ON CONVERGENCE RATE OF WIENER-ITO EXPANSION FOR GENERALIZED RANDOM VARIABLES

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Abstract. In this paper we present a new result about the estimate of the cutoff error of the Wiener-Ito expansion for a generalized random variable. As an application, we use the result to obtain an error estimate for the finite element approximation of the stochastic Helmholtz equation.

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1. Introduction

In the past few years there have been growing interests in numerical methods for stochastic partial differential equations (SPDEs) [1, 3, 2, 5, 8, 9, 10, 11, 13, 14]. One of the important topics is the numerical approximation of SPDEs where some of the coefficients are random variables. Some of the interesting approaches are spectral finite element methods using formal Hermite polynomial chaos [9, 13] , *hp* and *hk* finite element methods using tensor product of the space of random variables and Sobolev space [2] and the finite element method with Wick product variational formulation [11]. In all of the above approaches, the errors of numerical solutions are generated by two sources: the finite element approximation error and the cutoff error of series expansion of the solution as a random variable. Thus to control the overall error of the numerical solution, it is essential to balance the errors from both of the two sources. As demonstrated in [2], [3] and [11], the estimate of the first error can be obtained in the same way as the deterministic case while the estimate of the second error depends on the estimate of the cutoff error of random variables by using either the Karhunen-Loeve expansion or the Wiener-Ito chaos expansion.

The main result of this paper is an estimate for the cutoff error of the Wiener-Ito expansions for generalized random variables in Kondratiev norms . The first such an estimate belongs to Benth and Gjerde [3]. Based on the estimate they stablished a framework for the error estimates of finite element approximations of SPDEs. In this paper, we shall derive a

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new, improved error estimate. An immediate application of this result is obtaining improved error estimates for the finite element approximations of SPDEs.

The paper is organized as follows. In the next section we provide a brief mathematical background of the generalized random variables following the outline given by [11]. Then in Section 3, we prove the main result of the paper. Finally in Section 4, we apply the result in Section 3 to obtain error estimates for the finite element approximations of stochastic Helmholtz equations.

2. Preliminaries

Let $\mathcal{S}$ denote the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing C^∞ functions on $\mathbb{R}^d$. The dual space $\mathcal{S}'$ equipped with the weak-star topology is the space of tempered distributions. By the Bochner-Minlos theorem there exists a unique probability measure μ on the members of family $\mathcal{B}(\mathcal{S}')$ of Borel subsets of $\mathcal{S}'$ such that

$$E[e^{i(\cdot,\phi)}] := \int_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu(\omega) = e^{\frac{-\|\phi\|_0^2}{2}}$$

where $\|\phi\|_0 = (\phi,\phi) = \int_{\mathbb{R}^d} \phi(x)^2 dx$. The triplet $(\mathcal{S}', B, \mu)$ forms our basic probability space.

We will use the following multi-index notation. Let $\mathcal{T} = \mathbb{N}_0^{\mathbb{N}_c}$ denote the set of multi-indices $\alpha = (\alpha_1, \alpha_2, \cdots)$ where $\alpha \in \mathbb{N}_0$ and only finitely many $\alpha_i \neq 0$. For each $\alpha, \beta \in \mathcal{T}$ we define the usual operations $\alpha + \beta = (\alpha_1 + \beta_1, \cdots)$, $\alpha! = \alpha_1!\alpha_2!\cdots$, and $|\alpha| := \Sigma_j \alpha_j$.

For each $\alpha \in \mathcal{T}$ define the stochastic variable

$$H_\alpha(\omega) = \prod_{j=1}^{\infty} h_{\alpha_j}(< \omega, \eta_j >),$$

where h_n denotes the Hermite polynomial

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}) \quad (n \in N),$$

and the family $\{\eta_j\}_{j=1}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$. This orthonormal family is constructed from the Hermite functions

$$\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} e^{-x^2/2} h_{n-1}(\sqrt{2}x) \quad (x \in R,\ n \in N)$$

in the following way: let $\delta = (\delta_1, \cdots, \delta_d) \in \mathbb{N}_0^d$ be the d-dimensional multi-indices and let $\{\delta^{(i)}\}(i \in \mathbb{N})$ be some fixed ordering of these multi-indices such that $i < j \Rightarrow |\delta^{(i)}| \leq |\delta^{(j)}|$. Then we define η_j as the tensor product

$$\eta_j := \xi_{\delta_1^{(j)}} \otimes \cdots \otimes \xi_{\delta_d^{(j)}} \quad (j \in \mathbb{N}).$$

The family $\{\xi_n\}_{n=1}^{\infty}$ is a subset of $\mathcal{S}(\mathbb{R}^d)$ and forms an orthonormal basis for $L^2(\mathbb{R}^d)$. The following theorem can be found in [7].

