

Contents

Solutions

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Solutions to Selected Problems of the Web Chapters

Chapter 14

14.1.P p. ??
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- The two 5 by 5 matrices comprised of zeros everywhere, except for a 1 in positions (1,5) or (1,2) both have the 5 fold eigenvalue zero, but a 1-dimensional kernel, or only 4 corresponding linearly independent eigenvectors. According to Chapter 9.1 or 9.1.D they cannot be diagonalized.
- For a non-diagonalizable example, expand the solution of problem 1. to 12 by 12. For a diagonalizable triangular matrix, use an arbitrary 12 by 12 upper triangular matrix with distinct diagonal entries. Such a matrix has 12 distinct eigenvalues and is therefore diagonalizable.
- Look at the eigenspace for $\lambda = -1$, or at the kernel of $A + I = \begin{pmatrix} 1 & 0 & 1 \\ -3 & 0 & -3 \\ -1 & 0 & -1 \end{pmatrix}$. It has dimension two, so there are two linearly independent eigenvectors for $\lambda = -1$. Now compute $\text{trace}(A) = -3 = -1 - 1 + \lambda_3$, making $\lambda = -1$ a triple eigenvalue for A . Therefore the Jordan normal form of A is $J = \text{diag}(J(-1, 2), J(-1, 1))$.

7. For Example 4: $\begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix}$;
 for Problem 5: $\begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix}$.

- We can find the eigenvalues of $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ from $\det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ -b & a - \lambda \end{pmatrix} = (a - \lambda)^2 + b^2 = 0$ as $\lambda_{1,2} = a \pm bi$. We find the complex eigenvectors by looking at $A - (a + bi)I = \begin{pmatrix} -bi & b \\ -b & -bi \end{pmatrix}$ as $\begin{pmatrix} 1 \\ i \end{pmatrix}$, and from $A - (a - bi)I = \begin{pmatrix} bi & b \\ -b & bi \end{pmatrix}$ as $\begin{pmatrix} -1 \\ i \end{pmatrix}$. Now form the eigenvector column matrix $X = \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}$ with $X^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$. Then $X^{-1}AX = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} = \begin{pmatrix} a + bi & 0 \\ 0 & a - bi \end{pmatrix}$.
- Since $\text{trace}(C) = -6 - 2 + 10 = 2 = \lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_1 = -1$ and $\lambda_2 = 2$ are given, we must have $\lambda_3 = 1$. Therefore C has three

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distinct eigenvalues and is diagonalizable according to Chapter 9. The Jordan normal form of C is $\text{diag}(-1, 2, 1)$.

13. Let the size of A be n by n . 4 foldness of an eigenvalue refers to the algebraic multiplicity; therefore $n = 4$. Since $\text{geom. mult.} \leq \text{alg. mult.}$, we may have 1, or 2, or 3, or 4 linearly independent eigenvectors for this single eigenvalue. Giving rise to the follow-

ing Jordan diagrams in turn:

•
• in case
•
•
•
• or
• •

of geometric multiplicity 1, or

• • in case of geometric multiplicity 2,
• •
or • • • in case of geometric multiplicity 3, or to • • • • if A is diagonalizable.

15. Here $n = 6$ and $\text{diag}(J(\lambda, n_1), J(\lambda, n_2), \dots, J(\lambda, n_k))$ with $k \leq 6$ and $\sum_{i=1}^k n_i = 6$.

17. The Jordan normal form J of A has three separate Jordan blocks for λ . One of these must be 4 by 4, while the other two have sizes that will add up to 8. Therefore J will contain the string $J(\lambda, 4), J(\lambda, n_1), J(\lambda, n_2)$ with $n_1 + n_2 = 4$ and $n_i \geq 1$. This gives us two possibilities: $n_1 = 1, n_2 = 3$ or $n_1 = 2 = n_2$. The third possible case $n_1 = 3$ and $n_2 = 3$ is only a reordering of the first mentioned one. So for λ there are precisely two variations possible in the Jordan normal form for A .

19. The maximal index must be six since there must be seven linearly independent eigenvectors for our 12 fold eigenvalue. The minimal index is two, provided 5 Jordan blocks of size 2 occur together with two 1 by 1 Jordan blocks. Any index in between 2 and 6 can also occur.

21. (a) A 's Jordan normal form has one 2 by 2 block and 1 by 1 blocks else.

