + R^{\cdot, n}_Y,
\tag{18}
\]

where

\[
R^{\cdot, n}_Y = R^{W, n}_Y + R^{B, n}_Y .
\tag{19}
\]

Equation (18) is said to be the reference equation for the solution \( y_t \) of the BDSDEs.

Next we derive the reference equation for the solution \( z_t \) of the BDSDE (4). For this purpose, we multiply by \( \Delta W_{t_{n+1}} \) on both sides of (7) and take the conditional expectation \( \mathbb{E}_t^{y_{t_n}} [\cdot] \) to obtain

\[
- \mathbb{E}_t^{y_{t_n}} [y_{t_{n+1}} \Delta W_{t_{n+1}}] = \int_{t_n}^{t_{n+1}} \mathbb{E}_t^{y_{t_n}} [f(s, y_s) \Delta W_{t_{n+1}}] ds
\]

\[
+ \int_{t_n}^{t_{n+1}} \mathbb{E}_t^{y_{t_n}} [g(s, y_s) \Delta W_{t_{n+1}}] dB_s
\]

\[
- \int_{t_n}^{t_{n+1}} \mathbb{E}_t^{y_{t_n}} [z_s] ds.
\tag{20}
\]

Next, approximating the first two integrals on the right hand side of (20) with the right point formula, and the last integral with the left-point formula leads to

\[
\int_{t_n}^{t_{n+1}} \mathbb{E}_t^{y_{t_n}} [f(s, y_s) \Delta W_{t_{n+1}}] ds
\]

\[
= \Delta t_n \mathbb{E}_t^{y_{t_n}} [f(t_{n+1}, y_{t_{n+1}}) \Delta W_{t_{n+1}}] + R^{W, n}_Y,
\tag{21}
\]

\[
- \int_{t_n}^{t_{n+1}} \mathbb{E}_t^{y_{t_n}} [z_s] ds = - \Delta t_n \mathbb{E}_t^{y_{t_n}} [z_s] + R^{B, n}_Y,
\tag{22}
\]

and

\[
\int_{t_n}^{t_{n+1}} \mathbb{E}_t^{y_{t_n}} [g(s, y_s) \Delta W_{t_{n+1}}] dB_s
\]

\[
= \mathbb{E}_t^{y_{t_n}} [g(t_{n+1}, y_{t_{n+1}}) \Delta W_{t_{n+1}}] \Delta B_{t_{n+1}} + R^{B, n}_Y,
\tag{23}
\]
where \( z_{t_n}^{x,n} = \mathbb{E}^x_z[t_n], \)

\[
R_{z_1}^{W,n} = \int_{t_n}^{t_{n+1}} \mathbb{E}^x_{t_n}[f(s, y_s) - f(t_{n+1}, y_{t_{n+1}})] \Delta W_{t_{n+1}}] ds,
\]

(24)

and

\[
R_{z_2}^{W,n} = - \int_{t_n}^{t_{n+1}} [\mathbb{E}^x_{t_n}[z_s] - z_{t_n}^{x,n}] ds,
\]

(25)

Inserting (21), (22) and (23) into (20) yields the reference equation for \( z_t \) as follows.

\[
\Delta t_n z_{t_n}^{x,n} = \mathbb{E}^x_{t_n}[y_{t_{n+1}} - z_{t_n}^{x,n}] + \Delta t_n M_{t_n} + \Delta B_{t_{n+1}} \cdot N_{t_n},
\]

(27)

where

\[
R_n^z = R_{z_1}^{W,n} + R_{z_2}^{W,n} + R_{z}^{B,n}.
\]

(28)

(18) and (27) are essential in the derivation of our numerical schemes for the BDSDE (4).