Theorem 1. *(Wiener-Ito chaos expansion theorem) Every $f \in L^2(\mu)$ has a unique Wiener-Ito chaos expansion*

$$f(\omega) = \sum_{\alpha \in \mathcal{T}} c_\alpha H_\alpha(\omega) \quad \text{where } c_\alpha \in \mathbb{R}. \tag{1}$$

In addition, the family $\{H_\alpha \sqrt{\alpha!}\}_{\alpha \in \mathcal{T}}$ constitutes an orthonormal basis for $L^2(\mu)$ and we have that

$$\|f\|^2_{L^2(\mu)} = \sum_{\alpha \in \mathcal{T}} c_\alpha^2 \alpha!$$

for every $f \in L^2(\mu)$.

Let V be any real Hilbert space and $\rho \in [-1, 1]$, $k \in \mathbb{R}$. Then the stochastic Hilbert space $(S)^{\rho,k,V}$ is defined as the set of all (formal) sums

$$f = \sum_{\alpha \in \mathcal{T}} f_\alpha H_\alpha, \quad \text{where } f_\alpha \in V \text{ for all } \alpha \in \mathcal{T}. \tag{2}$$

such that the norm

$$\|f\|_{\rho,k,V} = \left(\sum_{\alpha \in \mathcal{T}} \|f_\alpha\|^2_V (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} \right)^{1/2} \tag{3}$$

is finite. The weights are defined as $(2\mathbb{N})^{k\alpha} := \prod_{j=1}^{\infty} (2j)^{k\alpha_j}$.

Notice that the norm $\| \cdot \|_{\rho,k,V}$ is well defined by the inner product $(\cdot, \cdot)_{\rho,k,V}$ defined as

$$(f, g)_{\rho,k,V} = \sum_{\alpha \in \mathcal{T}} (f_\alpha, g_\alpha)(\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha}$$

for $f = \sum_{\alpha \in \mathcal{T}} f_\alpha H_\alpha$ and $g = \sum_{\alpha \in \mathcal{T}} g_\alpha H_\alpha$ given in $\mathcal{S}^{\rho,k,V}$.

For $f, g \in \mathcal{S}^{\rho,p,V}$, definite the Wick product of f and g as follows.

$$f \diamond g := \sum_{\gamma \in \mathcal{T}} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) H_\gamma. \tag{4}$$

To ensure that the operator $g \mapsto g \diamond f$ is bounded and continuous on $\mathcal{S}^{-1,k,0}$, we introduce the Banach spaces $\mathcal{F}_l(D)$:

$$\mathcal{F}_l(D) := \{ f(x) = \sum_{\alpha} f_\alpha(x) H_\alpha : f_\alpha \text{measurable}, \ \|f\|_{l,*} < \infty \} \tag{5}$$

where D is an open subset of $\mathbb{R}^d$ and $\|f\|_{l,*}$ is defined as

$$\|f\|_{l,*} = \sup_{x \in D} \left(\sum_{\alpha} |f_\alpha| (2\mathbb{N})^{l\alpha} \right).$$

Let $H^m(D)$ be the usual Sobolev spaces and

$$H_0^1(D) := \{v; v \in H^1(D)\text{and } v = 0 \text{ on } \partial D.$$

When $V = H^m(D)$, we denote the norm $\|\cdot\|_{\rho,k,V}$ by $\|\cdot\|_{\rho,k,m}$. The following result is proved in [12]

Proposition 1. *(Vage inequality) Let $D \subset \mathbb{R}^d$ be an open set and $l \in \mathbb{R}$. Then for $l \geq \frac{k}{2}$ $g \mapsto f \diamond g$ defines a continuous operator on $\mathcal{S}^{-1,k,0}$. Further more we have*

$$\|f \diamond g\|_{-1,k,0} \leq \|f\|_{l,*}\|g\|_{-1,k,0}. \tag{6}$$

3. Approximation of generalized random variables

Introduce the set of multi-indices

$$A_{n,k} = \{\alpha \in \mathbb{N}_0^k \mid \alpha_k \neq 0, \alpha_1 + \cdots + \alpha_k = n\}$$

and

$$\bar{A}_{n,k} = \{\alpha \in \mathbb{N}_0^k \mid \alpha_l = 0,\ l > k, \alpha_1 + \cdots + \alpha_k = n\}$$

where $n, k \in \mathbb{N}$. For $N, K \in \mathbb{N}$ and the generalized random variable f definied in (2), we define the finite dimensional approximation

$$\Phi^{N,K} := c_0 + \sum_{n=1}^{N} \sum_{k=1}^{K} \sum_{\alpha \in A_{n,k}} c_\alpha H_\alpha.$$

First we prove the following lemma.