(b) A 's Jordan normal form has one 3 by 3 block and 1 by 1 blocks else, or it has two 2 by 2 blocks and 1 by 1 blocks else.

(c) There are three Jordan blocks for λ . Their possible sizes are 1, 1, 4; or 1, 2, 3; or 2, 2, 2, respectively.

23. (a) Simply multiply $J^{-1} \cdot J$ out for the matrix J^{-1} as given and you should obtain I .

(b) $J(\lambda, k)^{-1}$ is tridiagonal with its n fold eigenvalue $\frac{1}{\lambda}$ appearing on the diagonal. Since $\text{rank}(J(\lambda, k)^{-1} - \frac{1}{\lambda}I) = k - 1$ from the formula for the inverse in part (a), $J(\lambda, k)^{-1}$ must be similar to a single Jordan block of the form $J(\frac{1}{\lambda}, k)$.

And if $X^{-1}AX = J$ is a Jordan form matrix, then $J^{-1} = X^{-1}A^{-1}X$, or A^{-1} and J^{-1} are also similar, hence have the same Jordan normal form. But if $J = \text{diag}(J_i)$, then $J^{-1} = \text{diag}(J_i^{-1})$, so the Jordan normal form of the inverse of a nonsingular matrix A has the same Jordan normal form structure as A , except instead of the eigenvalues λ_i for A it has the eigenvalues $\frac{1}{\lambda_i}$.

25. We have $\text{trace}(A) = \lambda_1 + \lambda_2 = 4$ and $\det(A) = \lambda_1\lambda_2 = 4$, so that $\lambda_1 = \lambda_2 = 2$ is a double eigenvalue for A . Looking at the kernel of $A - 2I$ we find its dimension equal to 1. Therefore A is not diagonalizable with the Jordan normal form $J = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$.

27. (a) Every matrix $A = A(\alpha)$ has only one linearly independent eigenvector for its double eigenvalue λ since $\text{rank}(A - \lambda I) = \text{rank} \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} = 1$. Therefore each $A(\lambda)$ is

similar to $J(\lambda, 2) = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$. If $\alpha = 0$, then $A = \lambda I = \text{diag}(J(\lambda, 1), J(\lambda, 1))$ is not similar to $J(\lambda, 2)$.

(b) Analogous to part (a), if $\prod \alpha_i \neq 0$, then $\text{rank}(A - \lambda I) = 1$ and A is similar to a single Jordan block $J(\lambda, k)$. If at least one off diagonal entry $\lambda_j = 0$, then $\text{rank}(A - \lambda I) \geq 2$ and A is similar to a Jordan form with at least two Jordan blocks for λ .

(c) Both $B - \lambda I$ and $C - \lambda I$ have rank 1 for their double eigenvalue λ . Hence both B and C have the same Jordan normal form

$J = J(\lambda, 2) = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$. I.e., $X^{-1}BX = J$ and $Y^{-1}CY = J$ for two matrices X and Y , making $C = YX^{-1}BXY^{-1} = (XY^{-1})^{-1}B(XY^{-1})$ similar.

29. (a) $Ae_k = ae_k$.

(b) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ and $B =$

$\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ are both diagonaliz-

able, but $\begin{pmatrix} -3 & 0 & 0 & 0 \\ \alpha & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is not for any $\alpha \neq 0$.

(c) In our triangular matrix examples, nonzero entries in columns and rows that have the same diagonal entry seem to determine diagonalizability.

31. (a) $A = \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix}$ has one complex eigenvalue $\lambda = i$ and one real eigenvalue $\lambda = 1$.

(b), (c) $A = \begin{pmatrix} -1+i & 3 \\ i & 2-i \end{pmatrix}$ has determinant $(-1+i)(2-i) - 3i = -1$ and trace 1. Thus its characteristic polynomial is $f_A(x) = x^2 + x - 1$ with the roots $x_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} \in \mathbb{R}$. These are the eigenvalues of $A \notin \mathbb{R}^{n,n}$.

14.2.P p. ??

