Contents

Solutions to Selected Problems of the Web Chapters

Chapter 14

14.1.P **p. ??**
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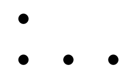
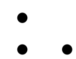
1. The two 5 by 5 matrices comprised of zeros everywhere, except for a 1 in positions (1,5) or (1,2) both have the 5 fold eigenvalue zero, but a 1-dimensional kernel, or only 4 corresponding linearly independent eigenvectors. According to Chapter 9.1 or 9.1.D they cannot be diagonalized.

3. For a non-diagonalizable example, expand the solution of problem 1. to 12 by 12. For a diagonalizable triangular matrix, use an arbitrary 12 by 12 upper triangular matrix with distinct diagonal entries. Such a matrix has 12 distinct eigenvalues and is therefore diagonalizable.

5. Look at the eigenspace for $\lambda = -1$, or at the kernel of $A + I = \begin{pmatrix} 1 & 0 & 1 \\ -3 & 0 & -3 \\ -1 & 0 & -1 \end{pmatrix}$. It has dimension two, so there are two linearly independent eigenvectors for $\lambda = -1$. Now compute $\text{trace}(A) = -3 = -1 - 1 + \lambda_3$, making $\lambda = -1$ a triple eigenvalue for A. Therefore the Jordan normal form of A is $J = \text{diag}(J(-1, 2), J(-1, 1))$.

7. For Example 4:  ;

for Problem 5:  .

9. We can find the eigenvalues of $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ from $\det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ -b & a-\lambda \end{pmatrix} = (a - \lambda)^2 + b^2 = 0$ as $\lambda_{1,2} = a \pm bi$. We find the complex eigenvectors by looking at $A - (a + bi)I = \begin{pmatrix} -bi & b \\ -b & -bi \end{pmatrix}$ as $\begin{pmatrix} 1 \\ i \end{pmatrix}$, and from $A - (a - bi)I = \begin{pmatrix} bi & b \\ -b & bi \end{pmatrix}$ as $\begin{pmatrix} -1 \\ i \end{pmatrix}$. Now form the eigenvector column matrix $X = \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}$ with $X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$. Then $X^{-1}AX = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} = \begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix}$.

11. Since $\text{trace}(C) = -6 - 2 + 10 = 2 = \lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_1 = -1$ and $\lambda_2 = 2$ are given, we must have $\lambda_3 = 1$. Therefore C has three distinct eigenvalues and is diagonalizable according to Chapter 9. The Jordan normal form of C is $\text{diag}(-1, 2, 1)$.

13. Let the size of A be n by n. 4 foldness of an eigenvalue refers to the algebraic multiplicity; therefore $n = 4$. Since geom. mult. $\leq$ alg. mult., we may have 1, or 2, or 3, or 4 linearly independent eigenvectors for this single eigenvalue. Giving rise to the following Jordan diagrams in turn: $\begin{array}{c}\bullet\\\bullet\\\bullet\\\bullet\end{array}$ in case of geometric multiplicity 1, or $\begin{array}{cc}\bullet & \\ \bullet & \\ \bullet & \bullet\end{array}$ or $\begin{array}{cc}\bullet & \bullet\\ \bullet & \bullet\end{array}$ in case of geometric multiplicity 2, or $\begin{array}{ccc}\bullet & & \\ \bullet & \bullet & \bullet\end{array}$ in case of geometric multiplicity 3, or to $\bullet \quad \bullet \quad \bullet \quad \bullet$ if A is diagonalizable.

15. Here $n = 6$ and $\text{diag}(J(\lambda, n_1), J(\lambda, n_2), ..., J(\lambda, n_k))$ with $k \leq 6$ and $\sum_{i=1}^{k} n_i = 6$.

17. The Jordan normal form J of A has three separate Jordan blocks for λ. One of these must be 4 by 4, while the other two have sizes that will add up to 8. Therefore J will contain the string $J(\lambda, 4), J(\lambda, n_1), J(\lambda, n_2))$ with $n_1 + n_2 = 4$ and $n_i \geq 1$. This gives us two possibilities: $n_1 = 1$, $n_2 = 3$ or $n_1 = 2 = n_2$. The third possible case $n_1 = 3$ and $n_2 = 3$ is only a reordering of the first mentioned one. So for λ there are precisely two variations possible in the Jordan normal form for A.