Lemma 1.

$$\sum_{\alpha \in \bar{A}_{n,k}} (2N)^{-\alpha\tau} \leq \frac{2^{-n\tau}}{\prod_{j=2}^{k}\left(1 - \frac{1}{j^\tau}\right)}. \tag{7}$$

Proof. The proof is by induction. The estimate is clearly true for $k = 1$. For $k = 2$ we have

$$\begin{aligned}
\sum_{\alpha \in \bar{A}_{n,k}} (2N)^{-\alpha\tau} &= \sum_{\alpha_1+\alpha_2=n} 2^{-\alpha_1\tau}(2\times 2)^{-\alpha_2\tau} = \sum_{i=0}^{n} 2^{-(n-i)\tau}(2\times 2)^{-i\tau} \\
&= 2^{-n\tau}\sum_{i=0}^{n} 2^{-i\tau} = \frac{2^{-n\tau}(1-2^{-n\tau})}{1-2^{-\tau}} \\
&\leq \frac{2^{-n\tau}}{1-\frac{1}{2^\tau}}.
\end{aligned}$$

Thus (7) is also valid for $k = 2$. Assume that (7) is true for $k = p$. Then

$$\begin{aligned}
\sum_{\alpha\in\bar{A}_{n,p+1}} (2N)^{-\alpha\tau} &= \sum_{i=0}^{p+1} \sum_{\alpha\in\bar{A}_{n-i,p}} (2N)^{-\alpha\tau} (2(p+1))^{-i\tau} \\
&\le \sum_{i=0}^{p+1} \frac{2^{(n-i)\tau}2^{-i\tau}}{\prod_{j=2}^{p}\left(1-\frac{1}{j^\tau}\right)}(p+1)^{-i\tau} \\
&= \frac{2^{-n\tau}}{\prod_{j=2}^{p}\left(1-\frac{1}{j^\tau}\right)}\sum_{i=0}^{p+1}(p+1)^{-\tau i} = \frac{2^{-n\tau}(1-(1+p)^{-\tau(p+1)})}{\prod_{j=2}^{p}\left(1-\frac{1}{j^\tau}\right)\left(1-\frac{1}{(p+1)^\tau}\right)} \\
&\le \frac{2^{-n\tau}}{\prod_{j=2}^{p+1}\left(1-\frac{1}{j^\tau}\right)}.
\end{aligned}$$

The proof is complete. □

Lemma 2.

(8) $$c_1(\tau) := \left(\prod_{j=2}^{\infty}\left(1-\frac{1}{j^\tau}\right)\right)^{-1} \le e^{\frac{2}{\tau-1}}$$

and

(9) $$c_2(\tau) := \left(\prod_{j=2}^{\infty}\left(1-\frac{1}{(2j)^\tau}\right)\right)^{-1} \le e^{\frac{1}{2^{(\tau-1)}(\tau-1)}}.$$

Proof. We only prove (8). The proof of (9) is similar. We have that

$$\begin{aligned}
\ln\prod_{j=2}^{\infty}\left(1-\frac{1}{j^\tau}\right) &= \sum_{j=2}^{\infty}\ln\left(1-\frac{1}{j^\tau}\right) \ge -\sum_{j=2}^{\infty}\frac{2}{j^\tau} \\
&\ge -2\int_1^\infty \frac{1}{x^\tau}dx = -\frac{2}{\tau-1}.
\end{aligned}$$

Thus

$$\left(\prod_{j=2}^{\infty}\left(1-\frac{1}{j^\tau}\right)\right)^{-1} \le e^{\frac{2}{\tau-1}}.$$

□

We are now ready to prove the main result of the paper.