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1. We can use vector iteration of Section

9.1 with $x = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$ for example and

obtain $\begin{pmatrix} \vdots & \vdots & \vdots \\ x & Ax & A^2x & A^3x & A^4x \\ \vdots & \vdots & \vdots \end{pmatrix} =$

$\begin{pmatrix} 0 & -10 & -10 & -20 & -60 \\ 2 & -1 & -2 & 6 & -8 \\ -1 & 3 & 6 & 2 & 24 \\ 1 & -3 & 4 & -2 & -4 \end{pmatrix}$ with

the reduced row echelon form $\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Therefore according

to Section 9.1, the roots of the polynomial $x^3 - 2x - 4$ are three eigenvalues of A . The roots are 2 (found by guessing) and $-1 \pm i$ (then found by long division and the quadratic formula). The fourth root is 2 from the trace condition. A row reduction of $A - 2I_4$ shows this matrix to have rank 3. Thus A has only one eigenvector for its double real eigenvalue 2. A therefore has the real Jordan normal form $J = \text{diag}(J(2, 2), J(-1, 1, 2))$.

To start out with computing $\det(A - \lambda I_4)$ seems much harder.

3. For the real matrix A all complex eigenvalues such as λ and μ come doubly as λ, μ and as $\bar{\lambda}, \bar{\mu}$. Thus the real principal subspace associated with λ is 6 dimensional, and the one for μ has dimension 4. Since the size of A is 13 by 13, this makes ν have algebraic multiplicity 3 as a root of the characteristic polynomial of A . Finally ν can have the geometric multiplicities 1, 2, or 3.

The possible real Jordan forms of A consist of any combination from the following three Jordan block groups: for $\lambda = 2 + 3i$:

$J(2, 3, 6), \text{diag}(J(2, 3, 4), J(2, 3, 2)),$ or $\text{diag}(J(2, 3, 2), J(2, 3, 2), J(2, 3, 2));$

for $\mu = 1 - 2i$: $J(1, -2, 4)$ or $\text{diag}(J(1, -2, 2), J(1, -2, 2));$

and for $\nu = 7$: $J(7, 3), \text{diag}(J(7, 2), J(7, 1)),$ or $\text{diag}(7, 7, 7)$.

5. (a) $Je_k = \lambda e_k$.

(b) The index of λ is k .

(c) Since $(J - \lambda I)^k = O_k$ and $(J - \lambda I)^{k-1}$ is the matrix of all zeros except for a 1 in position $(k,1)$, and since a principal vector of index k for A must lie in the kernel of $(J - \lambda I)^k$ but not in the kernel of $(J - \lambda I)^{k-1}$, this principal vector must be a multiple of e_1 .

(d) $e_{k-j+1} \in P_j(\lambda)$.

7. $vA =$

$$\begin{pmatrix} 1 & z & z^2 & \dots & z^{n-1} \end{pmatrix} \begin{pmatrix} 0 & & & & -a_0 \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & -a_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} z & z^2 & \dots & -a_0 - a_1z - \dots - a_{n-1}z^{n-1} \end{pmatrix}$$

$$= z \begin{pmatrix} 1 & z & z^2 & \dots & z^{n-1} \end{pmatrix} \text{ since } z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0 = 0.$$

9. The given matrix $B = \begin{pmatrix} -2 & 2i \\ 1 & 3 \end{pmatrix}$ has two distinct complex eigenvalues since its characteristic polynomial is $\lambda^2 - \lambda - 6 - 2i$. Thus B is similar over \mathbb{C} to a complex diagonal matrix $\text{diag}(\lambda_1, \lambda_2)$, or $X^{-1}BX = \text{diag}(\lambda_1, \lambda_2)$. Thus $B = (X \text{diag}(\lambda_1, \lambda_2)X^{-1})$ expresses B as the product of two complex symmetric matrices.

11. $A = \begin{pmatrix} -2 & 1 \\ 4 & -4 \end{pmatrix} =$

$$= \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix}.$$

13. Note that

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} a & -b \\ -b & a \end{pmatrix} \text{ and that } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

14.3.P p. ??

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1. (a) For Example 9 we use the product rule of calculus twice in each component function, abbreviate $\cos t$ by C and $\sin t$ by S , and obtain :

$$x'(t) = ce^t \begin{pmatrix} 3S + C + 3tS - tC \\ -tS \\ S + C + tS \\ S - tC \end{pmatrix} +$$

$$ce^t \begin{pmatrix} 3C - C + 3S - C + 3tC + tS \\ -S - tC \\ C - S + S + tC \\ C - C + tS \end{pmatrix} =$$

$$= ce^t \begin{pmatrix} 5S + 3C + 4tS + 2tC \\ -S - tC - tS \\ S + 2C + tS + tC \\ S - tC + tS \end{pmatrix}.$$