19. The maximal index must be six since there must be seven linearly independent eigenvectors for our 12 fold eigenvalue. The minimal index is two, provided 5 Jordan blocks of size 2 occur together with two 1 by 1 Jordan blocks. Any index in between 2 and 6 can also occur.

21. (a) A's Jordan normal form has one 2 by 2 block and 1 by 1 blocks else.
(b) A's Jordan normal form has one 3 by 3 block and 1 by 1 blocks else, or it has two 2 by 2 blocks and 1 by 1 blocks else.
(c) There are three Jordan blocks for λ. Their possible sizes are 1, 1, 4; or 1, 2, 3; or 2, 2, 2, respectively.

23. (a) Simply multiply $J^{-1} \cdot J$ out for the matrix J^{-1} as given and you should obtain I.
(b) $J(\lambda, k)^{-1}$ is tridiagonal with its n fold eigenvalue $\frac{1}{\lambda}$ appearing on the diagonal. Since $\text{rank}(J(\lambda, k)^{-1} - \frac{1}{\lambda}I) = k - 1$ from the formula for the inverse in part (a), $J(\lambda, k)^{-1}$ must be similar to a single Jordan block of the form $J\left(\frac{1}{\lambda}, k\right)$.
And if $X^{-1}AX = J$ is a Jordan form matrix, then $J^{-1} = X^{-1}A^{-1}X$, or A^{-1} and J^{-1} are also similar, hence have the same Jordan normal form. But if $J = \text{diag}(J_i)$, then $J^{-1} = \text{diag}(J_i^{-1})$, so the Jordan normal form of the inverse of a nonsingular matrix A has the same Jordan normal form structure as A, except instead of the eigenvalues λ_i for A it has the eigenvalues $\frac{1}{\lambda_i}$.

25. We have $\text{trace}(A) = \lambda_1 + \lambda_2 = 4$ and $\det(A) = \lambda_1\lambda_2 = 4$, so that $\lambda_1 = \lambda_2 = 2$ is a double eigenvalue for A. Looking at the kernel of $A - 2I$ we find its dimension equal to 1. Therefore A is not diagonalizable with the Jordan normal form $J = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$.

27. (a) Every matrix $A = A(\alpha)$ has only one linearly independent eigenvector for its double eigenvalue λ since $\text{rank}(A - \lambda I) = \text{rank}\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} = 1$. Therefore each $A(\lambda)$ is similar to $J(\lambda, 2) = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$. If $\alpha = 0$, then $A = \lambda I = \text{diag}(J(\lambda, 1), J(\lambda, 1))$ is not similar to $J(\lambda, 2)$.
(b) Analogous to part (a), if $\prod \alpha_i \neq 0$, then $\text{rank}(A - \lambda I) = 1$ and A is similar to a single Jordan block $J(\lambda, k)$. If at least one off diagonal entry $\lambda_j = 0$, then $\text{rank}(A - \lambda I) \geq 2$ and A is similar to a Jordan form with at least two Jordan blocks for λ.
(c) Both $B - \lambda I$ and $C - \lambda I$ have rank 1 for their double eigenvalue λ. Hence both B and C have the same Jordan normal form

$J = J(\lambda, 2) = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$. I.e., $X^{-1}BX = J$ and $Y^{-1}CY = J$ for two matrices X and Y, making $C = YX^{-1}BXY^{-1} = (XY^{-1})^{-1}B(XY^{-1})$ similar.

29. (a) $Ae_k = ae_k$.
(b) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ are both diagonalizable, but $\begin{pmatrix} -3 & 0 & 0 & 0 \\ \alpha & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is not for any $\alpha \neq 0$.
(c) In our triangular matrix examples, nonzero entries in columns and rows that have the same diagonal entry seem to determine diagonalizability.

31. (a) $A = \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix}$ has one complex eigenvalue $\lambda = i$ and one real eigenvalue $\lambda = 1$.
(b), (c) $A = \begin{pmatrix} -1+i & 3 \\ i & 2-i \end{pmatrix}$ has determinant $(-1+i)(2-i) - 3i = -1$ and trace 1. Thus its characteristic polynomial is $f_A(x) = x^2 + x - 1$ with the roots $x_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} \in \mathbb{R}$. These are the eigenvalues of $A \notin \mathbb{R}^{n,n}$.