3.2. First order numerical scheme

Dropping the error terms \( R_n^z \) and \( R_n^z \) in the two reference equations (18) and (27), we obtain the numerical algorithm for solving BDSDE (4) as follows. Given the random variable \( y^{N_n} \), for \( n = N_T - 1, N_T - 2, \ldots, 1, 0 \), solve for \( y^n \) and \( z^n \) backward from

\[
y^n = \mathbb{E}^x_{t_n}[y^{n+1}] + \Delta t_n \cdot M_{t_n} + \Delta B_{t_{n+1}} \cdot N_{t_n},
\]

(29)

\[
\Delta t_n z^n = \mathbb{E}^x_{t_n}[y^{n+1}] + \Delta t_n \cdot M_{t_n} + \Delta B_{t_{n+1}} \cdot N_{t_n}.
\]

(30)

where

\[
M_{t_n} = \mathbb{E}^x_{t_n}[f(t_{n+1}, y^{n+1})]
\]

\[
\frac{1}{2} \mathbb{E}^x_{t_n}[g_y'(t_{n+1}, y^{n+1})]
\]

\[
\frac{1}{2} \mathbb{E}^x_{t_n}[g_y'(t_{n+1}, y^{n+1}) \cdot g(t_{n+1}, y^{n+1})]
\]

and

\[
N_{t_n} = \mathbb{E}^x_{t_n}[g(t_{n+1}, y^{n+1})]
\]

\[
- \frac{1}{2} \mathbb{E}^x_{t_n}[g_y'(t_{n+1}, y^{n+1}) \Delta B_{t_{n+1}}]
\]

\[
+ \frac{1}{2} \mathbb{E}^x_{t_n}[g_y'(t_{n+1}, y^{n+1}) \cdot g(t_{n+1}, y^{n+1}) \Delta B_{t_{n+1}}]
\]

\((y^n, z^n)\) is the approximate solution for \((y, z)\) at \( t = t_n, n = 0, \ldots, N_T - 1.\)
4. Error estimate

In this section, we derive the error estimates for numerical scheme (29)–(30). Throughout the rest of the paper, we shall use $C$ to denote a generic constant whose value may vary in various appearances.

4.1. Error estimates with respect to the truncation errors

In this subsection, we derive error estimates for the approximate solutions $y^n$ and $z^n$ of (29)–(30) with respect to the truncation errors. This may also be considered as a stability analysis of the algorithm. To this end, we subtract (29) and (30) from (18) and (27) respectively to obtain the error equations for $y_t$ and $z_t$:

$$
\begin{align*}
\text{error estimate for } y_t & : e^n_y = E_{t_n}^x [e^n_y] + \Delta t_n \cdot \left( E_{t_n}^x [e^n_f] + \frac{1}{2} E_{t_n}^y [e^n_{g_y}] - \frac{1}{2} E_{t_n}^y [e^n_{g'_y}] \right) \\
& + \Delta B_{t_{n+1}} \cdot \left( E_{t_n}^y [e^n_y] + \frac{1}{2} E_{t_n}^y [e^n_{g_y}] \Delta B_{t_{n+1}} \right) + R^n_y,
\end{align*}
$$

and

$$
\begin{align*}
\text{error estimate for } z_t & : e^n_z = E_{t_n}^x [e^n_z] + \Delta t_n \cdot \left( E_{t_n}^x [e^n_f] + \frac{1}{2} E_{t_n}^z [e^n_{g_z}] - \frac{1}{2} E_{t_n}^z [e^n_{g'_z}] \right) \\
& + \Delta B_{t_{n+1}} \cdot \left( E_{t_n}^z [e^n_z] + \frac{1}{2} E_{t_n}^z [e^n_{g_z}] \Delta B_{t_{n+1}} \right) + R^n_z,
\end{align*}
$$

where

$$
\begin{align*}
& e^n_y = y_t - y^n, \\
& e^n_z = z_t - z^n, \\
& e^n_{y+1} = y_{t_{n+1}} - y^{n+1}, \\
& e^n_{z+1} = z_{t_{n+1}} - z^{n+1}, \\
& e^n_{y+1} = f(t_{n+1}, y_{t_{n+1}}) - f(t_n, y^n), \\
& e^n_{g_y} = g_y(t_{n+1}, y_{t_{n+1}}) - g_y(t_n, y^n), \\
& e^n_{g'_y} = g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}}) - g'_y(t_n, y^n) \cdot g(t_n, y^n), \\
& e^n_{g_z} = g_z(t_{n+1}, y_{t_{n+1}}) - g_z(t_n, y^n).
\end{align*}
$$