And

$$Ax = ce^t \begin{pmatrix} 4 & 7 & -1 & -6 \\ -2 & -3 & 2 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

$$\cdot \begin{pmatrix} 3S + C + 3tS - tC \\ -tS \\ S + C + tS \\ S - tC \end{pmatrix} = ce^t.$$

$$\begin{pmatrix} 12S + 4C + 12tS - 4tC - 7tS - S - C - tS - 6S + 6tC \\ -6S - 2C - 6tS + 2tC + 3tS + 2S + 2C + 2tS + 3S - 3tC \\ -tS + 2S + 2C + 2tS - S + tC \\ tS + S - tC \end{pmatrix}$$

$$= ce^t \begin{pmatrix} 5S + 3C + 4tS + 2tC \\ -S - tC - tS \\ S + 2C + tS + tC \\ S - tC + tS \end{pmatrix} \text{ as before.}$$

(b) In Example 10 we have computed the solution as $x(t) = ce^{-t}(1 + t) + ke^{3t}$. Consequently $x'(t) = -cte^{-t} + 3ke^{3t}$, $x''(t) = -ce^{-t} + tce^{-t} + 9ke^{3t}$, and $x'''(t) = 2ce^{-t} - tce^{-t} + 27ke^{3t}$. Therefore $x''' - x'' - 5x' - 3x = 0$.

7. The solution $w(t)$ of $w'(t) = \text{diag}(J(2, 2), J(-1, 1, 2))w(t)$ in terms of the Jordan basis \mathcal{U} of A is given according to

$$w(t) = \begin{pmatrix} ce^{2t} \\ tce^{2t} \\ ke^{-t} \sin t \\ ke^{-t} \cos t \end{pmatrix}.$$

Since the solution $x(t)$ of $x' = Ax$ is related to w and the Jordan normal form $J = U^{-1}AU$ of A according to the formula $x = Uw$, we need to compute the eigenvector/principal vector matrix U for A and J next.

With $B = A - 2I$ we compute $B^2 =$

$$\begin{pmatrix} -12 & -10 & -36 & 14 \\ -14 & -20 & -42 & 8 \\ 14 & 20 & 42 & -8 \\ 2 & 10 & 6 & 6 \end{pmatrix} \text{ and its RREF}$$

$$\begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{ Clearly } x^{(2)} =$$

$\begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \in \ker((A - 2I)^2)$ is a principal vector for $\lambda = 2$ of order two since

$$(A - 2I)x^{(2)} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix} = x^{(1)}$$
 is an eigen-

vector of A .

For the complex root $\lambda = -1 + i$ we look at the kernel of $A - (-1 + i)I$ via the RREF of

this matrix: $\begin{pmatrix} 1 & 0 & 0 & -1 + i \\ 0 & 1 & 0 & i \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \end{pmatrix}$. A com-

plex eigenvector for $\lambda = -1 + i$ is the vector

$$\begin{pmatrix} 1 - i \\ -i \\ i \\ 1 \end{pmatrix}.$$

Thus $U = \begin{pmatrix} 2 & 3 & 1 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ trans-

forms A to its Jordan form J . Note that the columns of U contain the principal vector chain in descending order for $\lambda = 2$ first and then the real and the complex part vectors for the complex eigenvalue follow.

Therefore $x(t) = \begin{pmatrix} 2 & 3 & 1 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} w(t)$

$= Uw(t)$ solves the original DE $x' = Ax$. Here $x(t)$ computes to be

$$\begin{pmatrix} ce^{2t}(2 + 3t) + ke^{-t}(\sin t - \cos t) \\ -ce^{2t} - ke^{-t} \cos t \\ -cte^{2t} + ke^{-t} \cos t \\ ce^{2t} + ke^{-t} \sin t \end{pmatrix}.$$

9. [The solution to this problem goes along the same path as that of Problem 7 above (and built on that of Problem 1 of 14.2.P). For brevity, here are the main steps of a solution:]

B has the double complex conjugate eigenvalue pair $1 \pm 2i$.