14.2.P p. ??
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1. We can use vector iteration of Section 9.1 with $x = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$ for example and obtain $\begin{pmatrix} \vdots & \vdots & & \vdots \\ x & Ax & A^2x & A^3x & A^4x \\ \vdots & \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} 0 & -10 & -10 & -20 & -60 \\ 2 & -1 & -2 & 6 & -8 \\ -1 & 3 & 6 & 2 & 24 \\ 1 & -3 & 4 & -2 & -4 \end{pmatrix}$ with the reduced row echelon form $\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Therefore according to Section 9.1, the roots of the polynomial $x^3 - 2x - 4$ are three eigenvalues of A. The roots are 2 (found by guessing) and $-1 \pm i$ (then found by long division and the quadratic formula). The fourth root is 2 from the trace condition. A row reduction of $A - 2I_4$ shows this matrix to have rank 3. Thus A has only one eigenvector for its double real eigenvalue 2. A therefore has the real Jordan normal form $J = \text{diag}(J(2,2), J(-1,1,2))$.
To start out with computing $\det(A - \lambda I_4)$ seems much harder.

3. For the real matrix A all complex eigenvalues such as λ and μ come doubly as λ, μ and as $\overline{\lambda}, \overline{\mu}$. Thus the real principal subspace associated with λ is 6 dimensional, and the one for μ has dimension 4. Since the size of A is 13 by 13, this makes ν have algebraic multiplicity 3 as a root of the characteristic polynomial of A. Finally ν can have the geometric multiplicities 1, 2, or 3.
The possible real Jordan forms of A consist of any combination from the following three Jordan block groups: for $\lambda = 2 + 3i$: $J(2,3,6)$, $\text{diag}(J(2,3,4), J(2,3,2))$, or $\text{diag}(J(2,3,2), J(2,3,2), J(2,3,2))$;
for $\mu = 1 - 2i$: $J(1,-2,4)$ or $\text{diag}(J(1,-2,2), J(1,-2,2))$;
and for $\nu = 7$: $J(7,3)$, $\text{diag}(J(7,2), J(7,1))$, or $\text{diag}(7,7,7)$.

5. (a) $Je_k = \lambda e_k$.
(b) The index of λ is k.
(c) Since $(J - \lambda I)^k = O_k$ and $(J - \lambda I)^{k-1}$ is the matrix of all zeros except for a 1 in position (k,1), and since a principal vector of index k for A must lie in the kernel of $(J - \lambda I)^k$ but not in the kernel of $(J - \lambda I)^{k-1}$, this principal vector must be a multiple of e_1.
(d) $e_{k-j+1} \in P_j(\lambda)$.

7. $vA =$

$$\begin{pmatrix} 1 & z & z^2 & \dots & z^{n-1} \end{pmatrix} \begin{pmatrix} 0 & & & -a_0 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & 1 & -a_{n-1} \end{pmatrix}$$

$= \begin{pmatrix} z & z^2 & \dots & -a_0 - a_1 z - \dots - a_{n-1}z^{n-1} \end{pmatrix}$ $= z \begin{pmatrix} 1 & z & z^2 & \dots & z^{n-1} \end{pmatrix}$ since $z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0 = 0$.

9. The given matrix $B = \begin{pmatrix} -2 & 2i \\ 1 & 3 \end{pmatrix}$ has two distinct complex eigenvalues since its characteristic polynomial is $\lambda^2 - \lambda - 6 - 2i$. Thus B is similar over $\mathbb{C}$ to a complex diagonal matrix diag(λ_1, λ_2), or $X^{-1}BX$ =diag(λ_1, λ_2). Thus $B = (X \text{ diag}(\lambda_1, \lambda_2)X^T)(X^{-T}IX^{-1})$ expresses B as the product of two complex symmetric matrices.