Theorem 2. *Let $p > 0$ be given and assume that $\tau > 1$. Then for any $\Phi \in S^{\rho,-p+\tau,V}$*

(10) $$\|\Phi - \Phi^{N,K}\|_{\rho,-p,V} \le \|\Phi\|_{\rho,-p+\tau,V}\sqrt{A(\tau)\frac{1}{K^{\tau-1}} + B(\tau)\frac{1}{2^{\tau N}}}$$

where

$$A(\tau) = e^{\frac{2}{\tau-1}} \frac{\tau}{\tau-1}$$
$$B(\tau) = e^{\frac{1}{2^{\tau-1}(\tau-1)}} \frac{1}{2^\tau(\tau-1)}.$$

Proof. Let

$$c_{n,k} := \sum_{\alpha \in A_{n,k}} c_\alpha H_\alpha.$$

We have that

$$\begin{aligned}
\Phi - \Phi^{N,K} &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} c_{n,k} - \sum_{n=1}^{N} \sum_{k=1}^{K} c_{n,k} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{K} c_{n,k} + \sum_{n=1}^{\infty} \sum_{k=K+1}^{\infty} c_{n,k} - \sum_{n=1}^{N} \sum_{k=1}^{K} c_{n,k} \\
&= \sum_{n=N+1}^{\infty} \sum_{k=1}^{K} c_{n,k} - \sum_{k=K+1}^{\infty} \sum_{n=1}^{\infty} c_{n,k}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\Phi - \Phi^{N,K}\|_{\rho,-p,V} &= \sum_{n=N+1}^{\infty} \sum_{k=1}^{K} \sum_{\alpha \in A_{n,k}} \|c_\alpha\|_V^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{-\alpha p} \\
&+ \sum_{k=K+1}^{\infty} \sum_{n=1}^{\infty} \sum_{\alpha \in A_{n,k}} \|c_\alpha\|_V^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{-\alpha p} \\
&= \sum_{n=N+1}^{\infty} \sum_{k=1}^{K} \sum_{\alpha \in A_{n,k}} \|c_\alpha\|_V^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{-\alpha(p-\tau)} (2\mathbb{N})^{-\alpha\tau} \\
&+ \sum_{k=K+1}^{\infty} \sum_{n=1}^{\infty} \sum_{\alpha \in A_{n,k}} \|c_\alpha\|_V^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{-\alpha(p-\tau)} (2\mathbb{N})^{-\alpha\tau} \\
&\le \|\Phi - \Phi^{N,K}\|_{\rho,-p+\tau,V} \left(\sum_{n=N+1}^{\infty} \sum_{k=1}^{K} \sum_{\alpha \in A_{n,k}} (2\mathbb{N})^{-\alpha\tau} + \sum_{k=K+1}^{\infty} \sum_{n=1}^{\infty} \sum_{\alpha \in A_{n,k}} (2\mathbb{N})^{-\alpha\tau} \right).
\end{aligned}$$

Let

$$I_{N,K} = \sum_{n=N+1}^{\infty} \sum_{k=1}^{K} \sum_{\alpha \in A_{n,k}} (2\mathbb{N})^{-\alpha\tau}$$

and

$$I_K = \sum_{k=K+1}^{\infty} \sum_{n=1}^{\infty} \sum_{\alpha \in A_{n,k}} (2\mathbb{N})^{-\alpha\tau}.$$

We first estimate $I_{N,K}$. Using the result of Lemma 1, we have that

$$\begin{aligned}
I_{N,K} &= \sum_{n=N+1}^{\infty} \sum_{k=1}^{K} \sum_{i=1}^{k} \left(\sum_{\alpha \in \bar{A}_{n-i,k-1}} (2\mathbb{N})^{-\alpha\tau} \right) (2k)^{-i\tau} \\
&\leq \sum_{n=N+1}^{\infty} \sum_{k=1}^{K} \sum_{i=1}^{k} \frac{2^{-(n-i)\tau}}{\prod_{j=2}^{k-1} \left(1 - \frac{1}{j^\tau}\right)} (2k)^{-i\tau} \\
&= \sum_{n=N+1}^{\infty} \sum_{k=1}^{K} \sum_{i=1}^{k} \frac{2^{-n\tau}}{\prod_{j=2}^{k-1} \left(1 - \frac{1}{j^\tau}\right)} k^{-i\tau} \\
&= \sum_{n=N+1}^{\infty} \sum_{k=1}^{K} \frac{2^{-n\tau}}{\prod_{j=2}^{k} \left(1 - \frac{1}{j^\tau}\right)} k^{-\tau}(1 - k^{-k\tau}) \\
&\leq \frac{1}{\prod_{j=2}^{\infty} \left(1 - \frac{1}{j^\tau}\right)} \sum_{n=N+1}^{\infty} 2^{-n\tau} \sum_{k=1}^{K} \frac{1}{k^\tau} \\
&\leq (1 + \frac{1}{\tau - 1}) e^{\frac{2}{\tau-1}} \frac{1}{2^{N\tau}} = A(\tau) \frac{1}{2^{N\tau}}.
\end{aligned}$$