The RREF of $(B - (1 + 2i)I)^2$ is

$$\begin{pmatrix} 1 & 0 & 1 + i & -4 - 2i \\ 0 & 1 & 1 & -5 + i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } x^{(2)} =$$

$$\begin{pmatrix} 4 + 2i \\ 5 - i \\ 0 \\ 1 \end{pmatrix}$$

is a complex principal vector of

$$\text{order 2 for } 1 + 2i \text{ since } (B - (1 + 2i)I)x^{(2)} = \begin{pmatrix} 5 - i \\ 12 - 5i \\ 12 - 5i \\ 5 - i \end{pmatrix} =: x^{(1)} \in \ker(B - (1 + 2i)I)$$

is an eigenvector for $1 + 2i$.

From (??) we obtain the real Jordan basis for B from the real and imaginary parts vectors of $x^{(2)}$ and $x^{(1)}$: $X =$

$$\begin{pmatrix} 4 & 2 & 5 & -1 \\ 5 & -1 & 12 & -5 \\ 0 & 0 & 12 & -5 \\ 1 & 0 & 5 & -1 \end{pmatrix}$$

transforms B to its real Jordan normal form $J = X^{-1}BX = J(1, 2, 4)$ that is comprised of a single 4-dimensional real Jordan block for $1 + 2i$.

Next, according to (??), the solution to the Jordan form DE $w'(t) = J(1, 2, 4)w(t)$ is

$$\text{given as } w(t) = ce^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \\ t \sin(2t) \\ t \cos(2t) \end{pmatrix}.$$

$$x(t) = Xw(t) = ce^t \begin{pmatrix} 4S + 2C + 5tS - tC \\ 5S - C + 12tS - 5tC \\ 12tS - 5tC \\ S + 5tS - tC \end{pmatrix}$$

solves the original DE $x'(t) = Bx(t)$. Here we have abbreviated $\sin(2t)$ by S and $\cos(2t)$ by C .

11. Following Section 14.3 (b), we form the associated 5th degree polynomial $p(r) = r^5 - 5r^4 + 5r^3 + 5r^2 - 6r$. Its roots are $-1, 0, 1, 2,$ and 3 . The associated transposed compan-

$$\text{ion matrix } C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 6 & -5 & -5 & 5 \end{pmatrix}$$

is diagonalizable since p has distinct roots. Thus $J = \text{diag}(-1, 0, 1, 2, 3) = X^{-1}CX$ for the matrix X with the eigenvectors of C as

its columns. For $\lambda = -1$, an eigenvector is $x_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$; for $\lambda = 2$ it is $x_2 = e_1$; for $\lambda = 1$ it is x_3 , the vector of all ones; for $\lambda = 2$ it is $x_4 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{pmatrix}$; and for $\lambda = 3$ we have $x_5 = \begin{pmatrix} 1 \\ 3 \\ 9 \\ 27 \\ 81 \end{pmatrix}$ as the corresponding eigenvector of C .

The Jordan normal form solution $w(t)$ of $w'(t) = Jw(t)$ has the totally separated form $w(t) = \begin{pmatrix} ce^{-t} \\ d \\ ke^t \\ \ell e^{2t} \\ me^{3t} \end{pmatrix}$ since J is diagonal. And

the solution to the original 5th order DE is equal to the first entry $x(t) = y_1(t)$ of $y(t) = Xw(t)$, or to $x(t) = ce^{-t} + d + ke^t + \ell e^{2t} + me^{3t}$ for arbitrary constants c, d, k, ℓ, m , since the first row of X is the vector of all ones.

14.R p. ??
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1. (a) See Theorem 7.2.
- (b) See the Determinant Proposition, part (12), in Section 8.1.
- (c), ..., (h) Similar matrices have the same characteristic and minimal polynomials.