11. $A = \begin{pmatrix} -2 & 1 \\ 4 & -4 \end{pmatrix} =$

$$= \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix}.$$

13. Note that $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$ and that $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

14.3.P **p. ??**
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1. (a) For Example 9 we use the product rule of calculus twice in each component function, abbreviate $\cos t$ by C and $\sin t$ by S, and obtain :

$$x'(t) = ce^t \begin{pmatrix} 3S + C + 3tS - tC \\ -tS \\ S + C + tS \\ S - tC \end{pmatrix} + ce^t \begin{pmatrix} 3C - C + 3S - C + 3tC + tS \\ -S - tC \\ C - S + S + tC \\ C - C + tS \end{pmatrix} =$$

$$= ce^t \begin{pmatrix} 5S + 3C + 4tS + 2tC \\ -S - tC - tS \\ S + 2C + tS + tC \\ S - tC + tS \end{pmatrix}.$$

And

$$Ax = ce^t \begin{pmatrix} 4 & 7 & -1 & -6 \\ -2 & -3 & 2 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 3S + C + 3tS - tC \\ -tS \\ S + C + tS \\ S - tC \end{pmatrix} = ce^t \cdot$$

$$\begin{pmatrix} 12S + 4C + 12tS - 4tC - 7tS - S - C - tS - 6S + 6tC \\ -6S - 2C - 6tS + 2tC + 3tS + 2S + 2C + 2tS + 3S - 3tC \\ -tS + 2S + 2C + 2tS - S + tC \\ tS + S - tC \end{pmatrix}$$

$$= ce^t \begin{pmatrix} 5S + 3C + 4tS + 2tC \\ -S - tC - tS \\ S + 2C + tS + tC \\ S - tC + tS \end{pmatrix} \text{ as before.}$$

(b) In Example 10 we have computed the solution as $x(t) = ce^{-t}(1 + t) + ke^{3t}$. Consequently $x'(t) = -cte^{-t} + 3ke^{3t}$, $x''(t) = -ce^{-t} + cte^{-t} + 9ke^{3t}$, and $x'''(t) = 2ce^{-t} - cte^{-t} + 27ke^{3t}$. Therefore $x''' - x'' - 5x' - 3x = 0$.

7. The solution $w(t)$ of $w'(t) =$ diag$(J(2,2), J(-1,1,2))w(t)$ in terms of the Jordan basis $\mathcal{U}$ of A is given according to (**??**) and (**??**) by $w(t) = \begin{pmatrix} ce^{2t} \\ cte^{2t} \\ ke^{-t}\sin t \\ ke^{-t}\cos t \end{pmatrix}$.

Since the solution $x(t)$ of $x' = Ax$ is related to w and the Jordan normal form $J = U^{-1}AU$ of A according to the formula $x = Uw$, we need to compute the eigenvector/principal vector matrix U for A and J next.

With $B = A - 2I$ we compute $B^2 = \begin{pmatrix} -12 & -10 & -36 & 14 \\ -14 & -20 & -42 & 8 \\ 14 & 20 & 42 & -8 \\ 2 & 10 & 6 & 6 \end{pmatrix}$ and its RREF $\begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Clearly $x^{(2)} =$

$\begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \in \ker((A-2I)^2)$ is a principal vector for $\lambda = 2$ of order two since $(A-2I)x^{(2)} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix} = x^{(1)}$ is an eigenvector of A.

For the complex root $\lambda = -1+i$ we look at the kernel of $A-(-1+i)I$ via the RREF of this matrix: $\begin{pmatrix} 1 & 0 & 0 & -1+i \\ 0 & 1 & 0 & i \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \end{pmatrix}$. A complex eigenvector for $\lambda = -1+i$ is the vector $\begin{pmatrix} 1-i \\ -i \\ i \\ 1 \end{pmatrix}$.

Thus $U = \begin{pmatrix} 2 & 3 & 1 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ transforms A to its Jordan form J. Note that the columns of U contain the principal vector chain in descending order for $\lambda = 2$ first and then the real and the complex part vectors for the complex eigenvalue follow.

Therefore $x(t) = \begin{pmatrix} 2 & 3 & 1 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} w(t)$ $= Uw(t)$ solves the original DE $x' = Ax$. Here $x(t)$ computes to be

$$\begin{pmatrix} ce^{2t}(2+3t) + ke^{-t}(\sin t - \cos t) \\ -ce^{2t} - ke^{-t}\cos t \\ -cte^{2t} + ke^{-t}\cos t \\ ce^{2t} + ke^{-t}\sin t \end{pmatrix}.$$

9. [The solution to this problem goes along the same path as that of Problem 7 above (and built on that of Problem 1 of 14.2.P). For brevity, here are the main steps of a solution:]

B has the double complex conjugate eigenvalue pair $1 \pm 2i$.