Next we estimate I_K. It is easy to see that

$$\sum_{n=1}^{\infty} \sum_{\alpha \in A_{n,k}} (2\mathbb{N})^{-\alpha\tau} = \sum_{\alpha_1, \cdots, \alpha_{k-1} \geq 0, \alpha_k \geq 1} \prod_{j=1}^{k} (2j)^{-\tau\alpha_j}.$$

Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{\alpha \in A_{n,k}} (2\mathbb{N})^{-\alpha\tau} &= \prod_{j=1}^{k-1} \sum_{\alpha_j=0}^{\infty} (2j)^{-\tau\alpha_j} \left(\sum_{\alpha_n=1}^{\infty} (2k)^{-\tau\alpha_n} \right) \\
&= \prod_{j=1}^{k-1} \frac{1}{1 - \frac{1}{(2j)^\tau}} \frac{1}{(2k)^\tau - 1}
\end{aligned}$$

Applying Lemma 2, we have that

$$I_K \leq e^{\frac{1}{2^{\tau-1}(\tau-1)}} \frac{1}{2^\tau} \sum_{k=K+1}^{\infty} \frac{1}{k^\tau} \leq e^{\frac{1}{2^{\tau-1}(\tau-1)}} \frac{1}{2^\tau} \int_K^\infty \frac{1}{x^\tau} dx = B(\tau) \frac{1}{K^{\tau-1}}.$$

The proof is complete. □

Remark Benth and Gjerde [3] obtain the first cutoff error for th Wiener-Ito expansion. There the estimate is

$$\|\Phi - \Phi^{N,K}\|_{\rho,-p,V} \le \|\Phi\|_{\rho,-p+\tau,V}\sqrt{A(\tau)\frac{1}{K^{\tau-1}} + B(\tau)(\frac{\tau}{2^{\tau}(\tau-1)})^{\tau N}} \tag{11}$$

where $\tau > \tau^* = 2^{\tau^*}(\tau^* - 1) > 1.5$. Clearly our estimate is an substantial improvement.

4. Finite element methods for stochastic Helmholtz equations

4.1. Variational formulation.

We consider the Helmholtz equation

$$\Delta u + k \diamond u = f \quad \text{in } D \tag{12}$$

$$u = 0 \quad \text{on } \partial D \tag{13}$$

where $k = k_0(x) + \sum_\alpha k_\alpha(x)H_\alpha(\omega)$ is a generalized random variable. For $u, v \in \mathcal{S}^{\rho,p,V}$, define a bilinear form

$$a(u,v) = (\nabla u, \nabla v)_{\rho,p,0} + (ku, v)_{\rho,p,0}. \tag{14}$$

We have the following continuity property and Garding inequality for a.

Proposition 2. *Assume that $k \in S^{-1,l,0}$ and $l \ge \frac{k}{2}$. Then there exist constants c_1, c_2 and c_3 such that*

$$\begin{aligned} (i) \quad & a(u,v) \le c_1\|u\|_{-1,p,1}\|v\|_{-1,p,1} \\ (ii) \quad & a(u,u) + c_2\|u\|_{-1,p,0} \ge c_3\|u\|_{-1,p,1}. \end{aligned}$$

Proof. (i) is a direct consequence of Proposition 1. To prove (ii) we let $\bar{k}_0 = \text{esssup}_{x\in D}|k_0(x)|$. Then by a result of Vage [12] (see also [11]), we have that

$$((k + c_2) \diamond u, u)_{-1,p,0} \ge (c_2 - \bar{k}_0 - 2^{k-2l})\|k\|_{*,l})\|u\|_{-1,p,0}.$$

Choosing c_2 such that $c_2 - \bar{k}_0 - 2^{k-2l}\|k\|_{*,l} > 0$, we have that

$$a(u,v) + c_2\|k\|_{-1,p,0} \ge \|\nabla u\|_{-1,p,0} + c_2 - \bar{k}_0 - 2^{k-2l})\|k\|_{*,l})\|u\|_{-1,p,0} \ge c_3\|u\|_{-1,p,1}$$

where $c_3 = \min\{1, c_2 - \bar{k}_0 - 2^{k-2l}\|k\|_{*,l}\}$. The proof is complete. □

Remark For deterministic constant wave number k, the Garding inequality ensures existence of unique solution except for countable many k. However it is not clear if this is the case when k is a random field. Nevertheless, Garding inequality is essential in proving the existence and rate of convergence of the finite element approximation for the Helmholtz equation.