**Theorem 1.** Assume that $f$, $g$, $g'_y$ and $g'_z$ are all Lipschitz continuous and $g$ is bounded. Then we have the following error estimates for our numerical scheme (29)–(30):

$$
\begin{align*}
\max_{0 \leq n \leq N_t-1} E[(e^n_y)^2] & \leq C \left\{ E[(N_y^n)^2] + \sum_{n=0}^{N_t-1} \left( E[(R^n_y)^2] + \frac{(E[R^n_{y_{\tiny{y}}}]^2)}{\Delta t_n} \right) \right\}, \\
\max_{0 \leq n \leq N_t-1} \Delta t_n E[(e^n_z)^2] & \leq C \left\{ E[(N_y^n)^2] + \sum_{n=0}^{N_t-1} \left( E[(R^n_z)^2] + \frac{(E[R^n_{z_y}]^2)}{\Delta t_n} \right) \right\} + \max_{0 \leq n \leq N_t-1} \frac{E[(R^n_z)^2]}{\Delta t_n},
\end{align*}
$$

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where $R^W_n$, $R^e_n$ and $R_n$ are defined in (9), (31) and (32) respectively.

**Proof:** Under the assumptions of the theorem, one easily obtains the estimates

\[
\begin{align*}
\mathbb{E}_t^n[|e^{n+1}_y|^2] &\leq L \mathbb{E}_t^n[|e^{n+1}_y|^2], \\
\mathbb{E}_t^n[|e^{n+1}_g|^2] &\leq L \mathbb{E}_t^n[|e^{n+1}_g|^2], \\
\mathbb{E}_t^n[|e_g^{n+1}|^2] &\leq L \mathbb{E}_t^n[|e_g^{n+1}|^2], \\
\mathbb{E}_t^n[|e_g^{n+1}|^2] &\leq L \mathbb{E}_t^n[|e_g^{n+1}|^2].
\end{align*}
\]  
(35)

Let

\[H^{n+1}_y = \mathbb{E}_t^n[e^{n+1}_y] + \frac{1}{2} \mathbb{E}_t^n[e_{g}^{n+1}] - \frac{1}{2} \mathbb{E}_t^n[e_{g}^{n+1}] \Delta B_{n+1},\]

and

\[G^{n+1}_y = \mathbb{E}_t^n[e^{n+1}_g] - \frac{1}{2} \mathbb{E}_t^n[e_{g}^{n+1}] \Delta B_{n+1} + \frac{1}{2} \mathbb{E}_t^n[e_{g}^{n+1}] \Delta B_{n+1}.\]

By (35) and Cauchy’s inequality, we derive that

\[E[(H^{n+1}_y)^2] \leq CE \left[ \mathbb{E}_t^n[(e^{n+1}_y)^2] \right]\]

(36)

and

\[E[(G^{n+1}_y)^2] \leq CE \left[ \mathbb{E}_t^n[(e^{n+1}_y)^2] + \Delta t_n \mathbb{E}_t^n[(e^{n+1}_y)^2] \right].\]

(37)