The RREF of $(B-(1+2i)I)^2$ is $\begin{pmatrix} 1 & 0 & 1+i & -4-2i \\ 0 & 1 & 1 & -5+i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $x^{(2)} = \begin{pmatrix} 4+2i \\ 5-i \\ 0 \\ 1 \end{pmatrix}$ is a complex principal vector of order 2 for $1+2i$ since $(B-(1+2i)I)x^{(2)} = \begin{pmatrix} 5-i \\ 12-5i \\ 12-5i \\ 5-i \end{pmatrix} =: x^{(1)} \in \ker(B-(1+2i)I)$ is an eigenvector for $1+2i$.

From (**??**) we obtain the real Jordan basis for B from the real and imaginary parts vectors of $x^{(2)}$ and $x^{(1)}$: $X = \begin{pmatrix} 4 & 2 & 5 & -1 \\ 5 & -1 & 12 & -5 \\ 0 & 0 & 12 & -5 \\ 1 & 0 & 5 & -1 \end{pmatrix}$ transforms B to its real Jordan normal form $J = X^{-1}BX = J(1,2,4)$ that is comprised of a single 4-dimensional real Jordan block for $1+2i$.

Next, according to (**??**), the solution to the Jordan form DE $w'(t) = J(1,2,4)w(t)$ is given as $w(t) = ce^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \\ t\sin(2t) \\ t\cos(2t) \end{pmatrix}$. Finally

$$x(t) = Xw(t) = ce^t \begin{pmatrix} 4S + 2C + 5tS - tC \\ 5S - C + 12tS - 5tC \\ 12tS - 5tC \\ S + 5tS - tC \end{pmatrix}$$

solves the original DE $x'(t) = Bx(t)$. Here we have abbreviated $\sin(2t)$ by S and $\cos(2t)$ by C.

11. Following Section 14.3 (b), we form the associated 5^{th} degree polynomial $p(r) = r^5 - 5r^4 + 5r^3 + 5r^2 - 6r$. Its roots are -1, 0, 1, 2, and 3. The associated transposed companion matrix $C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 6 & -5 & -5 & 5 \end{pmatrix}$ is diagonalizable since p has distinct roots. Thus $J = \text{diag}(-1,0,1,2,3) = X^{-1}CX$ for the matrix X with the eigenvectors of C as its columns. For $\lambda = -1$, an eigenvector is $x_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$; for $\lambda = 2$ it is $x_2 = e_1$; for $\lambda = 1$ it is x_3, the vector of all ones; for $\lambda = 2$ it is $x_4 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{pmatrix}$; and for $\lambda = 3$ we have $x_5 = \begin{pmatrix} 1 \\ 3 \\ 9 \\ 27 \\ 81 \end{pmatrix}$ as the corresponding eigenvector of C.

The Jordan normal form solution $w(t)$ of $w'(t) = Jw(t)$ has the totally separated form $w(t) = \begin{pmatrix} ce^{-t} \\ d \\ ke^{t} \\ \ell e^{2t} \\ me^{3t} \end{pmatrix}$ since J is diagonal. And the solution to the original 5^{th} order DE is equal to the first entry $x(t) = y_1(t)$ of $y(t) = Xw(t)$, or to $x(t) = ce^{-t} + d + ke^{t} + \ell e^{2t} + me^{3t}$ for arbitrary constants c, d, k, ℓ, m, since the first row of X is the vector of all ones.

14.R **p. ??**

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1. (a) See Theorem 7.2.
(b) See the Determinant Proposition, part (12), in Section 8.1.
(c), ..., (h) Similar matrices have the same characteristic and minimal polynomials.

3. (a) Each individual Jordan block is associated with a principal vector of order equal to the size of the block.
(b) If $\lambda \in \mathbb{R}$, then the above statement remains true. But if $\lambda \notin \mathbb{R}$, then the index of $\lambda = a + bi$ is equal to half the size m of the largest real Jordan block $J(a, b, 2m)$ associated with λ.

5. Since $(A - \lambda I)^k x = 0$ and $(A - \lambda I)^{k-1} x \neq 0$, we likewise have $(A - \lambda I)^{k-2} y = (A - \lambda I)^k x = 0$ and $(A - \lambda I)^{k-3} y = (A - \lambda I)^{k-1} x \neq 0$, provided $k > 2$.

7. 1 by 1 matrices are the only such matrices.

9. $A = \text{diag}(J(0,3), J(-2,4))$ and $B = \text{diag}(J(0,3), J(0,2), J(-2,4), J(-2,3))$ for example, or any matrices similar to these.