4.2. **Finite element approximations.** Assume that D is a polygonal domain. A regular triangulation of D is a finite collection of open triangles $\{\mathcal{T}_i\}_{i=1}^M$ such that
(i). $\mathcal{T}_i \cap \mathcal{T}_j = \{\}$ if $i \neq j$ and $\cup \bar{\mathcal{T}}_i = \bar{D}$.
(ii). For $i \neq j$, $\mathcal{T}_i$ and $\mathcal{T}_j$ is either
 (a). empty or
 (b). a common side of $\mathcal{T}_i$ and $\mathcal{T}_j$ or
 (c). a common edge of element $\mathcal{T}_i$ and $\mathcal{T}_j$.

With the triangular partition of D, the finite element subspace $V_h \subset H_0^1(D)$ is defined as the set of piecewise linear functions

$$V_h := \{v_h \in C(\bar{D}), v_h = 0 \text{ on } \partial D, v_h \mid_{\mathcal{T}_i} \text{ is a linear function}\}.$$

We assume that the following approximation property holds for V_h.

$$\inf_{v_h \in V_h} \|u - v_h\|_{H_0^1(D)} \leq Ch\|u\|_{H^2(D)} \quad \forall u \in H^2(D) \tag{15}$$

Now define a finite dimensional subspace $V_h^{N,K}$ of $H_0^1(D) \times \mathcal{S}'$ as

$$V_h^{N,K} := \{c_0^h(x) + \sum_{n=1}^{N}\sum_{k=1}^{K} c_\alpha^h H_\alpha\}$$

where $c_\alpha \in V_h$. For $\Phi = \sum_{\alpha \in \mathcal{T}} c_\alpha H_\alpha$, let

$$\Phi_h^{N,K} = c_0^h(x) + \sum_{n=1}^{N}\sum_{k=1}^{K} c_\alpha^h H_\alpha$$

where c_α^h are the projections of c_α from $H_0^1(D)$ to $V_h^{N,K}$. The following result is a direct consequence of Theorem 2 and (15) (see [3] for a proof).

Proposition 3.

$$\|\Phi - \Phi_h^{N,K}\|_{-1,p,1} \leq \sqrt{A(\tau)\frac{1}{K^{\tau-1}} + B(\tau)\frac{1}{2^{\tau N}}}\|\Phi\|_{\rho,-p+\tau,1} + Ch\|\Phi\|_{\rho,-p,1}. \tag{16}$$

The finite element approximation for (12)- (13) is to seek $u_h^{N,K} \in V_h^{N,K}$ such that

$$a(u_h^{N,K}, v) = (f, v), \quad \forall v \in V_h^{N,K}. \tag{17}$$

Theorem 3. *Assume that there exists a unique solution u for (12)- (13). Then there exists $h_0 > 0$ such that for $h < h_0$, (17) has a unique solution and*

$$\|u - u_h^{N,K}\|_{-1,p,1} \leq \sqrt{A(\tau)\frac{1}{K^{\tau-1}} + B(\tau)\frac{1}{2^{\tau N}}}\|\Phi\|_{\rho,-p+\tau,1} + Ch\|\Phi\|_{\rho,-p,2}$$

where C is a constant independent of h.

Proof. Using the Garding inequality, continuity property of a (Proposition 2)we can prove (see [4] for technical details)

$$\|u - u_h^{N,K}\|_{-1,p,1} \leq C \inf_{v \in V_h^{N,K}} \|u - v\|_{-1,p,2}.$$

The result of the theorem then follows from Proposition 3. □

Remark As pointed out in [2] and [8], a drawback of the Wick product is that higher order statistics do not have much effect on the solutions of SPDEs, which is generally not the case for nonlinear problems. However the Wick product is still a useful tool to study SPDEs under certain circumstances. We refer [7] for detailed analysis and [11] for numerical experiments on Wick product.

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