3. (a) Each individual Jordan block is associated with a principal vector of order equal to the size of the block.
- (b) If $\lambda \in \mathbb{R}$, then the above statement remains true. But if $\lambda \notin \mathbb{R}$, then the index of $\lambda = a + bi$ is equal to half the size m of the largest real Jordan block $J(a, b, 2m)$ associated with λ .
5. Since $(A - \lambda I)^k x = 0$ and $(A - \lambda I)^{k-1} x \neq 0$, we likewise have $(A - \lambda I)^{k-2} y = (A - \lambda I)^k x = 0$ and $(A - \lambda I)^{k-3} y = (A - \lambda I)^{k-1} x \neq 0$, provided $k > 2$.
7. 1 by 1 matrices are the only such matrices.
9. $A = \text{diag}(J(0, 3), J(-2, 4))$ and $B = \text{diag}(J(0, 3), J(0, 2), J(-2, 4), J(-2, 3))$ for example, or any matrices similar to these.
11. To find the minimal sized matrix B with the same minimal polynomial $(x - 2)^4(x + 1)^5x$ as J , we must only allow one properly sized Jordan block for each distinct eigenvalue, or B must be similar to $\text{diag}(J(2, 4), J(-1, 5), J(0, 1))$. This B has size $4 + 5 + 1 = 10$ by 10.
13. Clearly A^2 is similar to J^2 if A is to J , which we assume to be the Jordan normal form of A . If all eigenvalues of A are nonzero, i.e., if A is nonsingular, then A^2 has the same Jordan structure as A for the squared eigenvalues of A from the previous problem. If A is singular, i.e., if the Jordan normal form J of A has a Jordan block of the form $J(0, k)$, then $J(0, k)^2$ is similar to the matrix $\text{diag}(J(0, k/2), J(0, k/2))$ if k is even, and to $\text{diag}(J(0, (k + 1)/2), J(0, (k - 1)/2))$ if k is odd. This specifies the Jordan form of A^2 in terms of the Jordan form J of A completely.

Appendix D

A.D.P p. ??
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1. $f(x, y) = x^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1/2 \\ 0 & 1/2 & 4 \end{pmatrix} = x^T S x$.
But S is not positive definite since $e_2^T S e_2 =$

-1 and therefore the bilinear form f does not define a norm on \mathbb{R}^3 .

3. (a) All example functions, except (5) are bilinear forms.

(b) We need to check whether $\dots(x, x) > 0$ for all $x \neq 0$:

In (1), $h(e_1, e_1) = 0$, i.e., h does not define an inner product.

In (2) the defining matrix $\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$ can be replaced by the symmetric matrix $\begin{pmatrix} 4 & 3/2 \\ 3/2 & 1 \end{pmatrix}$ which has two positive real eigenvalues, hence is positive definite. This makes h an inner product.

In (3) $\ell(e_2, e_2) = -1$ contradicts ℓ being an inner product.

In (4) the defining matrix is positive definite and thus m is an inner product.

In (6) $p(e_1, e_1) = 0$, hence p cannot be an inner product.

5. If the weight function $w > 0$ and $f \neq 0$, then $\langle f, f \rangle = \int_0^1 f^2 w \, dx > 0$ since $f^2 w > 0$ on at least one subinterval of $[0, 1]$, establishing property (d) of an inner product. The other defining properties (a), (b), and (c) follow readily from the linearity of integrals.

7. We take $w(x) := x + 1 > 0$ on $[0, 1]$. Then $\int_0^1 f(x)g(x)w(x) \, dx = \int_0^1 (x-1/2)(x+1) \, dx \neq 0$ and thus with respect to the inner product $\langle f, g \rangle = \int_0^1 f g w \, dx$ the two given functions f and g are not perpendicular.

9. First level Gram-Schmidt:

$$v_1 = f = 1;$$

$$v_2 = \langle v_1, v_1 \rangle g - \langle g, v_1 \rangle v_1 = x - 1/2;$$

$$v_3 = \langle v_1, v_1 \rangle h - \langle h, v_1 \rangle v_1 = x^2 - 1/3;$$

Second level Gram-Schmidt:

$$v_3 = \langle v_2, v_2 \rangle v_3 - \langle v_3, v_2 \rangle v_2 = \frac{x^2}{12} - \frac{3x}{40} + \frac{7}{720}.$$

Students should check that the finally computed v_i are mutually orthogonal.

11. We verify conditions (1), (2), and (4) of Proposition 4:

(1) and (2) are obvious; for (4) we use the triangle inequality for numbers n times:

$$\begin{aligned} \|x + y\|_1 &= \sum_i |x_i + y_i| \leq \sum_i (|x_i| + |y_i|) = \\ &= \sum_i |x_i| + \sum_i |y_i| = \|x\|_1 + \|y\|_1. \end{aligned}$$

$\|x\|_1$ is not an induced norm, because it violates the parallelogram law: For $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have $x + y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $x - y = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ with $\|x\|_1^2 = 4$, $\|y\|_1^2 = 1$, $\|x + y\|_1^2 = 9$, and $\|x - y\|_1^2 = 9$. Therefore $\frac{1}{2}(\|x + y\|_1^2 + \|x - y\|_1^2) = 9 \neq 5 = \|x\|_1^2 + \|y\|_1^2$.