11. To find the minimal sized matrix B with the same minimal polynomial $(x-2)^4(x+1)^5 x$ as J, we must only allow one properly sized Jordan block for each distinct eigenvalue, or B must be similar to $\text{diag}(J(2,4), J(-1,5), J(0,1))$. This B has size $4 + 5 + 1 = 10$ by 10.

13. Clearly A^2 is similar to J^2 if A is to J, which we assume to be the Jordan normal form of A. If all eigenvalues of A are nonzero, i.e., if A is nonsingular, then A^2 has the same Jordan structure as A for the squared eigenvalues of A from the previous problem. If A is singular, i.e., if the Jordan normal form J of A has a Jordan block of the form $J(0,k)$, then $J(0,k)^2$ is similar to the matrix $\text{diag}(J(0,k/2), J(0,k/2))$ if k is even, and to $\text{diag}(J(0,(k+1)/2), J(0,(k-1)/2))$ if k is odd. This specifies the Jordan form of A^2 in terms of the Jordan form J of A completely.

Appendix D

A.D.P **p. ??**

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1. $f(x,y) = x^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1/2 \\ 0 & 1/2 & 4 \end{pmatrix} = x^T S x$.
But S is not positive definite since $e_2^T S e_2 =$

-1 and therefore the bilinear form f does not define a norm on $\mathbb{R}^3$.

3. (a) All example functions, except (5) are bilinear forms.
(b) We need to check whether ...$(x, x) > 0$ for all $x \neq 0$:
In (1), $h(e_1, e_1) = 0$, i.e., h does not define an inner product.
In (2) the defining matrix $\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$ can be replaced by the symmetric matrix $\begin{pmatrix} 4 & 3/2 \\ 3/2 & 1 \end{pmatrix}$ which has two positive real eigenvalues, hence is positive definite. This makes h an inner product.
In (3) $\ell(e_2, e_2) = -1$ contradicts ℓ being an inner product.
In (4) the defining matrix is positive definite and thus m is an inner product.
In (6) $p(e_1, e_1) = 0$, hence p cannot be an inner product.

5. If the weight function $w > 0$ and $f \neq 0$, then $\langle f, f\rangle = \int_0^1 f^2 w\ dx > 0$ since $f^2 w > 0$ on at least one subinterval of $[0, 1]$, establishing property (d) of an inner product. The other defining properties (a), (b), and (c) follow readily from the linearity of integrals.

7. We take $w(x) := x + 1 > 0$ on $[0, 1)$. Then $\int_0^1 f(x)g(x)w(x)\ dx = \int_0^1 (x-1/2)(x+1)\ dx \neq 0$ and thus with respect to the inner product $\langle f, g\rangle = \int_0^1 fgw\ dx$ the two given functions f and g are not perpendicular.

9. First level Gram–Schmidt:
$v_1 = f = 1$;
$v_2 = \langle v_1, v_1\rangle g - \langle g, v_1\rangle v_1 = x - 1/2$;
$v_3 = \langle v_1, v_1\rangle h - \langle h, v_1\rangle v_1 = x^2 - 1/3$;
Second level Gram–Schmidt:
$v_3 = \langle v_2, v_2\rangle v_3 - \langle v_3, v_2\rangle v_2 = \frac{x^2}{12} - \frac{3x}{40} + \frac{7}{720}$.
Students should check that the finally computed v_i are mutually orthogonal.

11. We verify conditions (1), (2), and (4) of Proposition 4:
(1) and (2) are obvious; for (4) we use the triangle inequality for numbers n times:

$$\|x+y\|_1 = \sum_i |x_i + y_i| \leq \sum_i (|x_i| + |y_i|) = \sum |x_i| + \sum_i |y_i| = \|x\|_1 + \|y\|_1.$$

$\|x\|_1$ is not an induced norm, because it violates the parallelogram law: For $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have $x + y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $x - y = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ with $\|x\|_1^2 = 4$, $\|y\|_1^2 = 1$, $\|x+y\|_1^2 = 9$, and $\|x-y\|_1^2 = 9$. Therefore $\frac{1}{2}\left(\|x + y\|_1^2 + \|x - y\|_1^2\right) = 9 \neq 5 = \|x\|_1^2 + \|y\|_1^2$.