Then it follows from (31) and estimates (36) and (37) that

\[
E[(e^{n+1}_y)^2] = E \left[ \left( \mathbb{E}_t^n[e_{g}^{n+1}] \right)^2 \right. \\
+ \left. \left[ \Delta t_n \cdot (H^{n+1}_y) + \Delta B_{n+1} \cdot (G^{n+1}_y) + R^e_n \right]^2 \right] \\
+ 2 \mathbb{E}_t^n[e_{g}^{n+1}] \cdot \left[ \Delta t_n \cdot (H^{n+1}_y) + \Delta B_{n+1} \cdot (G^{n+1}_y) + R^e_n \right] \\
\leq E \left[ \left( \mathbb{E}_t^n[e_{g}^{n+1}] \right)^2 + 3 \Delta t_n \left( \mathbb{E}_t^n[e_{g}^{n+1}] \right)^2 \right] \\
+ 3 \left( \Delta B_{n+1} \right)^2 \cdot \left( \mathbb{E}_t^n[e_{g}^{n+1}] \right)^2 + \Delta t_n \Delta t_n \mathbb{E}_t^n[e_{g}^{n+1}]^2 \\
+ 3 \left( \Delta B_{n+1} \right)^2 \cdot \left( \mathbb{E}_t^n[e_{g}^{n+1}] \right)^2 + C \Delta t_n \mathbb{E}_t^n[e_{g}^{n+1}]^2 \\
+ 2 \mathbb{E}_t^n[e_{g}^{n+1}] \cdot (\Delta B_{n+1} \cdot (G^{n+1}_y) + R^e_n) \right].
\]

(38)

Notice that

\[
E \left[ \left. \mathbb{E}_t^n[e_{g}^{n+1}] \cdot \Delta B_{n+1} \cdot (G^{n+1}_y) \right\] \\
= 0 + E[\mathbb{E}_t^n[e_{g}^{n+1}] \cdot \Delta B_{n+1} \cdot (G^{n+1}_y)] \\
\leq C \Delta t_n \cdot E \left[ \left. \mathbb{E}_t^n[(e^{n+1}_y)^2] \right\] .
\]

(39)

By the property of the backward Itô integral, we find

\[
E \left[ \left. \mathbb{E}_t^n[e_{g}^{n+1}] \right\] \cdot R^W_n \right] = 0.
\]

Thus

\[
2E \left[ \left. \mathbb{E}_t^n[e_{g}^{n+1}] \right\] \cdot R^e_n \right] \\
= 2E \left[ \left. \mathbb{E}_t^n[e_{g}^{n+1}] \right\] \cdot E[R^W_n] \\
\leq \Delta t_n E[(e^{n+1}_y)^2] + \frac{(E[R^W_n])^2}{\Delta t_n}.
\]

(40)
Combining (38), (39) and (40) leads to
\[
E[(e_y^2)^2] \leq E[(e_y^{n+1})^2] + C\Delta t_n \cdot E[(e_y^n)^2] + \Delta t_n E[(e_y^{n+1})^2]
+ C \cdot E[(R^y_\Delta)^2] + 2E \left[ E[(e_y^{n+1}) \cdot R^y_\Delta] \right]
\leq (1 + C\Delta t_n)E[(e_y^{n+1})^2] + C \cdot E[(R^y_\Delta)^2] + \frac{(E[R^W_\Delta]^2)}{\Delta t_n}.
\]

From the discrete Grönwall inequality, we have the estimate
\[
\max_{0 \leq n \leq N_T-1} E[(e_y^2)^2] \leq C \{ \max_{0 \leq n \leq N_T-1} E[(e_y^n)^2] + \sum_{n=0}^{N_T-1} (E[(R^y_\Delta)^2] + \frac{(E[R^W_\Delta]^2)}{\Delta t_n}) \},
\]  \tag{41}

which is the estimate (33).

Squaring both sides of (32) and using the Cauchy inequality, we have that
\[
(\Delta t_n)^2 (e_y^2) \leq 4 \{ (E[e_y^n\Delta W_{\Delta t_n}])^2 + C(\Delta t)^2 (E[e_y^{n+1} \Delta W_{\Delta t_n}])^2
+ C(\Delta B_{\Delta t_n})^2 (E[e_y^{n+1} \Delta W_{\Delta t_n}])^2 + (R^y_\Delta)^2 \},
\]

Applying the Schwartz inequality on the three expectations of the right hand side of the above inequality and then taking the expectation, we obtain
\[
(\Delta t_n)^2 E[(e_y^2)^2] \leq C(\Delta t_n)E[(e_y^{n+1})^2] + E[(R^y_\Delta)^2]). \tag{42}
\]