13. (a) From the parallelogram identity $\|v\|^2 = \frac{1}{2}(\|u + v\|^2 + \|u - v\|^2) - \|u\|^2 = \frac{1}{2}(36 + 25) - 16 = 15.5$, or $\|v\| = \sqrt{15.5}$. Note that $d(u, v) = \|u - v\| = 5$ as given.

(b) If $\|u\| = 7$, then from part (a) $\|v\|^2 = \frac{61}{2} - 49 < 0$ which is impossible.

15. If $x = \sum_{i=1}^k \alpha_i u_i$ then $\|x\|^2 = \langle \sum_{i=1}^k \alpha_i u_i, \sum_{j=1}^k \alpha_j u_j \rangle = \sum_{i=1}^k \alpha_i \langle u_i, \sum_{j=1}^k \alpha_j u_j \rangle = \sum_{i=1}^k \alpha_i \langle u_i, \alpha_i u_i \rangle = \sum_{i=1}^k |\alpha_i|^2$. And $\langle x, u_j \rangle = \langle \sum_{i=1}^k \alpha_i u_i, u_j \rangle = \alpha_j$ completes one direction of the statement.

Conversely assume that V has an ONB of the form $\{u_1, \dots, u_k, u_{k+1}, \dots, u_n\}$ where the first k basis vectors are as given. Then for any $x \in V$ we have $x = \sum_{i=1}^n \alpha_i u_i$. Consequently $\|x\|^2 = \langle x, x \rangle = \langle \sum_{i=1}^n \alpha_i u_i, \sum_{j=1}^n \alpha_j u_j \rangle = \sum_{i=1}^n |\alpha_i|^2$. If we assume that $\|x\|^2 = \sum_{i=1}^k |\alpha_i|^2$, then $\alpha_j = 0$ for $j = k + 1, \dots, n$, or $x \in \text{span}\{u_1, \dots, u_k\}$.

17. (a) $d(x, y) = \|x - y\| = | -1 | \|x - y\| = \|y - x\| = d(y, x)$.

(b) $d(x, y) = \|x - y\| = \|(x - z) - (y - z)\| \leq \|x - z\| + \|y - z\| = d(x, z) + d(y, z)$.

(c) $\|x - y\|_2 = \sqrt{1^2 + 4^2 + 5^2 + 0^2} = \sqrt{52}$.

$\|x - y\|_\infty = \max\{|x_i - y_i|\} = 5$.

19. (a) $\|x\| = 4$, $\|y\| = \sqrt{6}$.
 (b) $d(x, y) = \|x - y\| = \sqrt{20}$.
 (c) $\|y - x\| = d(x, y) = \sqrt{20}$ from part (b).
 (d) $\|x + y\|^2 = 24$.
 (e) $\cos \angle(x, y) = \frac{x^*y}{\|x\| \|y\|} = \frac{1 - 3i}{4\sqrt{6}}$ and
 $\angle(x, y) = \text{complex arccos} \left(\frac{1-3i}{4\sqrt{6}} \right)$. [Complex trigonometric functions are defined over \mathbb{C} in books and courses on complex analysis.]
21. (a) Write out $f(p+r, q)$, $f(p, q)$, and $f(r, q)$ and compare. likewise for $f(p, q+s)$.
 (b) Since a polynomial of degree 0, 1, or 2 can have at most two zeros unless it is the zero polynomial, the induced function

$N(p) := \sqrt{p^2(1) + 2p^2(2) + 3p^2(3)} : \mathcal{P}_2 \rightarrow \mathbb{R}$ satisfies property (d) of Definition 1. The properties (a) through (c) are obviously true. Thus N is a norm on \mathcal{P}_2 by Proposition 4.
 (c) N is no longer definite for polynomials of degree exceeding 3 since such polynomials may have the zeros 1, 2, and 3 without being the zero function themselves. So $N(p) = 0$ does no longer imply $p = 0 \in \mathcal{P}_m$ for $m > 2$.
 (d) $N(2x^2 - 3x) = \|2x_2 - 3x\| = \sqrt{(2-3)^2 + 2(6-6)^2 + 3(18-9)^2} = \sqrt{28}$.
 $1-x^2-x^3+4x \in \mathcal{P}_3$ and $1-x^2-x^3+4x \notin \mathcal{P}_2$. But N is no longer a norm in \mathcal{P}_3 ; therefore asking for the length or the “induced norm” of the second polynomial makes no sense.

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