13. (a) From the parellelogram identity $\|v\|^2 = \frac{1}{2}\left(\|u + v\|^2 + \|u - v\|^2\right) - \|u\|^2 = \frac{1}{2}(36 + 25) - 16 = 15.5$, or $\|v\| = \sqrt{15.5}$. Note that $d(u, v) = \|u - v\| = 5$ as given.
(b) If $\|u\| = 7$, then from part (a) $\|v\|^2 = \frac{61}{2} - 49 < 0$ which is impossible.

15. If $x = \sum_{i=1}^k \alpha_i u_i$ then $\|x\|^2 = \langle \sum_{i=1}^k \alpha_i u, \sum_{j=1}^k \alpha_j u_j\rangle = \sum_{i=1}^k \alpha_i \langle u_i, \sum_{j=1}^k \alpha_j u_j\rangle = \sum_{i=1}^k \alpha_i \langle u_i, \alpha_i u_i\rangle = \sum_{i=1}^k |\alpha_i|^2$. And $\langle x, u_j\rangle = \langle \sum_{i=1}^k \alpha_i u_i, u_j\rangle = \alpha_j$ completes one direction of the statement.
Conversely assume that V has an ONB of the form $\{u_1, ..., u_k, u_{k+1}, ..., u_n\}$ where the first k basis vectors are as given. Then for any $x \in V$ we have $x = \sum_{i=1}^n \alpha_i u_i$. Consequently $\|x\|^2 = \langle x, x\rangle = \langle \sum_{i=1}^n \alpha_i u, \sum_{j=1}^n \alpha_j u_j\rangle = \sum_{i=1}^n |\alpha_i|^2$. If we assume that $\|x\|^2 = \sum_{i=1}^k |\alpha_i|^2$, then $\alpha_j = 0$ for $j = k + 1, ..., n$, or $x \in \text{span}\{u_1, ..., u_k\}$.

17. (a) $d(x, y) = \|x - y\| = |-1|\ \|x - y\| = \|y - x\| = d(y, x)$.
(b) $d(x, y) = \|x-y\| = \|(x-z)-(y-z)\| \leq \|x - z\| + \|y - z\| = d(x, z) + d(y, z)$.
(c) $\|x - y\|_2 = \sqrt{1^2 + 4^2 + 5^2 + 0^2} = \sqrt{52}$.
$\|x - y\|_\infty = \max\{|x_i - y_i|\} = 5$.

19. (a) $\|x\| = 4,\ \|y\| = \sqrt{6}$.
(b) $d(x,y) = \|x-y\| = \sqrt{20}$.
(c) $\|y-x\| = d(x,y) = \sqrt{20}$ from part (b).
(d) $\|x+y\|^2 = 24$.
(e) $\cos\angle(x,y) = \dfrac{x^*y}{\|x\|\,\|y\|} = \dfrac{1-3i}{4\sqrt{6}}$ and $\angle(x,y) = \text{complex arccos}\left(\frac{1-3i}{4\sqrt{6}}\right)$. [Complex trigonometric functions are defined over $\mathbb{C}$ in books and courses on complex analysis.]

21. (a) Write out $f(p+r,q)$, $f(p,q)$, and $f(r,q)$ and compare. likewise for $f(p,q+s)$.
(b) Since a polynomial of degree 0, 1, or 2 can have at most two zeros unless it is the zero polynomial, the induced function $N(p) := \sqrt{p^2(1)+2p^2(2)+3p^2(3)} : \mathcal{P}_2 \to \mathbb{R}$ satisfies property (d) of Definition 1. The properties (a) through (c) are obviously true. Thus N is a norm on $\mathcal{P}_2$ by Proposition 4.
(c) N is no longer definite for polynomials of degree exceeding 3 since such polynomials may have the zeros 1, 2, and 3 without being the zero function themselves. So $N(p) = 0$ does no longer imply $p = 0 \in \mathcal{P}_m$ for $m > 2$.
(d) $N(2x^2-3x) = \|2x_2-3x\| = \sqrt{(2-3)^2+2(6-6)^2+3(18-9)^2} = \sqrt{28}$. $1-x^2-x^3+4x \in \mathcal{P}_3$ and $1-x^2-x^3+4x \notin \mathcal{P}_2$. But N is no longer a norm in $\mathcal{P}_3$; therefore asking for the length or the "induced norm" of the second polynomial makes no sense.

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