From (41) and (42), we derive the estimate
\[
\max_{0 \leq n \leq N_T-1} \Delta t_n E[(e_y^2)^2] \leq C \{ \max_{0 \leq n \leq N_T-1} E[(e_y^n)^2] + \sum_{n=0}^{N_T-1} (E[(R^y_\Delta)^2] + \frac{(E[R^W_\Delta]^2)}{\Delta t_n})
+ \max_{0 \leq n \leq N_T-1} \frac{E[(R^y_\Delta)^2]}{\Delta t_n} \},
\]

which is estimate (34). \square

4.2. Error estimates

Theorem 1 indicates that the error of the numerical scheme (29)-(30) is controlled by the truncation errors \(R^W_\Delta\), \(R^y_\Delta\) and \(R^W_\Delta\). In this subsection, we will apply the estimates (33) and (34) to obtain the error estimates for the scheme defined by (29) and (30) through the estimations of \(R^W_\Delta\), \(R^y_\Delta\) and \(R^W_\Delta\).

First we state a regularity result for the exact solution \((y_t, z_t)\) of (4) which can be obtained using the standard techniques of SDEs, FBSDEs and BDSDEs [16, 20, 22].

**Lemma 2.** For bounded \(f, g \text{ and } h\), we have
\[
E[(y^t_x - y^{s}_x)^2 + E[(z^2_x)] \leq C|t - s|, \tag{43}
\]

and for bounded function \(\Psi\) with bounded second-order derivatives we have
\[
(E[\Psi(t, y_t) - \Psi(s, y_s))]^2 \leq C(t - s)^2, \tag{44}
\]

where \(C\) is a constant independent of \(s\) and \(t\). \square
We have the following estimates about the truncated errors $R_{W,n}^t$, $R_{g,1}^t$ and $R_{g,2}^t$.

**Lemma 3.** Assume that $f$ and its first and second derivatives, as well as $g$ and its first and second derivatives, are all continuous and uniformly bounded. Then we have the following estimates for the truncated errors $R_{W,n}^t$, $R_{g,1}^t$ and $R_{g,2}^t$ defined by (9), (19) and (28), respectively.

(i). $\mathbb{E}[R_{W,n}^t]^2 \leq C(\Delta t)^4$, \hfill (45)

(ii). $\mathbb{E}[(R_{g,1}^t)^2] \leq C(\Delta t)^3$, \hfill (46)

(iii). $\mathbb{E}[(R_{g,2}^t)^2] \leq C(\Delta t)^3$. \hfill (47)

Here $C$ is a constant independent of $\Delta t$.

**Proof:** (i). By the definition of $R_{W,n}^t$ (see Equation (9)), the Schwartz inequality and Lemma 2, we have that

\[
\mathbb{E}[R_{W,n}^t]^2 = \mathbb{E}\left[\int_{t_n}^{t_{n+1}} |f(s, y_s) - f(t_{n+1}, y_{t_n})| ds\right]^2
\]

\[
= \left(\int_{t_n}^{t_{n+1}} \mathbb{E}[f(s, y_s) - f(t_{n+1}, y_{t_n})]^2 ds\right)^2
\]

\[
\leq \Delta t_n \int_{t_n}^{t_{n+1}} (\mathbb{E}[f(s, y_s) - f(t_{n+1}, y_{t_n})])^2 ds
\]

\[
\leq C \Delta t_n \int_{t_n}^{t_{n+1}} (t_{n+1} - s)^2 ds
\]

\[
\leq C(\Delta t_n)^4.
\]

This proves (i).

(ii). By the definition of $R_{g,1}^t$ (see Equation (19)), it suffices to estimate $R_{W,n}^t$ defined in (9) and $R_{g,2}^t$ defined in (15). For $R_{W,n}^t$, it follows by the Schwartz inequality, Jensen’s inequality and Lemma 2 that

\[
\mathbb{E}[R_{W,n}^t]^2 = \mathbb{E}\left[\int_{t_n}^{t_{n+1}} |f(s, y_s) - f(t_{n+1}, y_{t_n})| ds\right]^2
\]

\[
\leq \Delta t_n \mathbb{E}\left[\int_{t_n}^{t_{n+1}} |f(s, y_s) - f(t_{n+1}, y_{t_n})|^2 ds\right]
\]

\[
\leq \Delta t_n \mathbb{E}\left[\int_{t_n}^{t_{n+1}} |f(s, y_s) - f(t_{n+1}, y_{t_n})|^4 ds\right]
\]

\[
= \Delta t_n \mathbb{E}\left[\int_{t_n}^{t_{n+1}} (f(s, y_s) - f(t_{n+1}, y_{t_n}))^2 ds\right]
\]

\[
\leq C \Delta t_n \int_{t_n}^{t_{n+1}} ((\Delta t_n)^2 + (y_s - y_{t_n})^2) ds
\]

\[
\leq C(\Delta t_n)^3.
\]

To estimate $R_{g,2}^t$, we need estimations for $R_{g,1}^t$ defined in (11) and $R_{g,2}^t$ defined in (13). From the definition of $R_{g,1}$ and Lemma 2 one easily obtains

\[
\mathbb{E}[(R_{g,1})^2] \leq C(\Delta t_n)^2.
\]

(49)
By Itô’s isometry and (43) in Lemma 2, for \(t_{n+1} \leq s \leq t_n\), we have that
\[
E[\int_s^{t_{n+1}} (\hat{g}'_B(r,y_r) - \hat{g}'_B(t_{n+1},y_{t_{n+1}})) d\hat{B}_r]^2
= \int_s^{t_{n+1}} E[(\hat{g}'_B(r,y_r) - \hat{g}'_B(t_{n+1},y_{t_{n+1}}))^2] dr
\leq C(\Delta t_n)^2.
\]
Similarly,
\[
E[\int_s^{t_{n+1}} (\hat{g}'_y(r,y_r) - \hat{g}'_y(t_{n+1},y_{t_{n+1}})) g(t_{n+1},y_{t_{n+1}}) d\hat{B}_r]^2 \leq C(\Delta t_n)^2.
\]
The above two estimates yield
\[
E[\langle R_{y2} \rangle] \leq (\Delta t_n)^2. \tag{50}
\]

From (49) and (50) we have that
\[
E[\langle R_{y,n}^R \rangle] = E[\langle \mathbb{E}^{\infty,-}_0 R_{y1}(s) R_{y2}(s) d\hat{B}_s \rangle]^2
\leq E[\mathbb{E}^{\infty,-}_0 \int_{t_n}^{t_{n+1}} (R_{y1}(s) + R_{y2}(s))^2 d\hat{B}_s]^2]
\leq E[\int_{t_n}^{t_{n+1}} (R_{y1}(s) + R_{y2}(s))^2 ds]
\leq (\Delta t_n)^3. \tag{51}
\]
The estimates (48) and (51) lead to
\[
E[\langle R_{y}^n \rangle] = E[\langle R_{y,n}^R + R_{y,n}^W \rangle^2] \leq C(\Delta t_n)^3. \tag{52}
\]

(iii) To estimate \(R_{z}^n\) defined by (28), we need estimations for \(R_{z1}^W\), \(R_{z2}^W\) and \(R_{z}^R\) respectively defined in (24), (25) and (26). For \(R_{z1}^W\), using the Schwartz inequality and Lemma 2, we have that
\[
E[\langle R_{z1}^W \rangle^2] = E[\int_{t_n}^{t_{n+1}} \mathbb{E}^{\infty,-}_0 |(f(s,y_s) - f(t_{n+1},y_{t_{n+1}}))\Delta W_{t_{n+1}}|^2 ds]
\leq (\Delta t_n)^2 E[\int_{t_n}^{t_{n+1}} \mathbb{E}^{\infty,-}_0 [(f(s,y_s) - f(t_{n+1},y_{t_{n+1}}))^2] ds]
\leq C(\Delta t_n)^4.
\]
Also, using the theory of variation[20, 22] for SDEs and by (5), we have the estimate
\[
E[\langle R_{z2}^W \rangle] = E[\int_{t_n}^{t_{n+1}} \mathbb{E}^{\infty,-}_0 |\hat{z}_s - \mathbb{E}^{\infty,-}_s [\hat{z}_s]|^2 ds] \leq C(\Delta t_n)^3. \tag{53}
\]
Following similar arguments as above and in (ii), one obtains
\[ E[(R^n_{\cdot n})^2] \leq C(\Delta t)^3. \]

Now the estimate (47) for \( R^n_{\cdot n} \) follows directly from the above estimates. □

Combing the results of Theorem 1 and Lemma 3, we have the main result of the
paper.

**Theorem 2.** Under the conditions of Theorem 1 and Lemma 3, we have the error esti-
mates
\[
\max_{0 \leq n \leq N-1} E[(e^n_{\cdot n})^2] \leq C(\Delta t)^2,
\]
\[
\max_{0 \leq n \leq N-1} E[(e^n_{\cdot n})^2] \leq C\Delta t.
\] (53)

**Proof:** The results are direct consequences of Theorem 1 and Lemma 3. □

**Remark** One still needs to evaluate the expectations in the scheme (29)–(30), and
an obvious approach would be to use the Monte Carlo method. However, when the
density functions are known, high order quadratures are more efficient than the Monte
Carlo method. In the case of our numerical experiments, we shall use Gauss-Hermite
quadrature, since the random variables involved are normally distributed. We refer to
[30] for a comparative error analysis when approximating the expectations using the
Monte Carlo method or Gauss-type quadratures.

5. Numerical experiments

The aim of this section is to demonstrate the effectiveness and accuracy of our
method for solving the BDSDE (4), which is also the numerical approximation of back-
ward SPDE (2). We will compare our first order scheme with the half-order scheme in
[3]. We also provide a comparison between our numerical scheme and a standard finite
difference scheme for SPDEs.

Consider the FBDSDE
\[
X_t^{t,x} = x + W(T) - W(t),
\]
\[
y_t^{t,x} = \sin(X_t^{t,x} + B(T) - B(s) + B(s)) + \int_t^T \frac{1}{2} \cos(X_t^{t,x} + B(T) - B(s))
\]
\[ + B(s)\frac{1}{2} - \sin(x + B(T) - B(s)) + u_s(x)\frac{1}{2} + u_s(x)\overline{\Delta B}_s. \]

The exact solution is \( y_s = \sin(X_s^{t,x} + B(T) - B(s))\), \( z_s = \cos(X_s^{t,x} + B(T) - B(s)) \).

The corresponding SPDE is
\[
u_s(x) = \sin(x + B(T) - B(s)) + \int_t^T \frac{1}{2} \frac{\partial^2}{\partial x^2} u_s(x) + \frac{3}{8} u_s(x) ds
\]
\[ + \int_t^T \frac{1}{2} \cos(x + B(T) - B(s)) \]
\[ - \sin(x + B(T) - B(s)) + u_s(x)\overline{\Delta B}_s. \]
4. Our theoretical convergence order for $u_t(x)$ is $\frac{1}{2}$. However, this particular example gives us first order convergence for $z_t$. We believe that this is an exception rather than the norm.

6. Conclusion

In this paper, we have constructed a first order numerical scheme for backward doubly stochastic differential equations. The main idea is to use the Itô-Taylor expansion for forward and backward stochastic differentials to construct high order quadratures for the stochastic integrals involving both backward and forward Brownian motions. The particular scheme we constructed in the paper is first order, but high order schemes can also be developed simply by working with the appropriate high order Itô-Taylor expansions. Because of the relation between FBDSDEs and Zakai equations, our scheme can also be used to find numerical solutions of the Zakai equation. Thus our algorithm provides a numerical method of solving nonlinear filtering problems. First order schemes already exist for Zakai equations, but our method is more general, since it covers the cases when Zakai equations are nonlinear. In future work we plan to carry out rigorous error analysis to evaluate the conditional expectations in the numerical schemes with the Monte Carlo method and Gaussian quadrature.

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